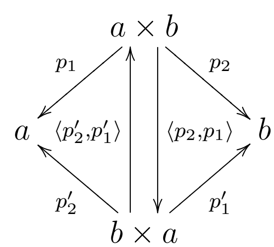
COSC 5P05 - Introduction to Lambda-Calculus

Term Test 2

Question 1 (10 marks): Let $\mathbb{C}$ be a category with products. Show that $a \times b \cong b \times a$ for all objects a and b of $\mathbb{C}$.

Solution: Consider the following diagram:



We have

$$\begin{aligned} p_1 \circ \langle p_2', p_1' \rangle \circ \langle p_2, p_1 \rangle &= p_2' \circ \langle p_2, p_1 \rangle \\ &= p_1, \\ p_2 \circ \langle p_2', p_1' \rangle \circ \langle p_2, p_1 \rangle &= p_1' \circ \langle p_2, p_1 \rangle \\ &= p_2, \end{aligned}$$

so that from the uniqueness of the product morphism $\langle p_2', p_1' \rangle \circ \langle p_2, p_1 \rangle = \mathrm{id}_{a \times b}$ follows. The equation $\langle p_2, p_1 \rangle \circ \langle p_2', p_1' \rangle = \mathrm{id}_{b \times a}$ can be shown analogously.

Question 2 (10 marks): Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be categories. Show that one can define a category $\mathbb{C}_1 \times \mathbb{C}_2$ whose objects are pairs (a, b) of objects a from $\mathbb{C}_1$ and b from $\mathbb{C}_2$ and whose morphisms are pairs of morphisms from $\mathbb{C}_1$ and

$\mathbb{C}_2$, that is $(f, g) \in (\mathbb{C}_1 \times \mathbb{C}_2)[(a, b), (c, d)]$ with $f \in \mathbb{C}_1[a, c]$ and $g \in \mathbb{C}_2[b, d]$.

Solution: Suppose $(f, g) : (a, b) \to (c, d)$ and $(h, k) : (c, d) \to (e, f)$ with $f : a \to c$, $g : b \to d$, $h : c \to e$ and $k : d \to f$. Then define $(h, k) \circ (f, g) = (h \circ f, k \circ g)$. Then we have

$$\begin{aligned}
(f, g) \circ (\mathrm{id}_a, \mathrm{id}_b) &= (f \circ \mathrm{id}_a, g \circ \mathrm{id}_b) \\
&= (f, g), \\
(\mathrm{id}_c, \mathrm{id}_d) \circ (f, g) &= (\mathrm{id}_c \circ f, \mathrm{id} \circ g) \\
&= (f, g),
\end{aligned}$$

i.e., $(\mathrm{id}_a, \mathrm{id}_b)$ is the identity on (a, b). Associativity is shown as follows

$$\begin{aligned}
((l, m) \circ (h, k)) \circ (f, g) &= (l \circ h, m \circ k) \circ (f, g) \\
&= (l \circ h \circ f, m \circ k \circ g), \\
(l, m) \circ ((h, k) \circ (f, g)) &= (l, m) \circ (h \circ f, k \circ g) \\
&= (l \circ h \circ f, m \circ k \circ g).
\end{aligned}$$

The last lines of each computation above is correct because composition in $\mathbb{C}_1$ and $\mathbb{C}_2$ is associative so that no brackets are needed.