COSC 5P05 - Introduction to Lambda-Calculus

Term Test 1

Question 1 (5 marks): Perform the following substitutions:

1. $[\lambda \mathtt{z} : \mathtt{A}.(\mathtt{u}\ \mathtt{z})/\mathtt{x}]\lambda \mathtt{y} : \mathtt{A} \to \mathtt{A}.(\mathtt{y}\ \mathtt{x})$,
2. $[\lambda \mathtt{z} : \mathtt{A}.(\mathtt{u}\ \mathtt{z})/\mathtt{y}]\lambda \mathtt{y} : \mathtt{A} \to \mathtt{A}.(\mathtt{y}\ \mathtt{x})$,
3. $[\lambda \mathtt{z} : \mathtt{A}.(\mathtt{u}\ \mathtt{z})/\mathtt{x}]\lambda \mathtt{u} : \mathtt{A} \to \mathtt{A}.(\mathtt{u}\ \mathtt{x})$.

Solution:

$$
\begin{aligned}
&[\lambda \mathtt{z} : \mathtt{A}.(\mathtt{u}\ \mathtt{z})/\mathtt{x}]\lambda \mathtt{y} : \mathtt{A} \to \mathtt{A}.(\mathtt{y}\ \mathtt{x})\\
&= \lambda \mathtt{y} : \mathtt{A} \to \mathtt{A}.(\mathtt{y}\ \lambda \mathtt{z} : \mathtt{A}.(\mathtt{u}\ \mathtt{z})),\\
&[\lambda \mathtt{z} : \mathtt{A}.(\mathtt{u}\ \mathtt{z})/\mathtt{y}]\lambda \mathtt{y} : \mathtt{A} \to \mathtt{A}.(\mathtt{y}\ \mathtt{x})\\
&= \lambda \mathtt{y} : \mathtt{A} \to \mathtt{A}.(\mathtt{y}\ \mathtt{x}),\\
&[\lambda \mathtt{z} : \mathtt{A}.(\mathtt{u}\ \mathtt{z})/\mathtt{x}]\lambda \mathtt{u} : \mathtt{A} \to \mathtt{A}.(\mathtt{u}\ \mathtt{x})\\
&= [\lambda \mathtt{z} : \mathtt{A}.(\mathtt{u}\ \mathtt{z})/\mathtt{x}]\lambda \mathtt{y} : \mathtt{A} \to \mathtt{A}.(\mathtt{y}\ \mathtt{x})\\
&= \lambda \mathtt{y} : \mathtt{A} \to \mathtt{A}.(\mathtt{y}\ \lambda \mathtt{z} : \mathtt{A}.(\mathtt{u}\ \mathtt{z})).
\end{aligned}
$$

Question 2 (8 marks): Find the normal form of the following λ-terms (show intermediate steps):

1. $\langle \mathtt{fst}(\mathtt{p}), \lambda \mathtt{x}{:}\mathtt{A}.\mathtt{x}\rangle$,

2. $(\lambda \mathtt{p}:\mathtt{A}\times\mathtt{B}.((\lambda \mathtt{x}:\mathtt{A}.\mathtt{x})\ \mathtt{fst}(\mathtt{p})))\ \langle \mathtt{y},\mathtt{z}\rangle$,

3. $(\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ ((\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ \mathtt{f})$.

Remark: In the reduction of the third term you will apply an η-rule twice.

Solution:
The first term is already in normal form and the others reduce as follows:

$$
\begin{aligned}
&(\lambda \mathtt{p}:\mathtt{A}\times\mathtt{B}.((\lambda \mathtt{x}:\mathtt{A}.\mathtt{x})\ \mathtt{fst}(\mathtt{p})))\ \langle \mathtt{y},\mathtt{z}\rangle\\
&\to (\lambda \mathtt{x}:\mathtt{A}.\mathtt{x})\ \mathtt{fst}(\langle \mathtt{y},\mathtt{z}\rangle)\\
&\to (\lambda \mathtt{x}:\mathtt{A}.\mathtt{x})\ \mathtt{y},\\
&\to \mathtt{y},\\
&(\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ ((\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ \mathtt{f})\\
&\to (\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ (\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{f}\ \mathtt{y})\ \mathtt{x}))\\
&\to \lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.(((\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{f}\ \mathtt{y})\ \mathtt{x}))\ \mathtt{y})\ \mathtt{x}))\\
&\to \lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\lambda \mathtt{z}:\mathtt{A}.((\mathtt{f}\ \mathtt{z})\ \mathtt{y}))\ \mathtt{x})\\
&\to \lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{f}\ \mathtt{x})\ \mathtt{y})\\
&\to \lambda \mathtt{x}:\mathtt{A}.(\mathtt{f}\ \mathtt{x})\\
&\to \mathtt{f}.
\end{aligned}
$$

Question 3 (7 marks): Write a λ-term

$$\mathtt{double}_{\mathtt{A}} : \mathtt{A}\to(\mathtt{A}\times\mathtt{A}),$$

so that $\mathtt{fst}(\mathtt{double}_{\mathtt{A}}\ \mathtt{x}) \to \mathtt{x}$ for all $\mathtt{x}:\mathtt{A}$. Compute the previous property explicitly (show intermediate steps).

Solution:
Define

$$\mathtt{double}_{\mathtt{A}} \equiv \lambda \mathtt{x}:\mathtt{A}.\langle \mathtt{x},\mathtt{x}\rangle.$$

Then we have

$$\begin{aligned}
&\texttt{fst}(\texttt{double}_{\texttt{A}}\ \texttt{x}) \\
&\rightarrow \texttt{fst}(\lambda \texttt{x} \colon \texttt{A}.\langle \texttt{x}, \texttt{x} \rangle\ \texttt{x}) \\
&\rightarrow \texttt{fst}(\langle \texttt{x}, \texttt{x} \rangle) \\
&\rightarrow \texttt{x}.
\end{aligned}$$