A Representation Theorem for Boolean Contact Algebras

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Abstract

We prove a representation theorem for Boolean contact algebras which implies that the axioms for the Region Connection Calculus [23] (RCC) are complete for the class of subalgebras of the algebras of regular closed sets of weakly regular connected T_1 spaces.

Key words: Qualitative spatial reasoning, Region Connection Calculus, Boolean contact algebras, topological representation

1 Introduction

The main aim of the present paper is to establish a representation theorem for certain *contact algebras* which have arisen in various fields such as qualitative spatial reasoning, proximity theory, and ontology. The common ground is what has become known as *mereotopology*, based, among others, on Whitehead's notion of connection [32], Leśniewski's mereology [13, 14], and "pointless geometry" which originates with the works of de Laguna [5], Nicod [20], and Tarski [29]. Historically, standard (models for) mereotopological structures were collections of regular closed (or regular open) sets of topological spaces $\langle X, \tau \rangle$. Following Whitehead [32], two regular closed sets are said to be *in contact*, if they have a non-empty intersection; the primary example is the collection of all regular closed sets of the Euclidean plane. Algebraizations of mereotopological structures have been considered for some time and in various ways, see e.g. [1, 3, 12, 23, 27]. In a sequence of papers, Pratt and Schoop have investigated the first order theory of Boolean contact algebras of polygons in the plane and have obtained some very satisfying results [21, 22].

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From a different perspective, *proximity structures* have been investigated in a topological context since the 1950's; for an exhaustive treatment of proximity spaces we invite the reader to consult [19]. Proximity spaces are relational structures on families of sets that satisfy axioms which to some extent coincide with those for the connection structures mentioned earlier.

The search for representation theorems is motivated by a desire to relate general structures to – in some sense – more "familiar" ones, the most famous of such results being

- Each finite group is isomorphic to a group permutations [2].
- Each Boolean algebra is isomorphic to an algebra of sets [28].

Representation problems also play a part in the completeness of logical axioms: Having fixed a class of "standard models" of some theory, the non–existence of a standard model for a structure shows that the axiomatization is incomplete for this class.

Given the origin and original motivation for contact or proximity structures, several authors aimed at representing such structures as Boolean algebras of regular closed (or, equivalently, regular open) sets of some topological space where contact was defined in the Whiteheadian sense, but to our knowledge no such representation has as yet been found for general contact algebras or models of the popular Region Connection Calculus [4, 17, 18, 24, 27]; recently, Vakarelov et al. [31] have proved a representation result for a restricted class of RCC models, but the general problem still remained open [see also 30]. Since contact algebras are first order structures, and the Boolean algebra $\mathrm{RegCl}(X)$ of all regular closed subsets of a topological space is complete [see 11], we need to be content with substructures of $\mathrm{RegCl}(X)$ if we want to stay in the realm of first order logic. Thus, a Boolean algebra B endowed with some contact relation C will be called *representable* if there are a topological space X and a mapping $h : B \to \mathrm{RegCl}(X)$ such that h is a Boolean embedding and $xCy \iff h(x) \cap h(y) \neq \emptyset$.

2 Topological spaces

For any notion not explained here, we invite the reader to consult [9]. We will denote topological spaces by $\langle X, \tau \rangle$, where τ is the topology on X; for $x \subseteq X$, we let $\mathrm{cl}_\tau(x)$ be the τ-closure of x, and $\mathrm{int}_\tau(x)$ its τ-interior. If τ is understood, we will just speak of X as a topological space, and drop the subscripts from the operators. If $x, y \in \tau$, then x and y are called *separated*, if $\mathrm{cl}(x) \cap y = x \cap \mathrm{cl}(y) = \emptyset$. A non–empty open set x is called *connected* if it is not the union of two separated nonempty open sets. A set $u \subseteq X$ is called *regular open* if $u = \mathrm{int}(\mathrm{cl}(u))$, and *regular closed*, if $u = \mathrm{cl}(\mathrm{int}(u))$. The set complement of a regular open set is regular closed and

vice versa. $\mathrm{RegCl}(\langle X,\tau\rangle)$ is the collection of regular closed sets, and $\mathrm{RegOp}\langle X,\tau\rangle)$ the collection of regular open sets; we will sometimes just write $\mathrm{RegCl}(X)$ or $\mathrm{RegCl}(\tau)$ (respectively, $\mathrm{RegOp}(X)$ or $\mathrm{RegOp}(\tau)$) if no confusion can arise. It is well known that $\mathrm{RegCl}(X)$ is a complete Boolean algebra with the operations $a+b=a\cup b$, $a\cdot b=\mathrm{cl}(\mathrm{int}(a\cap b)))$, and $-a=\mathrm{cl}(X\setminus a)$. Note that we can have $a\cdot b=0$, while $a\cap b\neq\emptyset$. Similarly, $\mathrm{RegOp}(X)$ is a Boolean algebra with the operations $a+b=\mathrm{int}(\mathrm{cl}(a\cup b))$, $a\cdot b=a\cap b$, $-a=\mathrm{int}(X\setminus a)$.

Let C_τ be defined on $\mathrm{RegCl}(X)$ by $aCb\iff a\cap b\neq\emptyset$, and D_τ be defined on $\mathrm{RegOp}(X)$ by $aD_\tau b\iff \mathrm{cl}(a)\cap\mathrm{cl}(b)\neq\emptyset$. These relations will be our standard contact relations, and the following known result shows that it is not structurally important whether we work with $\mathrm{RegCl}(X)$ or $\mathrm{RegOp}(X)$:

Lemma 2.1. $\mathrm{RegCl}(X)$ *and* $\mathrm{RegOp}(X)$ *are isomorphic Boolean algebras, and the pairs* $\langle\mathrm{RegCl}(X),C_\tau\rangle$ *and* $\langle\mathrm{RegOp}(X),D_\tau\rangle$ *are isomorphic relational structures.*

Proof. It is easy to see that the assignment $f:\mathrm{RegOp}(X)\to\mathrm{RegCl}(X)$ defined by $u\mapsto\mathrm{cl}(u)$ has the desired properties. □

A space is *regular* if a point x and a closed set not containing x have disjoint open neighborhoods, and *semiregular* if it has a basis of regular open sets; regularity implies semiregularity, but not vice versa.

Since representations of regions will be regular closed sets, we could restrict ourselves to semiregular spaces. The following result shows that we do not lose anything in this case. First, define the *semi-regularization of* $\langle X,\tau\rangle$ as the topology $r(\tau)$ on X whose open basis is $\mathrm{RegOp}(\tau)$. Now,

Proposition 2.2. *Suppose that* $\langle X,\tau\rangle$ *is a topological space. Then,* $\mathrm{RegOp}(\tau)=\mathrm{RegOp}(r(\tau))$.

Proof. [See also 9, p. 84] Let $a\subseteq X$. Then,

$$\mathrm{cl}_{r(\tau)}(a)=-\bigcup\{m\in\mathrm{RegOp}(\tau):m\cap\mathrm{cl}_\tau(a)=\emptyset\}\supseteq-\bigcup\{m\in\tau:m\cap\mathrm{cl}_\tau(a)=\emptyset\}=cl_\tau(a).$$

Let $a\in\mathrm{RegOp}(\tau)$. Then,

$$\begin{aligned}\mathrm{int}_{r(\tau)}\,\mathrm{cl}_{r(\tau)}(a)&=\mathrm{int}_{r(\tau)}\left(-\bigcup\{m\in\mathrm{RegOp}(\tau):m\cap\mathrm{cl}_\tau(a)=\emptyset\}\right),\\&=\bigcup\{t\in\mathrm{RegOp}(\tau):t\cap m=\emptyset\text{ for all }m\in\mathrm{RegOp}(\tau)\text{ with }m\cap\mathrm{cl}_\tau(a)=\emptyset\},\\&=a,\end{aligned}$$

since a and t are regular open, and thus, $t\subseteq\mathrm{cl}_\tau(a)$ implies $t\subseteq a$.

Conversely, let $a\in\mathrm{RegOp}(r(\tau))$. Then,

$$a=\mathrm{int}_{r(\tau)}\,\mathrm{cl}_{r(\tau)}(a)=\bigcup\{t\in\mathrm{RegOp}(\tau):t\subseteq\mathrm{cl}_{r(\tau)}(a)\}.$$

Now, $\operatorname{int}_\tau \operatorname{cl}_\tau(a) \in \operatorname{RegOp}(\tau)$, and thus, $a \subseteq \operatorname{int}_\tau \operatorname{cl}_\tau(a) \subseteq \operatorname{int}_{r(\tau)} \operatorname{cl}_{r(\tau)}(a) = a$.

If $a \in \operatorname{RegOp}(\tau)$, then, by the preceding consideration, $-_\tau a = -_{r(\tau)} a$, and thus, $\operatorname{cl}_\tau(a) = \operatorname{cl}_{r(\tau)}(a)$. This implies the claim. □

Together with Lemma 2.1, the following now is a straightforward consequence, and it shows that we can restrict our attention to semiregular spaces:

Lemma 2.3. *The enhanced Boolean algebras* $\langle \operatorname{RegCl}(\tau), C_\tau \rangle$ *and* $\langle \operatorname{RegCl}(r(\tau)), C_{r(\tau)} \rangle$ *are isomorphic.*

It is well known that X is regular, if and only if for each non–empty $u \in \tau$ and each $x \in u$ there is some $v \in \tau$ such that $x \in v \subseteq \operatorname{cl}(v) \subseteq u$. We call X *weakly regular* if it is semiregular and for each non–empty $u \in \tau$ there is some non–empty $v \in \tau$ such that $\operatorname{cl}(v) \subseteq u$. Weak regularity may be called a "pointless version" of regularity, and each regular space is weakly regular. It is worthy to mention that Li and Ying [16] present the related notion of an "inexhaustable space".

A topological space X is called *normal*, if any two disjoint closed sets can be separated by disjoint open sets. X is called *κ–normal*, if any two disjoint regular closed sets can be separated by disjoint open sets. Then, for T_1 spaces,

$$X \text{ is normal} \Rightarrow X \text{ is } \kappa\text{–normal} \Rightarrow X \text{ is regular} \Rightarrow X \text{ is weakly regular} \Rightarrow X \text{ is semiregular},$$

and none of the implications can be reversed: Shchepin [25] gives an example of a κ-normal space which is not normal, and of a regular space which is not κ-normal. Example 5.3, together with the Representation Theorem 5.4 exhibits a weakly regular T_1 space which is not T_2, and thus, it is not regular. Finally, here is an example of a connected, semiregular T_2 space which is not weakly regular in a very strong way:

Example 2.4. The *relatively prime integer topology* is defined as follows [see 26, Example 60, for details]: Let X be the set of positive integers, and for all $a, b \in X$ let $U_a(b) = \{b + n \cdot a : n \in \mathbb{Z}\} \cap X$. Consider the topology τ generated by the basis

$$\mathfrak{B} = \{U_a(b) : a, b \in X, \gcd(a, b) = 1\}.$$

This topology is T_2 and has a basis of regular open sets. Furthermore, the closures of any two non-empty open sets intersect, and it follows that X is connected and not weakly regular. □

This also shows that weak regularity is independent of the T_2 property.

3 Boolean contact algebras

In the sequel, $\langle B, +, \cdot, -, 0, 1\rangle$ will denote a Boolean algebra (BA); we will usually identify algebraic structures with their base set. B^+ is the set of all non-zero elements of B. Furthermore, if $a \in B$ and $T \subseteq B$, we let $a \leq T$ if and only if $a \leq b$ for all $b \in T$. If A and B are BAs, then A is called *dense in* B, if A is a subalgebra of B, and for every $b \in B^+$ there is some $a \in A^+$ such that $a \leq b$. If A is complete and dense in B, then $A = B$. For properties of BAs not explained here, the reader is invited to consult Koppelberg [11].

Lemma 3.1. *1. For each ideal I and each filter F such that $I \cap F = \emptyset$ there is some ultrafilter U of B such that $F \subseteq U$ and $U \cap I = \emptyset$.*

2. Suppose that $M \subseteq B$ such that

(a) $0 \notin M$.

(b) $y + z \in M \Rightarrow y \in M$ or $z \in M$ for all $y, z \in B$.

(c) If $x \in M$ and $x \leq y$, then $y \in M$.

Then, for each $x \in M$ there is an ultrafilter F such that $x \in F$ and $F \subseteq M$.

Proof. 1. is a well known consequence of the Boolean prime ideal theorem, see e.g. [11, p 33 ff]. 2. has previously been shown in the context of proximity structures [19, Lemma 5.7]; for completeness we provide a proof. Let $x \in M$, and suppose that F is a subset of M containing x which is maximal with respect to the property

$$y_0, \ldots, y_k \in F \text{ implies } y_0 \cdot \ldots \cdot y_k \in M. \tag{3.1}$$

The existence of such F is an easy consequence of Zorn's Lemma, and we show that F is an ultrafilter. Suppose that $a, b \in F$, and that $y_0, \ldots, y_k \in F$. Then, $y_0 \cdot \ldots \cdot y_k \cdot a \cdot b$ is a finite product of elements of F, and thus it is an element of M; the maximality of F now implies $a \cdot b \in F$. If $a \in F$, $a \leq b$, $y_0, \ldots, y_k \in F$, then $y_0 \cdot \ldots \cdot y_k \cdot a \in F$. By condition 2c above and $a \leq b$, we have $y_0 \cdot \ldots \cdot y_k \cdot b \in M$, and the maximality of F implies $b \in F$. Finally, assume that there is some $a \in B$ such that $a, -a \notin F$. Then, there are $y, z \in F$ such that $y \cdot a \notin M$ and $z \cdot -a \notin M$. By (3.1) and $y, z \in F$ we have $y \cdot z \in M$. On the other hand, $y \cdot a, z \cdot -a \notin M$ and condition 2c on M imply that $y \cdot z \cdot a \notin M$ and $y \cdot z \cdot -a \notin M$. Since $y \cdot z \cdot a + y \cdot z \cdot -a = y \cdot z \in M$, this contradicts condition 2b. □

A binary relation C on B is called a *contact relation* if it satisfies

C0. $aCb \Rightarrow a, b \neq 0$.

C1. $a \neq 0 \Rightarrow aCa$.

C2. C is symmetric.

C3. aCb and $b \leq c \Rightarrow aCc$.

C4. $aC(b+c) \Rightarrow aCb$ or aCc.

C5. $C(a) \subseteq C(b) \Rightarrow a \leq b$.

As shown in [30], in the presence of the other axioms we can replace C5 by

C5'. If $a \not\leq b$, there is some $c \in B$ such that aCc and $c(-C)b$.

If B is a Boolean algebra and C a contact relation on B, the pair $\langle B, C\rangle$ will be called a *Boolean contact algebra* (BCA). We will consider the following additional properties of C:

C6. $a(-C)b \Rightarrow (\exists c)[a(-C)c$ and $-c(-C)b]$ (The *strong axiom* of [19]).

C7. $a, b \neq 0$ and $a + b = 1 \Rightarrow aCb$ (C is *connecting*).

Let $aOb \iff a \cdot b \neq 0$. The following Lemma lists some easy properties of C. The proof is left to the reader:

Lemma 3.2. *1. O is a contact relation on B and $O \subseteq C$.*

2. $a(-C)b \Rightarrow a \cdot b = 0$.

3. If $a \notin \{0, 1\}$, there is some $b \neq 0$ such that $a(-C)b$.

4. aCb or $aCc \Rightarrow aC(b+c)$.

5. $a \leq b \Rightarrow C(a) \subseteq C(b)$.

The *Region Connection Calculus* (RCC) of Randell et al. [23] has received some prominence as a structure for qualitative spatial reasoning. RCC models are BCAs which satisfy C7. Indeed, the primary motivation for this paper was to find a standard topological representation for each RCC model.

The presence of the extensionality axiom C5 causes that the interesting BCAs have a not too simple structure. Recall that a Boolean algebra is called $finite{-}cofinite$ if every nonzero element is a finite sum of atoms or the complement of such an element. In particular, every finite BA is finite–cofinite.

Lemma 3.3. *1. If B is a finite–cofinite algebra, there is exactly one contact relation on B.*

2. If C satisfies C7, then B is atomless.

Proof. 1. We show that O is the only contact relation on B. Assume that C is a contact relation on B, and that $C \neq O$. Since O is the smallest contact relation on B by Lemma 3.2(1), there are $x, y \in B^+$ such that xCy and $x \cdot y = 0$. Since B is finite-cofinite, and the meet of two cofinite elements is always nonzero, we may assume by C4 that y is an atom; by C3 we may assume that $x = -y$, i.e. that x is the antiatom disjoint to y. But then, $x \neq 1$ and x is connected to every region, contradicting C5'.

2. This was shown in [7]. □

This shows that BCAs are not the best choice for reasoning with finite or discrete structures: The RCC models assume a continuous interpretation of the world since by Lemma 3.3(2) every region is infinitely divisible, and Lemma 3.3(1) tells us that finite BCAs are quite trivial. A way of coping with this situation is to omit C5 as in [15] and also weaken the other axioms such as in [6], see also [10].

For later use we mention the following construction of a BCA:

Example 3.4. Let L be a linear order with smallest element 0_L. Suppose that ∞ is a symbol not in L, and set $L^+ = L \cup \{\infty\}$ with $x < \infty$ for all $x \in L$. An *interval of* L is a set of the form $[s, t) = \{u \in L : s \leq u \lneq t\}$. $IntAlg(L)$ is the collection of all finite unions of intervals

$$[x_0^0, x_0^1) \cup [x_1^0, x_1^1) \cup \ldots \cup [x_{t(x)}^0, x_{t(x)}^1), \tag{3.2}$$

along with the empty set. It is well known that $IntAlg(L)$ is a Boolean algebra [see 11, p.10], called *the interval algebra of* L. Each nonzero $x \in IntAlg(L)$ can be written in the form (3.2) in such a way that $x_j^i \in L^+$, $x_j^0 < x_j^1 < x_{j+1}^0$, and the intervals $[x_j^0, x_j^1)$ are pairwise disjoint. The representation of x in this form is unique, and we call it the *standard representation.* In the sequel, we shall assume that all elements of $IntAlg(L)$ are in standard representation. For each $x \in IntAlg(L)^+$, we let

$$\mathrm{rel}(x) = \{x_j^0 : j \leq t(x)\} \cup \{x_j^1 : j \leq t(x)\}$$

be the set of *relevant points* of x.

Suppose that L is the rational unit interval $[0, 1)$ and that $B = IntAlg(L)$. Set

$$xCy \overset{\text{def}}{\Longleftrightarrow} x \cdot y \neq 0 \text{ or } \mathrm{rel}(x) \cap \mathrm{rel}(y) \neq \emptyset. \tag{3.3}$$

Proposition 3.5. *[8] $\langle B, C\rangle$ is a BCA which satisfies C6 and C7.*

We also require the following construction:

Proposition 3.6. *[8] Let B be atomless, F, G be distinct ultrafilters of B, and $D = C \cup (F \times G) \cup (G \times F)$. Then, D is a contact relation on B.*

The common standard models of Boolean contact algebras are substructures of the regular closed algebra $\mathrm{RegCl}(X)$ for some semiregular topological space $\langle X, \tau \rangle$ where the contact relation C_τ is defined by $aC_\tau b \iff a \cap b \neq \emptyset$. The following result exhibits the connections between the BCA axioms and topological properties of $\langle X, \tau \rangle$:

Proposition 3.7.

1. *C_τ satisfies C0 – C4.*
2. *C_τ satisfies C5 if and only if X is weakly regular.*
3. *C_τ satisfies C6 if and only if X is κ–normal.*
4. *C_τ satisfies C7 if and only if X is connected.*

Proof. Clearly, C_τ satisfies C0 – C4. For 2., suppose that C_τ satisfies C5, and let $u \neq \emptyset$ be open; since τ is semiregular, we may assume that u is in fact regular open. Then, $X \setminus u$ is regular closed and not equal to X. By C5' there is some non-empty regular closed v such that $v(-C_\tau)(X \setminus u)$, which implies $v \cap (X \setminus u) = \emptyset$, i.e. $v \subseteq u$.

Conversely, suppose that X is weakly regular, and that $a, b \in \mathrm{RegCl}(X)$ with $a \not\subseteq b$, i.e. $a \cap (X \setminus b) \neq \emptyset$. Since a is regular closed, we have in fact $\mathrm{int}(a) \cap (X \setminus b) \neq \emptyset$, and $\mathrm{int}(a) \cap (X \setminus b)$ is regular open. By weak regularity, there is some non-empty regular closed c such that $c \subseteq \mathrm{int}(a) \cap (X \setminus b)$, which shows that $C_\tau(a) \not\subseteq C_\tau(b)$.

For 3., suppose that X satisfies C6, and suppose that a, b are non-empty disjoint regular closed sets. By C6, there exists some regular closed c such that $a(-C_\tau)c$ and $-c(-C_\tau)b$. Then, $a \cap c = \mathrm{cl}(X \setminus c) \cap b = \emptyset$, and thus, $a \subseteq \mathrm{int}(-c)$ and $b \subset \mathrm{int}(c)$, showing that a and b can be separated. Conversely, suppose that X is κ–normal, and that $a, b \in \mathrm{RegCl}(X)^+$, $a(-C_\tau)b$. Then, $a \cap b = \emptyset$, and by κ–normality there are disjoint open u, v such that $a \subseteq u$ and $b \subseteq v$. Then, regularity of a implies $a \cap \mathrm{cl}(X \setminus u) = \emptyset$, and $\mathrm{cl}(u) \cap b = \emptyset$, i.e. $a(-C_\tau)(-u)$ and $u(-C_\tau)b$.

Finally, suppose that X satisfies C7, and suppose that a, b are non-empty, disjoint open sets whose union is X. Then, in particular, a and b are regular closed, and $a + b = X$. C7 implies that $aC_\tau b$, i.e. $a \cap b \neq \emptyset$, a contradiction.

Conversely, suppose that C_τ does not satisfy C7. Then, there are $a, b \in \mathrm{RegCl}(X)^+$ such that $a \cup b = X$ and $a \cap b = \emptyset$. It follows that a and b are open and closed, showing that X is not connected. $\square$

In the literature, standard representation spaces for contact algebras are usually assumed to be regular and T_1, but the previous result shows that this may not be a necessary condition. Indeed, Example 5.3 below will show that such spaces need not even be T_2.

4 Clusters

The points of Stone's topological representation space of a Boolean algebra B are the ultrafilters of B. The additional relation on B requires a different construction, which first was used in the theory of proximities, see [19]. As a preparation, we mention the following, which may be of independent interest: For each $a \in B^+$ we let $I_a = \{b : a(-C)b\}$.

Lemma 4.1. *Let $a \neq 0, 1$. Then I_a is a proper non–trivial ideal of B and $\sup I_a = -a$.*

Proof. Since $a \neq 1$, there is some $b \neq 0, 1$ such that $a(-C)b$ by C5', and thus, I_a is non–trivial. Since $a \neq 0$, we have aCa, and hence, I_a is proper. If $b, c \in I_a$, then $a(-C)b$ and $a(-C)c$, and C4 shows that $a(-C)(b + c)$. If $a(-C)b$ and $c \leq b$, then $a(-C)c$ by C3, and it follows that I_a is an ideal.

Observe that by Lemma 3.2(2), $-a$ is an upper bound of I_a. Let $b \leq s$ for all $b \in I_a$, and assume $-a \not\leq s$, i.e. $a + s \neq 1$. Since $a + s \notin \{0, 1\}$, there is some $d \in B^+$ such that $(a + s)(-C)d$ by C5'. By C4, $a(-C)d$ and $s(-C)d$. The first condition implies $d \in I_a$, and hence $d \leq s$, and the second condition implies $s \cdot d = 0$, which together imply $d = 0$, contradicting our choice of d. □

If $M \subseteq B$, we let I_M be the ideal of B generated by $\{I_a : a \in M\}$. Thus,

(4.1)
$$x \in I_M \iff (\exists y_0, \ldots, y_k \in M)(\exists p_0, \ldots, p_k)[x = p_0 + \ldots + p_k \text{ and } p_0(-C)y_0, \ldots, p_k(-C)y_k].$$

We also let $F_M = \{-x : x \in I_M\}$ be the filter containing the complements of the elements of I_M.

Now we are ready for our main definition: A non–empty subset Γ of B is called a *clan* if for all $x, y \in B$,

CL1. If $x, y \in \Gamma$ then xCy.

CL2. If $x + y \in \Gamma$ then $x \in \Gamma$ or $y \in \Gamma$.

CL3. If $x \in \Gamma$ and $x \leq y$, then $y \in \Gamma$.

Lemma 4.2. *1. If F, G are filters of B such that $F \times G \subseteq C$, then there is a clan containing $F \cup G$.*

2. *If Γ is a clan, U an ultrafilter, and $U \times \Gamma \subseteq C$, then $U \cup \Gamma$ is a clan.*

3. *If Γ is a clan and $x \in \Gamma$, then there is some ultrafilter U such that $x \in U \subseteq \Gamma$.*

Proof. 1. Since $F \times G \subseteq C$, we have $G \cap I_F = \emptyset$, and by 3.1(1), there is some ultrafilter U disjoint from I_F and containing G. Assume that $x \in F \cap I_U$. Then, there are $y_0, \ldots, y_k \in U$ and, for each $i \leq k$ some $p_i \in I_{y_i}$ such that $x = p_0 + \ldots + p_k$. Since U is a filter we have $y = y_0 \cdot \ldots \cdot y_k \in U$. By C3 and C4 we have $(p_0 + \ldots + p_k)(-C)y$, i.e. $x(-C)y$. On the other hand, $x \in F$ and $y \in U$ imply xCy, since $U \cap I_F = \emptyset$. Again by Lemma 3.1(1) there is some ultrafilter V disjoint from I_U and containing F. Then, $U \times V \subseteq C$, and clearly, $U \cup V$ is a clan.

Observe that this implies that

$$xCy \Rightarrow \text{ There is a clan } \Gamma \text{ such that } x, y \in \Gamma. \tag{4.2}$$

2. This follows immediately from CL1 – CL3.

3. Since Γ satisfies the conditions of Lemma 3.1(2), there is some ultrafilter U such that $x \in U \subseteq \Gamma$. □

Clearly, the class of clans on B is closed under union of chains, and thus each clan is contained in a maximal element which we call *cluster*. The set of all clusters in B will be denoted by $\mathrm{Clust}(B)$. A characterization of clusters is as follows:

Proposition 4.3. *A clan Γ is a cluster if and only if*

$$U \times \Gamma \subseteq C \Rightarrow U \subseteq \Gamma \tag{4.3}$$

for all ultrafilters U.

Proof. "$\Rightarrow$": If $U \times \Gamma \subseteq C$, then $U \cup \Gamma$ is a clan by Lemma 4.2(2), and the maximality of Γ implies $U \subseteq \Gamma$.

"$\Leftarrow$": Suppose that U and Γ satisfy (4.3), and assume that Δ is a clan such that $\Gamma \subsetneq \Delta$. Then, there is some $z \in \Delta \setminus \Gamma$, and, by 4.2(3) there is some ultrafilter U such that $z \in U \subseteq \Delta$. Since $U \cup \Gamma \subseteq \Delta$ and $\Delta \times \Delta \subseteq C$, we have, in particular, $U \times \Gamma \subseteq C$. By our hypothesis, we have $z \in U \subseteq \Gamma$, a contradiction. □

Lemma 4.4. *Let $\Gamma \in \mathit{Clust}(B)$. Then, $B \setminus \Gamma = I_\Gamma$.*

Proof. Assume that $x \in \Gamma \cap I_\Gamma$. Then, by (4.1), there are $y_0, \ldots, y_k \in \Gamma, p_0, \ldots, p_k \in B$ such that $x = p_0 + \ldots + p_k$ and $p_i(-C)y_i$ for all $i \leq k$. Since $x \in \Gamma$, we have $p_i \in \Gamma$ for some $i \leq k$ by CL2. Now, $p_i(-C)y_i$ contradicts CL1.

Next, assume that $x \notin \Gamma \cup I_\Gamma$. By Lemma 3.1(1), there is some ultrafilter U such that $x \in U$ and $U \cap I_\Gamma = \emptyset$. It is easy to see that $U \cup \Gamma$ satisfies CL2 – CL3; for CL1, let $x \in U$ and $y \in \Gamma$, and assume that $x(-C)y$. Since $y \in \Gamma$, we have $x \in I_y \subseteq I_\Gamma$, contradicting $x \notin I_\Gamma$. Thus, $U \subseteq \Gamma$ by the maximality of Γ. This contradicts $x \notin \Gamma$. □

The representation theorem of Vakarelov et al. [30] uses a different definition of cluster; let us call $\Gamma \subseteq B$ a *proximity cluster* (p-cluster), if it is a clan and furthermore satisfies

(P). $(\forall x)[\{x\} \times \Gamma \subseteq C \Rightarrow x \in \Gamma]$.

Lemma 4.5. *A clan Γ is a p-cluster if and only if*

$$F \times \Gamma \subseteq C \Rightarrow F \subseteq \Gamma \tag{4.4}$$

for all filters F.

Proof. "$\Rightarrow$": Suppose that F is a filter of B and $F \times \Gamma \subseteq C$. In particular, $\{x\} \times \Gamma \subseteq C$ for all $x \in F$, and hence, $F \subseteq \Gamma$ by (P).

"$\Leftarrow$: Suppose that $\{x\} \times \Gamma \subseteq C$, and let F be the principal filter generated by x. By C2 and C3 we have $\{y\} \times \Gamma \subseteq C$ for all $y \in F$, and thus, $F \subseteq \Gamma$. □

Clearly, each p-cluster is a cluster, but the converse need not hold in general:

Example 4.6. Suppose that $\langle B, C\rangle$ is the BCA of Example 3.4, and let $0 \lneq a \lneq b \lneq 1$. Furthermore, let F_a be the ultrafilter of B of all sets containing a, and F_b be the ultrafilter of B of all sets containing b. By Proposition 3.6, the relation $D = C \cup (F_a \times F_b) \cup (F_b \times F_a)$ is a contact relation on B. Let $\Gamma = F_a \cup F_b$; clearly, Γ is a clan, and we shall show that it is a cluster. Suppose that U is an ultrafilter of B and that $U \times \Gamma \subseteq D$. By (4.3) it is sufficient to prove $U \subseteq \Gamma$. Thus, assume that $x \in U$, $x \notin \Gamma$. Since U is an ultrafilter and no element below x can be in Γ by CL3, we can suppose that $x = [s, t)$. Since $a, b \notin x$ and xCy for all $y \in \Gamma = F_a \cup F_b$, we must have $x = [s, a) = [s, b)$, contradicting $a \neq b$. Hence, $U \subseteq \Gamma$. If $s \lneq a \lneq t \lneq b$ and $y = [s, a) \cup [t, b)$, then $\{y\} \times \Gamma \subseteq D$ and $y \notin \Gamma$. □

However, if C6 holds, then the two notions coincide:

Proposition 4.7. *Suppose that C satisfies C6. Then, each cluster is a p-cluster.*

Proof. Suppose $\{x\} \times \Gamma \subseteq C$; by Lemma 4.4 it is enough to show that $x \notin I_\Gamma$. Assume otherwise; then, there are $y_0, \ldots, y_n \in \Gamma$ and for each $i \leq n$ some p_i such that $p_i(-C)y_i$, and $x = p_0 + \ldots + p_n$. With C6, choose for each $1 \leq i \leq n$ some a_i such that $p_i(-C) - a_i$ and $a_i(-C)y_i$, and let $a = a_1 + \ldots + a_n$. Now, set $z_0 = y_0 \cdot -a$, and, for $1 \leq i \leq n$ set $z_i = y_0 \cdot a_i$. Then,

$$z_0 + \ldots + z_n = y_0 \cdot -a + y_0 \cdot (a_1 + \ldots a_n) = y_0 \cdot -a + y_0 \cdot a = y_0.$$

Our aim is to show that $z_i \notin \Gamma$ for all $i \leq n$ which contradicts $y_0 \in \Gamma$. First, we consider z_0. From $z_0 = y_0 \cdot -a = y_0 \cdot -a_1 \cdot \ldots \cdot -a_n$ we infer $z_0 \leq y_0$ and $z_0 \leq -a_i$ for $1 \leq i \leq n$. If $z_0 C p_0$, then $z_0 \leq y_0$ implies $p_0 C y_0$, contradicting $p_0 \in I_{y_0}$. If $z_0 C p_i$ for $1 \leq i \leq n$, then $p_i C - a_i$, contradicting the choice of a_i. Hence, $z_0(-C)(p_0 + \ldots + p_n)$, and $x \leq p_0 + \ldots + p_n$ now implies $x(-C)z_0$. From $\{x\} \times \Gamma \subseteq C$ we now obtain $z_0 \notin \Gamma$. If $1 \leq i \leq n$, then $z_i \leq a_i$ and $a_i(-C)y_i$ imply that $z_i(-C)y_i$, and hence, $y_i \in \Gamma$ shows that $z_i \notin \Gamma$. □

5 The Representation Theorem

Let $\langle B, C\rangle$ be a BCA, $X = \mathrm{Clust}(B)$ and $h : B \to 2^X$ be defined by the Stone–like assignment $h(x) = \{\Gamma \in \mathrm{Clust}(B) : x \in \Gamma\}$. Our first result shows that h is injective and preserves $+$:

Lemma 5.1. *1. $x \leq y \iff h(x) \subseteq h(y)$.*

2. $h(0) = \emptyset$, $h(1) = X$, and $h(x) \cup h(y) = h(x + y)$ for all $x, y \in B$.

Proof. 1. "$\Rightarrow$: Let $x \leq y$ and $x \in \Gamma$. By CL3 we have $y \in \Gamma$, and thus, $\Gamma \in h(y)$.

"$\Leftarrow$": Let $x \not\leq y$. C5 implies that there is some $z \in B$ such that xCz and $y(-C)z$. By (4.2), there is some cluster Γ containing both x and z, and $y(-C)z$ implies that $y \notin \Gamma$. Hence, $h(x) \not\subseteq h(y)$.

It may be worthy to note that, except for weak regularity, this is the only place in the proof of the Representation Theorem, where C5 is used.

2. Since 0 is not contained in any cluster by C0, and 1 is contained in every cluster by CL3, we have $h(0) = \emptyset$ and $h(1) = X$. 1. above implies that $h(x) \cup h(y) \subseteq h(x + y)$. Conversely, if $\Gamma \in h(x + y)$, then $x + y \in \Gamma$, and it follows from CL2 that $x \in \Gamma$ or $y \in \Gamma$, i.e. $\Gamma \in h(x) \cup h(y)$. □

Let τ be the topology on X which has the family $\mathfrak{A} = \{h(x) : x \in B\}$ as a basis for the closed sets. That $\mathfrak{A}$ is indeed a basis follows from Lemma 5.1(2). Thus, $\mathfrak{B} = \{X \setminus h(x) : x \in B\}$ is a basis for the open sets, and each open set U has the form

$$U = \bigcup\{X \setminus h(x) : x \in T\} = X \setminus \bigcap\{h(x) : x \in T\} \tag{5.1}$$

for some $T \subseteq B$.

The next Lemma will exhibit some properties of the topology τ which will be useful for the proof of the Representation Theorem:

Lemma 5.2. *1. $\langle X, \tau \rangle$ is a T_1 space.*

2. Let $M \subseteq X$. Then, $\mathrm{cl}(M) = \{\Gamma : \bigcap M \subseteq \Gamma\}$.

3. A is closed iff there is some $T \subseteq B$ such that $A = \{\Gamma : T \subseteq \Gamma\}$.

4. Let $\emptyset \neq T \subseteq B$ and $A = \{\Gamma : T \subseteq \Gamma\}$ be closed. Then,

(a) $\mathrm{int}(A) = \{\Gamma : (\exists y)[y \notin \Gamma \text{ and } -y \leq T]\}$.
(b) $\bigcap \mathrm{int}(A) = \{x : (\forall y)[y \leq T \Rightarrow y \leq x]\}$.
(c) Let $S = \{y : -y \leq T\}$. Then, $\mathrm{cl}(X \setminus A) = \{\Gamma : S \subseteq \Gamma\}$.

Proof. 1. Suppose that $\Gamma \in X$. Clearly, $\Gamma \in \bigcap\{h(y) : y \in \Gamma\}$. Let $\Delta \in \bigcap\{h(y) : y \in \Gamma\}$. If $y \in \Gamma$, then $\Delta \in h(y)$, i.e. $y \in \Delta$. The maximality of Γ shows that $\Gamma = \Delta$, and it follows that $\{\Gamma\} = \bigcap\{h(y) : y \in \Gamma\}$ is closed.

2. Since the sets $h(x)$ are a basis for the closed sets of τ, we have

$$\begin{aligned}
\mathrm{cl}(M) &= \bigcap\{h(y) : M \subseteq h(y)\} \\
&= \{\Gamma : (\forall y)[M \subseteq h(y) \Rightarrow y \in \Gamma]\} \\
&= \{\Gamma : y \in \bigcap M \Rightarrow y \in \Gamma\} \\
&= \{\Gamma : \bigcap M \subseteq \Gamma\}.
\end{aligned}$$

3. Since the sets of the form $h(x)$ are a basis for the closed sets of τ, A is closed if and only if there is some $T \subseteq B$ such that $A = \bigcap\{h(x) : x \in T\}$. Now,

$$\Gamma \in \bigcap\{h(x) : x \in T\} \iff (\forall x \in T)\Gamma \in h(x) \iff (\forall x \in T)x \in \Gamma \iff T \subseteq \Gamma.$$

4a. By definition of τ and A, we have

$$\begin{aligned}
\Gamma \in \mathrm{int}(A) &\iff (\exists y)[\Gamma \in X \setminus h(y) \subseteq A] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall \Delta)(y \notin \Delta \Rightarrow \Delta \in A)] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall \Delta)(y \in \Delta \text{ or } T \subseteq \Delta)] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall \Delta)(\forall t \in T)(y + t \in \Delta)] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall t \in T)(y + t \in \bigcap \mathrm{Clust}(B))] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall t \in T)(y + t = 1)] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } -y \leq T].
\end{aligned}$$

Observe that this implies

$$\Gamma \in \mathrm{int}(h(x)) \iff (\exists y)[y \notin \Gamma \text{ and } -y \leq x] \iff (\exists y)[y \notin \Gamma \text{ and } -x \leq y] \iff -x \notin \Gamma. \tag{5.2}$$

4b. Let $x \in \bigcap \mathrm{int}(A)$, $y \leq T$, and assume $y \not\leq x$, i.e. $-y + x \neq 1$. By C5' there is some $z \notin \{0, 1\}$ such that $z(-C)(-y + x)$, and by C4 we have $z(-C) - y$ and $z(-C)x$. Choose some Γ with $z \in \Gamma$. Then, $-y \notin \Gamma$ and $y \leq T$ shows that $\Gamma \in \mathrm{int}(A)$ by 4a. Since $z(-C)x$ implies $x \notin \Gamma$, this contradicts $x \in \bigcap \mathrm{int}(A)$.

Conversely, suppose that $z \leq T$ implies $z \leq x$, and let $\Gamma \in \mathrm{int}(A)$. Then, there is some y such that $y \notin \Gamma$ and $-y \leq T$. Hence, $-y \leq x$ and thus, $x \in \Gamma$.

4c: Consider the following:

$$\begin{aligned}
\Gamma \in \mathrm{cl}(X \setminus A) &\iff \Gamma \notin \mathrm{int}(A) \\
&\iff (\forall y)[y \notin \Gamma \Rightarrow -y \not\leq T] && \text{by 4a} \\
&\iff (\forall y)[-y \leq T \Rightarrow y \in \Gamma] \\
&\iff (\forall y)[y \in S \Rightarrow y \in \Gamma] \\
&\iff S \subseteq \Gamma.
\end{aligned}$$

Note that S is a filter: If $x, y \in S$, then $-x \leq T, -y \leq T$ implies $-x + -y \leq T$, and thus, $x \cdot y \in S$. If $x \leq z$, then $-z \leq -x \leq T$, and it follows that $z \in S$. □

The following example shows that $\langle X, \tau \rangle$ need not be a T_2 space:

Example 5.3. Let $\langle B, C \rangle$ be the algebra of Example 3.4, $0 \lneq a \lneq b \lneq c \lneq 1$ and F_a, F_b, F_c be the ultrafilters of B of all elements of B containing, respectively, a, b or c. Let $D = C \cup (F_a \times F_b) \cup (F_b \times F_a) \cup (F_a \times F_c) \cup (F_c \times F_a)$. By Proposition 3.6, D is a contact relation. As in Example 4.6, one can show that $\Gamma = F_a \cup F_b$ and $\Delta = F_a \cup F_c$ are clusters. Incidentally, this shows that an ultrafilter can be contained in two different clusters which is not possible for p-clusters. Assume that there are open sets u, v such that $\Gamma \in u$, $\Delta \in v$ and $u \cap v = \emptyset$. Since the sets $h(x)$ are a basis for the closed sets, there are x, y in B such that $\Gamma \notin h(x)$, $\Delta \notin h(y)$ and $h(x) + h(y) = X$. Since h is an embedding, the latter implies $x + y = 1$. On the other hand, $\Gamma \notin h(x)$ implies that $a \notin x$ and $b \notin x$, and $\Delta \notin h(y)$ implies that $a \notin y$ and $c \notin y$. Together, we obtain $a \notin x + y$, contradicting $x + y = 1$. □

We can now prove the Representation Theorem:

Proposition 5.4. *1. Each BCA B is isomorphic to a dense substructure of some $\langle \mathrm{RegCl}(X), C_\tau \rangle$ where τ is T_1 and weakly regular.*

2. B satisfies C7 if and only if X is connected.

Proof. 1. Let $X = \mathrm{Clust}(B)$. We show that h is a Boolean embedding into $\mathrm{RegCl}(X)$ such that $h[B]$ is a dense subalgebra of $\mathrm{RegCl}(X)$, and $xCy \iff h(x) \cap h(y) \neq \emptyset$. By Lemma 5.1, h is injective and preserves sums. From (4.2) and CL1 it is easily seen that $xCy \iff h(x) \cap h(y) \neq \emptyset$.

Next, we show that $h(x) \in \mathrm{RegCl}(X)$ and that h preserves complements. First, observe that for all $x \in B$,

(5.3)
$$\mathrm{cl}(X \setminus h(x)) = \bigcap\{h(y) : X \setminus h(x) \subseteq h(y)\} = \bigcap\{h(y) : h(x) \cup h(y) = X\} = \bigcap\{h(y) : x + y = 1\}.$$

If $\Gamma \in h(-x)$, then $-x \in \Gamma$, and, if $x + y = 1$, then $-x \leq y$ which implies $y \in \Gamma$. Hence, $\Gamma \in \mathrm{cl}(X \setminus h(x))$. Conversely, suppose that $\Gamma \in h(y)$ whenever $x + y = 1$; then, in particular, $\Gamma \in h(-x)$. It follows that $h(x) = \mathrm{cl}(X \setminus h(-x))$ for all $x \in B$. Now,

$$h(x) = \mathrm{cl}(X \setminus h(-x)) = \mathrm{cl}(X \setminus (\mathrm{cl}(X \setminus h(x)))) = \mathrm{cl}(\mathrm{int}(h(x)))$$

shows that $h(x)$ is regular closed and it follows from (5.3) that h preserves complements.

Now we show that $h[B]$ is dense in $\mathrm{RegCl}(X)$: Suppose that $\emptyset \neq A = \{\Gamma : T \subseteq \Gamma\}$ is regular closed. Then, by Lemma 5.2(4a), there are some $\Gamma \in \mathrm{int}(A)$ and some $y \in B$ such that $y \notin \Gamma$ and $-y \leq T$. Since $y \notin \Gamma$, we have $-y \in \Gamma$, i.e. $\emptyset \neq h(-y)$. If $\Delta \in h(-y)$,then, $-y \leq T$ implies $T \subseteq \Delta$. It follows that $\Delta \in A = \{\Gamma : T \subseteq \Gamma\}$, and hence, $\emptyset \neq h(-y) \subseteq A$. Therefore, $h[B]$ is is a dense subalgebra of $\mathrm{RegCl}(X)$. Note that this implies $h[B] = \mathrm{RegCl}(X)$ just in case B is complete.

Since τ is T_1 by Lemma 5.2(1), it remains to show that τ is weakly regular. Suppose that $\emptyset \neq u \in \tau$. Since the sets of the form $X \setminus h(y)$ are regular open and a basis for the open sets, we can assume that $X \setminus h(y) \subseteq u$ for some $y \neq 1$. By C5' there is some $z \neq 0$ such that $z(-C)y$. Hence, by CL1 no cluster contains both y and z, and it follows that $h(z) \cap h(y) = \emptyset$, i.e. $h(z) \subseteq X \setminus h(y)$.

2. Suppose that $M \neq \emptyset, X$ is clopen. By Lemma 5.2(4a) and (4c), there are filters F, G of B such that $M = \{\Gamma : F \subseteq \Gamma\}$, $X \setminus M = \{\Delta : G \subseteq \Delta\}$, and G is the set of upper bounds of the set $\{-x : x \in F\}$. If $x \in F$ and $y \in G$, then $-x \leq y$ by definition of G, and thus $x + y = 1$. It follows from C7 that xCy. Hence, $F \times G \subseteq C$, and by Lemma 4.2(1) there is some cluster Γ such that $F \cup G \subseteq \Gamma$. On the other hand,

$$F \cup G \subseteq \Gamma \iff F \subseteq \Gamma \text{ and } G \subseteq \Gamma \iff \Gamma \in M \text{ and } \Gamma \in X \setminus M,$$

which is not possible. □

Corollary 5.5. *Each RCC model is isomorphic to a substructure of some $\langle \mathrm{RegCl}(X), C_\tau \rangle$ for a connected weakly regular T_1 space X.*

Finally, having shown that $h(x)$ is regular closed, we can now prove

Proposition 5.6. *If $\langle Clust(B), \tau\rangle$ is a T_2 space, then B satisfies (P).*

Proof. First, observe that for any cluster Γ, the set $\{\mathrm{int}(h(y)) : y \notin \Gamma\}$ is a neighbourhood basis for Γ by (5.2) and the fact that $\mathrm{int}(h(y)) = \mathrm{Clust}(B) \setminus h(-y)$. Since $h(y)$ is regular closed, we obtain that $\{h(y) : -y \notin \Gamma\}$ is a basis for the closed neighbourhodds of Γ, i.e. each closed neighbourhood of Γ contains some $h(y)$ with $y \notin \Gamma$.

Recall that $F_\Gamma = \{y \in \Gamma : -y \notin \Gamma\} \subseteq \Gamma$. Let $x \in B$ such that $\Gamma \times \{x\} \subseteq C$. Then, in particular, $F_\Gamma \times \{x\} \subseteq C$, and it follows from Lemma 4.2(1) that there is some $\Delta \in X$ such that $F_\Gamma \cup \{x\} \subseteq \Delta$. Now, since X is T_2, $\{\Gamma\}$ is the intersection of all its closed neighborhoods, and thus, since each closed neighbourhood of Γ contains some $h(y)$ with $y \notin \Gamma$, we obtain $\{\Gamma\} = \bigcap\{h(y) : y \in F_\Gamma\} = \{\Delta : F_\Gamma \subseteq \Delta\}$. In other words, Γ is the only cluster containing F_Γ, and thus, $x \in \Gamma$. $\square$

6 Conclusion and outlook

We have provided a topological Representation Theorem for Boolean contact algebras which implied a representation of the Region Connection Calculus. As a consequence, one can observe that the common assumption of regularity and T_1 of the representation space is too strong and not forced by the axioms, since our spaces need not be regular. We do not even need any point separation axioms: The T_1 property of our representation space is a consequence of the fact that the clusters are pairwise incomparable under $\subseteq$. If we "double" the points of $\mathrm{Clust}(B)$ and choose the obvious topology, then the resulting space will still provide a representation for $\langle B, C\rangle$, but it will not even be T_0. Two interesting questions remain:

1. Is there a topological representation theorem for Boolean algebras where C only satisfies C0 – C4? If C5 is not satisfied, then regions cannot be distinguished by C anymore, and the mapping h above need not be injective.

2. Are there BCAs which do not have a regular representation space? Even though our spaces are not necessarily regular, it may be that by another method such spaces could be obtained.

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