Brock University

Department of Computer Science

On the direct scaling approach of eliciting aggregated fuzzy information: The psychophysical view.

G. Gediga, I. Düntsch, J. Adams-Webber
Technical Report # CS-04-06
March 2004

Brock University
Department of Computer Science
St. Catharines, Ontario
Canada L2S 3A1
www.cosc.brocku.ca

On the direct scaling approach of eliciting aggregated fuzzy information: The psychophysical view

Günther Gediga, Ivo Düntsch
Department of Computer Science
Brock University
St. Catharines, Ontario, Canada, L2S 3A1
{gediga|duentsch}@cosc.brocku.ca

Jack Adams-Webber
Department of Psychology
Brock University
St. Catharines, Ontario, Canada, L2S 3A1
jadams@spartan.ac.brocku.ca

***Abstract*— We describe an experiment in which we elicit aggregated fuzzy information, and link it to the "direct scaling" approach of S.S. Stevens [1]. We also re-analyze earlier experiments [2] and compare the results to the proposed scaling method. Our results seem to indicate that a compensatory operator is not always necessary to model aggregated fuzzy information, and that humans internally use a simple rescaling.**

I. Introduction

Fuzzy sets were introduced independently by Klaua [3] and Zadeh [4], and have become a popular tool for handling vague information: Given a set U and a property P, a *fuzzy set* is a function $f_P : U \to [0, 1]$. The number $f_P(x)$ may be taken as an indicator of the graded membership of x in the (extent of the) concept described by P, and can be considered as an attempt to quantify vagueness of meaning (see also [5] for earlier work in this direction).

Though one may have reservations about the usefulness of offering continuously many choices, see e.g. [6], these are widely used to indicate degrees of belief.

Applications of fuzzy set theory have been mainly industrial, see e.g. [7], Chapter 11, and the references therein, or the Fuzzy Application Library [8]. In the behavioral sciences, fuzzy sets have received a mixed reaction; a recent account can be found in [9].

In the present study, we shall be concerned with the conjunctive aggregation of fuzzy information from several dimensions as performed by humans: Given a stimulus x and a concept $\mathcal{A}$, respondents are asked for the degree of membership $\mu(x, \mathcal{A})$ of the stimulus x in the concept $\mathcal{A}$. Given two different concepts $\mathcal{A}, \mathcal{B}$, the question arises how the degrees of membership are combined, i.e. what, if any, is the relationship between $\mu(x, \mathcal{A}),\ \mu(x, \mathcal{B})$, and $\mu(x, \mathcal{A} \wedge \mathcal{B})$. The early works suggest three ways of doing this:

$\min\{\mu(x, \mathcal{A}), \mu(x, \mathcal{B})\}$	*Min-norm* [3], [4],
$\max\{0, \mu(x, \mathcal{A}) + \mu(x, \mathcal{B}) - 1\}$	*Łukasiewicz norm* [3],
$\mu(x, \mathcal{A}) \cdot \mu(x, \mathcal{B})$	*Product norm* [4].

These operators are members of a class of functions called *triangular norms* (t–norms), that has been accepted by the fuzzy set community as the proper generalization of the conjunction operator of classical propositional logic. A t–norm is a function $\triangle$: $[0, 1]^2 \to [0, 1]$ which satisfies

(1) $x \triangle y = y \triangle x,$

(2) $(x \triangle y) \triangle z = x \triangle (y \triangle z),$

(3) $x \triangle y \leq x \triangle z,\ \text{if } y \leq z,$

(4) $x \triangle 1 = x.$

Historically, t–norms precede fuzzy sets, see e.g. [10]. It is well known that the min–norm is the largest t–norm in the sense that for any t–norm $\triangle$,

(5) $x \triangle y \leq \min\{x, y\}.$

Disjunctive aggregation is often modeled by *t–conorms* which are functions $\triangledown : [0, 1]^2 \to [0, 1]$ which satisfy

(6) $x \triangledown y = y \triangledown x,$

(7) $(x \triangledown y) \triangledown z = x \triangledown (y \triangledown z),$

(8) $x \triangledown y \leq x \triangle z,\ \text{if } y \leq z,$

(9) $x \triangledown 0 = x.$

Observe that axiomatically t–norms and t–conorms only differ in their boundary conditions (4), (9). The t–conorms corresponding to the classical t–norms are

$\max\{\mu(x, \mathcal{A}), \mu(x, \mathcal{B})\}$	*Max-conorm*,
$\min\{1, \mu(x, \mathcal{A}) + \mu(x, \mathcal{B})\}$	*Łukasiewicz conorm*,
$\mu(x, \mathcal{A}) + \mu(x, \mathcal{B}) - \mu(x, \mathcal{A}) \cdot \mu(x, \mathcal{B})$	*Sum norm*.

While the definition of a t–norm is mathematically satisfying, the question remains whether humans form conjunctive phrases according to the laws of t–norms. From their concept of prototype theory, Osherson and Smith [11], [12] argue strongly that this is not the case. Empirical studies send mixed signals: While [13] finds some support for product norm, the results of [14] favor the min–norm. In a study of human perception of color category membership, Kay and McDaniel [15] show that the membership value of an object in a conjunctive class (orange) may have a higher value than both of its constituents (red and yellow), thus violating (5). The experiments of [16] and [17] found "…overwhelming support for averaging rules and practically none for any rule that can be represented as a t–norm" [17]. It is not clear, though, to what extent these results are really contradictory, owing to different elicitation of membership functions and experimental design.

II. Compensatory aggregation

Oden [13] recognizes that "…the characteristic of compensation is an important property because it means that errors of opposite polarity will tend to cancel each other out". In a widely noticed experiment, Zimmermann and Zysno [2] (ZZ) investigate these compensatory effects, and argue that compound membership can be described by a fuzzy aggregation operator which combines the product norm $\triangle_p$ and its dual conorm ∇_p by the weighted geometric mean

$$\begin{aligned} d &= (x_0 \triangle_p x_1)^\gamma \cdot (x_0 \nabla_p x_1)^{1-\gamma} \\ &= (x_0 \cdot x_1)^\gamma \cdot (1-(1-x_0)\cdot(1-x_1))^{1-\gamma}. \end{aligned}$$

It may be noted that one can find simple one–parameter aggregation operators based on other classical t–norms, and that the ZZ-aggregation is not necessarily "the best" in terms of explained variance. In Table I we compare the following four functions:

ZZ aggregation:

$$f_{ZZ}(x_0,x_1) = (x_0x_1)^\gamma \cdot (1-(1-x_0)(1-x_1))^{1-\gamma} \tag{10}$$

Rescaled product:

$$f_{PR}(x_0,x_1) = (x_0 \cdot x_1)^\gamma \tag{11}$$

$\min-\max$ aggregation:

$$f_{MM}(x_0,x_1) = \min(x_0,x_1)\gamma + \max(x_0,x_1)(1-\gamma) \tag{12}$$

Łukasiewicz aggregation:

$$\begin{aligned} f_{LU}(x_0,x_1) = \max(0,x_0+x_1-1)\gamma + \\ (1-\max(0,1-(x_0+x_1)))(1-\gamma) \end{aligned} \tag{13}$$

Throughout the study, parameter estimation was done by the Levenberg-Marquart-Algorithm using ordinary least-squares, with the individual membership estimates as independent variables, and aggregated membership as the dependent variable. The differences among the models are marginal, not only in numbers, but also with respect to the residuals of the first three regression equations shown in Figure 1.

TABLE I
One parameter aggregation

Function	Weight of $\triangle$ (γ)	∇ ($1-\gamma$)	Explained variance
ZZ aggregation	0.409	0.591	0.925
Rescaled product	0.494	–	0.872
$\min-\max$ [18]	0.674	0.326	0.942
Łukasiewicz	0.605	0.395	0.931

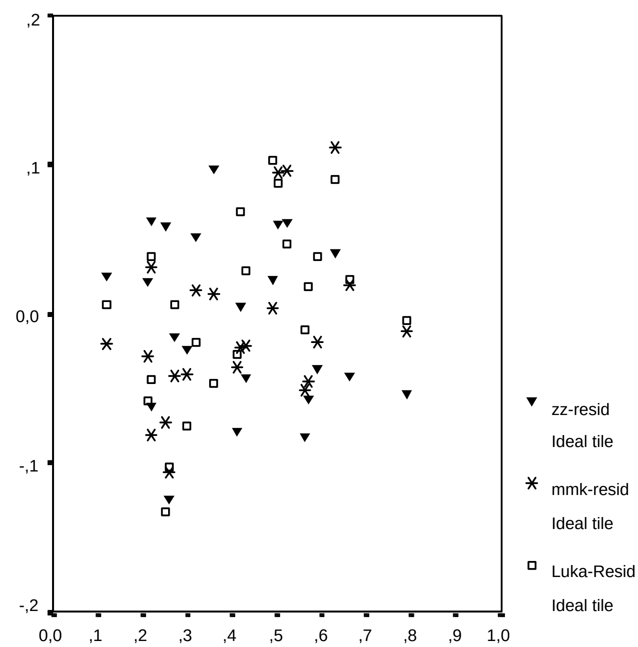

Fig. 1. Residuals

TABLE II
One parameter aggregation – a second data set

Function	Weight of $\triangle$ (γ)	∇ ($1-\gamma$)	Explained variance
ZZ aggregation	0.543	0.457	0.917
Rescaled product	0.598	–	0.939
$\min-\max$	0.924	0.076	0.924
Łukasiewicz	0.707	0.293	0.676

[2] offers a second data sets based on verbal descriptions of stimuli. The results of the four aggregation functions is presented in Table II. Once again the ZZ aggregation scheme seems not to be the best candidate for an aggregation operator. In both data sets the weighted min-max-rule is somewhat better than the ZZ aggregation; in the second data set, the rescaled product rule performs best, which is notable, because the rescaled product avoids the use of an "OR" part in the definition of the aggregation procedure

Finding aggregation operators is a pure model fitting act, and there is no sound theory or explanation why aggregation should work as it does. Furthermore, "The question of which interaction and union operator best reflects psychological reality still is open" [19] – even today, see [9].

Another aspect should be noted as well: Model fitting in [2] was done by using mean values of the responses. In the described experiment, this type of modeling is only adequate, if we assume that the individual ratings show large random variation and that the mean of the ratings reflects a "true" and reliable value.

The situation is even more complex. The regression approaches are based on non–linear functions which may cause problems for the proposed aggregation procedure: Since the models are applied to aggregated data, the non–linear relationship may be valid on an individual level, but the gauging of the fuzzyness is totally different among respondents. Suppose, for example, that the min–max-aggregation is valid on an individual level with the weights $\frac{2}{3}$ for the minimum and $\frac{1}{3}$ for the maximum – just as the empirical data indicate. Assuming

TABLE III

INDIVIDUAL AND AGGREGATED SCORES

	X	Y	aggregate observed	aggregate predicted
Subject 1	0.9	0.1	0.367	0.367
Subject 2	0.1	0.9	0.367	0.367
Mean	0.5	0.5	0.367	0.5

the data of Table III, the theory is obviously wrong when using mean values, although it is exact on the basis of individual data. Thus, a good fit of an aggregation operator using mean values does not mean that this aggregation operator fits well the given individual data. Indeed, our results reported below show a marked difference between the two models.

Niskanen [20] uses the ZZ data to compare several soft computing decision algorithms. The explained variances of these schemes are far better than the simple aggregation schemes presented above. However, all his algorithms use at least three parameters to describe the relationship between input and output. Thus, improved performance can be – at least partially – attributed to overfitting, and not to a better theory.

III. DIRECT SCALING

The elicitation procedure of [2] has been called "direct scaling" for fuzzy membership functions [21]. The psychologist S.S. Stevens introduced another "direct scaling" approach already in the 1950s [1], which has received little, if any, attention in the fuzzy set community: Let X be a dimension (loudness, length of a line, area of a rectangle...). When comparing a stimulus x with a standard X_0, the reported value of x by a human subject (in terms of another dimension) can be expressed by

$$S(x|X_0) = c \cdot x^{n(X)}$$

Here, $n(X)$ is a characteristic constant for the dimension-dimension transfer, e.g. $n(X) \approx \frac{2}{3}$ for area estimation in terms of length of a line. Assuming unit weights and identical power exponents for the stimuli under study, we define

$$\begin{aligned} s(x) &\stackrel{\text{def}}{=} S(x|X_0) = x^{n(X)}, \\ s(y) &\stackrel{\text{def}}{=} S(y|Y_0) = y^{m(Y)}, \\ s(x,y) &\stackrel{\text{def}}{=} S((x,y)|(X_0,Y_0)) = \left(x^{n(X)} y^{m(Y)}\right)^{\lambda} \\ &= (s(x)s(y))^{\lambda}, \end{aligned}$$

where $s(x,y)$ is the rescaled product. A weak version of the rescaled product model is the main effect scaling model:

$$s(x,y) \stackrel{\text{def}}{=} s(x|y) \cdot c_y = t(y|x) \cdot c_x.$$

Here, x and y are rescaled with unknown functions s and t and the aggregation is multiplicative. We discuss this approach in Section V-B.2.

One aim of the current investigation is to investigate whether Steven's approach (or a variation thereof) can be used to describe aggregated fuzzy memberships.

IV. THE EXPERIMENT

The experiment was performed using a WWW-based interface on a local server in a laboratory at Brock University[1].

For elicitation of the fuzzy membership functions we have used the same steps as [2] ("within factors"). Additionally, there are three independent groups ("between factors"), which vary the influence of verbal instruction within the process of membership generation, thus allowing for different possibilities of prototype description. In order to use "real" stimuli we have employed six elliptic forms and eight colours, which were expressed by a stimulus or by a verbal description, or both, resulting in three different groups. Prototypes were "circle" (shape) and "green" (color); a variant of the experiment had color "red". Elicitation of fuzzy membership was done by adopting the similarity view in the sense of [22]: Respondents were asked to indicate their agreement between the concepts and the prototype on a sliding scale ranging from 0% to 100%, resulting in $6 + 8 + 6 \cdot 8 = 186$ responses per participant. The sequence of the blocks, as well as the sequence of stimuli in each part, were chosen at random. We have always used stimuli for the concepts, and the following three prototype descriptions:

1) *Stimulus block*: Prototypes are
   - A black circle on a white background (for shape)
   - A green square (for color),
   - A green disk (for both).
2) *Verbal block*: Prototypes are
   - TEXT("The shape is a circle"),
   - TEXT("The color is green"),
   - TEXT("It is a green circle.")
3) *S+V block*: Prototypes are
   - A green disk + TEXT("The shapes of the stimuli are identical"),
   - A green disk + TEXT("The colours of the stimuli are identical"),
   - A green disk + TEXT("The stimuli are identical")

Respondents were students of Brock University; each was paid $10 for participating in the experiment. The initial group consisted of 60 respondents. In a second round of 25 respondents, the color "green" was substituted by "red". All in all, we collect 12,384 single observations. It turns out that there are only seven significant differences among the three prototype groups for "green" and none for "red". In Table IV we list the observed membership functions for the single concepts, and Table V gives the observed aggregated values.

[1] The interested reader can replicate the experiment at `www.xfrage.de/fuz/`

TABLE IV

MEAN VALUES OF SINGLE CONCEPTS (GREEN)

Green	C1	C2	C3	C4	C5	C6	C7
90.9	30.35	33.12	42.21	48.92	46.94	31.58	52.88

Circle	E80	E60	E40	E20	Line
90.18	48.86	40.89	33.94	27.6	12.61

TABLE V

OBSERVED AGGREGATION

	Green	C1	C2	C3	C4	C5	C6	C7
Circle	91,35	45.8	49.38	53.65	60.8	58.62	48.49	60.69
E80	59.71	29.47	32.72	35.33	43.54	42.38	29.96	42.81
E60	49.4	21.8	25.7	29.39	36.87	33.43	24.66	32.88
E40	43.28	20.53	22.41	24.85	31.29	27.41	22.26	30.71
E20	34.03	15.51	15.62	19.33	24.15	23.7	16.58	25.1
Line	19.65	8.47	9.31	10.14	13.78	16.61	8.58	12.86
	Red	C1	C2	C3	C4	C5	C6	C7
Circle	95.72	70.84	46.43	32.3	50.84	44.29	82.59	92.77
E80	67.46	49.39	30.9	23.52	38.75	32.75	57.55	61.61
E60	52.14	34.38	25.77	19.25	29.94	24.13	42.96	52.22
E40	39.88	28.83	20.14	14.42	22.23	18.54	34.57	37.88
E20	32.61	22.61	13.87	9.17	14.64	13.91	24.16	32.03
Line	15.9	10.91	6.97	3.83	7.26	5.19	11.26	12.07

V. MODEL FITTING

A. Compensatory aggregates and rescaled product norm

In a first step, it can be shown that none of the traditional t–norms fits the data. In a second analysis, we try to fit the four models (10) – (13), assuming that a single parameter governs the behavior of the subjects in every condition; as a check for this assumption, we estimate the parameter of the models in the "green" and "red" condition as well. Table VI shows the result for the aggregated data, i.e. the mean values of the responses to each stimulus/prototype pair, while Table VII shows the results for the individual data. Here, one parameter was estimated for every response; for each model we report the mean of the estimated values of γ.

TABLE VI

ONE PARAMETER AGGREGATION – AGGREGATED EXPERIMENTAL DATA

Color	Function	Weight of $\triangle$ (γ)	∇ $(1-\gamma)$	Explained variance
Green	ZZ aggregation	0.604	0.396	0.971
	Rescaled product	0.676	–	0.971
	min-max	0.876	0.124	0.814
	Łukasiewicz	0.671	0.329	0.903
Red	ZZ aggregation	0.653	0.363	0.954
	Rescaled product	0.697	–	0.957
	min-max	0.883	0.117	0.909
	Łukasiewicz	0.726	0.274	0.752

Table VI shows that the rescaled product and ZZ aggregation give the best result for the aggregated data in terms of explained variance. For the individual data, the weighted min-max aggregation consistently outperforms the other functions. The second best fit is shown by the rescaled product norm. The ZZ and Łukasiewicz approach do not seem to be the best choice for describing the data in this situation.

TABLE VII

ONE PARAMETER AGGREGATION – INDIVIDUAL EXPERIMENTAL DATA

Color	Function	Weight of $\triangle$ $(\bar{\gamma})$	$\nabla(1-\bar{\gamma})$	Explained variance
Both	ZZ aggregation	0.599	0.401	0.475
	Rescaled product	0.635	–	0.479
	min-max	0.809	0.191	0.519
	Łukasiewicz	0.833	0.167	0.427
Green	ZZ aggregation	0.591	0.409	0.449
	Rescaled product	0.632	–	0.454
	min-max	0.802	0.198	0.498
	Łukasiewicz	0.829	0.171	0.406
Red	ZZ aggregation	0.615	0.385	0.545
	Rescaled product	0.644	–	0.550
	min-max	0.826	0.174	0.580
	Łukasiewicz	0.842	0.158	0.486

B. Stevens' direct scaling

1) Is there a power law: In terms of aggregated data, the rescaled product as a first approximation of Stevens' law fits very well. There is no need for an additional fuzzy set based theory for aggregation, and there is no need for a compensating operator. Even the power constant of about 2/3 is predictable from Stevens' experiments in direct scaling of area estimation. Thus, by Occam's razor, this model would be the preferred one.

When fitting the aggregation equations to the individual data – resulting in one parameter per individuum - the goodness of fit index of the rescaled norm is much lower, and the min-max-rule seems to be the better choice.

The reduction of error within aggregated data is well known: Whereas the single values may be quite unstable because of uncontrollable influences, the mean (of 60 and more observations) is a quite noise free measure due to the law of large numbers.

Using means seems to put a bias on smooth functions such as the ZZ function or the rescaled product. Note that the min-max-rule (and Łukasiewicz as well) are limited by a function of min and max values of the one-dimensional scale values, whereas ZZ and rescaled product are not. Even if the individual aggregation functions reflects the functional form of (e.g.) a min-max rule, the relationship based on mean values may not.

The question whether "Stevens' law" or a min-max-rule governs the results is still open, and it seems to be dependent on the intended application of the results. If we want to describe aggregation in terms of reliable (aggregated) data, the rescaled product seems to be a good choice - if we want to describe aggregation for an individual, the min-max-rule seems to be more appropriate.

2) Are there only main effects: The rescaled product is a restrictive formulation of Stevens' approach, because it is assumed that the power exponent is identical for the stimuli under study. A weak version of Stevens' direct scaling approach is the main effect scaling model

$$s(x, y) = s(x|y) \cdot c_y = t(y|x) \cdot c_x.$$

The multiplicative relationship is a main feature of the direct scaling assumption in case of combined stimuli. This assumption can be used without assuming the validity of a power law (or even the simple version of a rescaled product). Since we have used a factorial design, we first predict the conditional functions $s(x|y)$ and $t(y|x)$, and then all elements in Table V by the marginals. The results reported in Table VIII and Table IX are quite satisfactory.

TABLE VIII

FIT OF THE CONDITIONAL FUNCTIONS IN THE MAIN EFFECT MODEL

Experiment	$s(form\|color)$	$t(color\|form)$
Green	0.890	0.959
Red	0.945	0.985

TABLE IX

FIT OF THE MAIN EFFECT MODEL

Experiment	Best t-Norm	Main effect model
Green	0.971	0.988
Red	0.957	0.990

3) Is there really a psycho–physical power law in fuzzy membership estimation: The power law in Stevens' psychophysical approach links parameters of a physical stimulus with parameters of human behavior. In every analysis above, we linked behavior parameters to other behavior parameters under the assumption that the behavior is governed by a power law using a relationship of the form $s(x,y) = g(u(x), v(y))$, assuming the g, u and v can be described by the power law. In case of the estimation of the conditional "form" function we can link the data to the parameter of the form. Since we use ellipses as stimuli, we may use the area of the ellipses as a scaling parameter.

Because $s(x|color) = c \cdot x^a$, we check the validity of the function $\ln(s(x|color)) = c + a \cdot \ln(x)$ by measuring x in percentages of the area of the circle given as the prototype. The badness of fit of this assumption is demonstrated in Figure 2.

Clearly, the power law fails as a description of a psycho–physical law in this context. A reason for this may be the fact that the fuzzy membership scale is bounded – a situation which is not present in the scaling experiments of Stevens. If we assume that the end–points of the fuzzy membership scale have to be reflected by an additional scaling function, it is quite natural to assume that there is a function which links the parameters of the physical world to the scaling. The complementary log-log transformation

$$\ln(-\ln(1 - s(x|color))) = c + a \cdot \ln(x)$$

is a good candidate for the linearization of the relationship, as Figure 3 demonstrates.

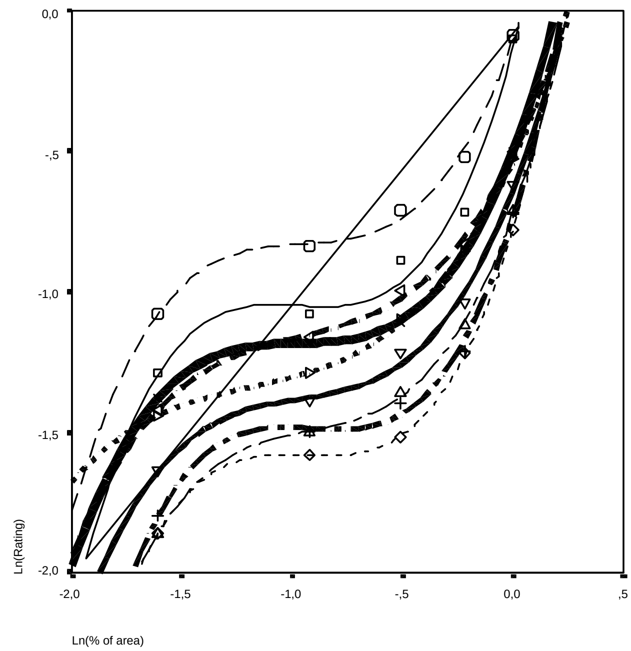

Fig. 2. Badness of fit of Stevens' law, given a physical dimension in fuzzy membership estimation

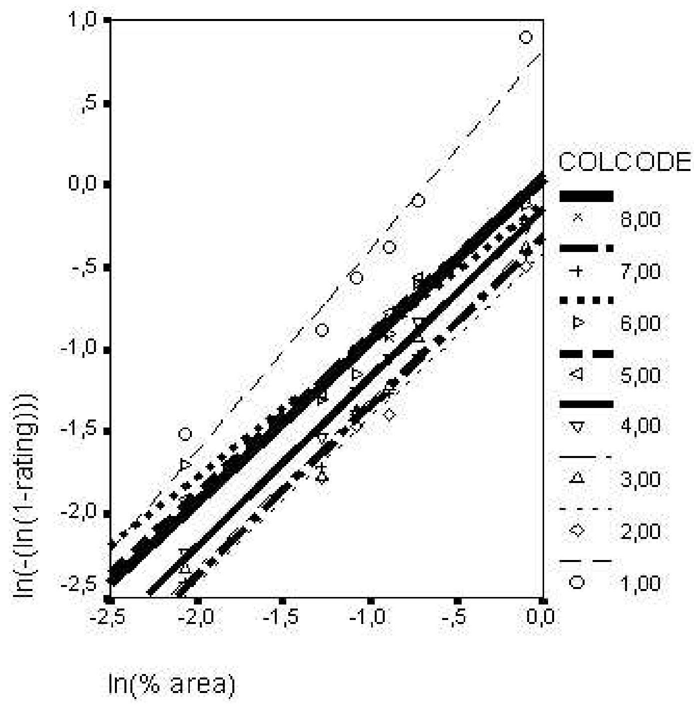

Fig. 3. Stevens' law including complementary log-log transformation as a link function (code 1= no color; 2..8 different colours)

VI. CONCLUSIONS

Using means of observation values as basis for a model is, for practical purposes, a simple and quick way to obtain results which are relatively error free. For decision making, however, it is necessary to model the behavior of the individual. Our results show that on the individual level there is some evidence that the min-max model (12) is preferable, whereas on the aggregated level the rescaled product (11) shows good success.

However, the high error variance in estimating the individual parameters shows that further research is necessary.

Because of the applicability and the earlier literature, we have concentrated on model building using the means of the individual observations.

Human aggregation of fuzzy information is – first of all – a scaling problem. The good performance of the rescaled product show that there is no need for a model to use information outside the intersection of the stimuli representation. One problem that arises is that the structural properties of the rescaled product are different to classical fuzzy operators.

As the main effect model holds, the estimation of fuzzy membership values for combined stimuli can be done without parametric and non–linear models, but it suffices to take the same amount of information using the marginals. Unfortunately, at present, we cannot offer a general solution for compound stimuli consisting of three or more parts.

The original Stevens' law does not hold for the direct scaling of fuzzy membership values. This drawback can be explained by the different scaling tasks: Whereas the original Stevens' task was the construction of a measurement line, the fuzzy membership values are restricted to the interval $[0, 1]$. The results of our experiment offer a link function – the complementary log-log transformation – which links the results of both scaling tasks. However, further experimental research has to be undertaken to validate this functional relationship.

Since our experimental conditions are rather artificial, the validity of the results should be tested in settings which mimic real-life situations.

Acknowledgments

Co-operation for this paper was supported by EC COST Action 274 "Theory and Applications of Relational Structures as Knowledge Instruments" (TARSKI; `http://www.tarski.org/`). Ivo Düntsch gratefully acknowledges support from the National Sciences and Engineering Research Council of Canada. Also, we are grateful to Linda Chang for her assistance in collecting data.

References

[1] S. S. Stevens, "On the psychophysical law," *Psychological Review*, vol. 64, pp. 153–181, 1957.

[2] H.-J. Zimmermann and P. Zysno, "Latent connectives in human decision making," *Fuzzy Sets and Systems*, vol. 4, pp. 37–51, 1980.

[3] Dieter Klaua, "Über einen Ansatz zur mehrwertigen Mengenlehre," *Monatsb. Deutsch. Akad. Wiss. Berlin*, vol. 7, pp. 859–867, 1965.

[4] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[5] C. I. Mosier, "A psychometric study of meaning," *Journal of Social Psychology*, vol. 13, pp. 123–140, 1941.

[6] G. A. Miller, "The magical number seven, plus or minus two: Some limits on our capacity for processing information," *Psychological Review*, vol. 63, no. 2, pp. 81–97, 1956.

[7] H.-J. Zimmermann, *Fuzzy set theory and its applications*, Kluwer, Boston, 3 edition, 1996.

[8] "Fuzzy application library," `http://www.fuzzytech.com/`, June 2003.

[9] M. Smithson and G. C. Oden, "Fuzzy set theory and applications in psychology," in *Practical Applications of Fuzzy Technologies*, H.-J. Zimmermann, Ed., Handbooks of Fuzzy Sets, chapter 17. Kluwer, Dordrecht, 1999.

[10] Erich Peter Klement, Radko Mesiar, and Endre Pap, *Triangular norms*, Kluwer, Dordrecht, 2000.

[11] D. N. Osherson and E. E. Smith, "Gradedness and conceptual combination," *Cognition*, vol. 12, pp. 299–318, 1982.

[12] D. N. Osherson and E. E. Smith, "On the adequacy of prototype theory as a theory of concepts," *Cognition*, vol. 9, pp. 35–58, 1981.

[13] Gregg C. Oden, "Integration of fuzzy logical information," *Journal of Experimental Psychology*, vol. 3, no. 4, pp. 565–575, 1977.

[14] U. Thole, H. J. Zimmermann, and P. Zysno, "On the suitability of minimum and product operators for the interpretation of fuzzy sets," *Fuzzy Sets and Systems*, vol. 2, pp. 167–180, 1979.

[15] Paul Kay and Chad K. McDaniel, "The linguistic significance of the meanings of basic color terms," *Language*, vol. 54, no. 3, pp. 610–646, 1978.

[16] Rami Zwick, David V. Budescu, and Thomas S. Wallsten, "An empirical study of the integration of linguistic probabilities," In Zetenyi [23], pp. 91–125.

[17] David V. Budescu, Rami Zwick, Thomas S. Wallsten, and Ido Erev, "Integration of linguistic probabilities," *International Journal of Man–Machine Studies*, vol. 33, pp. 657–676, 1990.

[18] L.F. Sugianto, "Optimal decision making with imprecise data," *International Journal of Fuzzy Systems*, vol. 3, pp. 569–576, 2001.

[19] Michael Smithson, *Fuzzy set analysis for behavioral and social sciences*, Springer–Verlag, 1987.

[20] Vesa A. Niskanen, "A soft multi-criteria decision-making approach to assessing the goodness of typical reasoning systems based on empirical data," *Fuzzy Sets and Systems*, vol. 131, no. 1, pp. 79–100, 2002.

[21] D. Dubois and H. Prade, "Fuzzy sets and probability: misunderstandings, bridges and gaps," in *Proceedings of the Second IEEE Conference on Fuzzy Systems*, 1993, pp. 1059–1068.

[22] Taner Bilgiç and I. Burhan Türkşen, "Measurement of membership functions: Theoretical and empirical work," in *Handbook of Fuzzy Sets and Systems*, D. Dubois and H. Prade, Eds., vol. 1, chapter 3, pp. 195–202. Kluwer, Dordrecht, 1999.

[23] T. Zetenyi, Ed., *Fuzzy sets in psychology*, North–Holland, Amsterdam, 1988.