Weak Contact Structures

Ivo Düntsch & Michael Winter*

Department of Computer Science,
Brock University,
St. Catharines, Ontario, Canada, L2S 3A1
{duentsch,mwinter}@brocku.ca

Abstract. In this paper we investigate weak contact relations C on a lattice L, in particular, the relation between various axioms for contact, and their connection to the algebraic structure of the lattice. Furthermore, we will study a notion of orthogonality which is motivated by a weak contact relation in an inner product space. Although this is clearly a spatial application, we will show that, in case L is distributive and C satisfies the orthogonality condition, the only weak contact relation on L is the overlap relation; in particular no RCC model satisfies this condition.

1 Introduction

Various hybrid algebraic/relational systems have been proposed for reasoning about spatial regions, among them the Region Connection Calculus (RCC) [12], proximity structures [13], adjacency relations [9], and others. In most cases, the underlying algebraic structure is a Boolean algebra $\langle B, +, \cdot, ^*, 0, 1\rangle$ whose non-zero elements are called *regions*. One also has binary relations P and C, respectively called *part of relation* and *contact relation*. P is the underlying partial order of the algebra and constitutes the *mereological* part of the structure [11], and C is often regarded as its *topological* part [14]. C is related to P in varying degrees of strength. The part common to most axiomatizations for such structures consists of four axioms:

C0. $(\forall x)\neg 0Cx$
C1. $(\forall x)[x \neq 0 \Rightarrow xCx]$
C2. $(\forall x)(\forall y)[xCy \Rightarrow yCx]$
C3. $(\forall x)(\forall y)(\forall z)[xCy \text{ and } yPz \Rightarrow xCz]$.

Axiom C3 prescribes only a very weak connection between C and P, which holds in the most common interpretations: If a region x is in contact with a region y, and y is a part of z, then x is in contact with z.

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A relation satisfying C0 – C3 will be called a *weak contact relation* (with respect to P). Note that these are indeed weaker than the contact relations of e.g. [2], which satisfy C0 – C3 and C5e below.

A *weak contact structure* is a tuple $\langle L, P, 0, 1, C\rangle$, where P is a bounded partial order on L (with smallest element 0 and largest element 1), and C is a weak contact relation (with respect to P). A *weak contact lattice* is a bounded lattice $\langle L, +, \cdot, 0, 1\rangle$ with a weak contact structure $\langle L, \leq, 0, 1, C\rangle$. We will usually denote a weak contact lattice by $\langle L, C\rangle$.

Some or all of the following additional axioms have appeared in various systems:

C4.	$(\forall x)(\forall y)(\forall z)[xC(y+z) \Rightarrow (xCy \text{ or } xCz)]$	(The sum axiom)
C5c.	$(\forall x)(\forall y)[(\forall z)(xCz \Rightarrow yCz) \Rightarrow xPy]$.	(The compatibility axiom)
C5e.	$(\forall x)(\forall y)[(\forall z)(xCz \Leftrightarrow yCz) \Rightarrow x = y]$.	(The extensionality axiom)
C5d.	$(\forall x \neq 1)(\exists y \neq 0)[x(-C)y]$.	(The disconnection axiom)
C6.	$(\forall x, y)[(x, y \neq 0 \wedge x + y = 1) \Rightarrow xCy]$	(The connection axiom)

If a weak contact relation satisfies one of the additional axioms, say Cx, we will denote this by $C \models Cx$.

It may be worthy of mention that C5d is equivalent to

$$(\forall x)[(\forall z)(xCz \Longleftrightarrow 1Cz) \Rightarrow x = 1] \tag{1.1}$$

C5e was one of the traditionally used axioms [14], and it is quite strong; it implies that each region is completely determined by those regions to which it is in contact. The difference between C5e and C5d is that in the former, C distinguishes any two non–zero elements, while with C5d, C distinguishes only 1 from any non–zero element, as can be seen from (1.1).

As we shall see below, C5c makes P definable from C, so that the primitive relation C suffices. Its strength also may be seen as a weakness, since on finite–cofinite Boolean algebras, and, in particular, finite Boolean algebras, the only relation C which satisfies C1 – C5c is the *overlap* relation, defined by

$$xOy \Longleftrightarrow (\exists z)[z \neq 0 \wedge xP\breve{\ } zPy]; \tag{1.2}$$

xOy means that x and y have a common non-zero part. If P is the underlying order $\leq$ of a lattice, then

$$xOy \Longleftrightarrow (\exists z)[z \neq 0 \wedge z \leq x \wedge z \leq y], \tag{1.3}$$

$$\Longleftrightarrow x \cdot y \neq 0. \tag{1.4}$$

If C satisfies, in addition, C6, then its underlying Boolean algebra is atomless [5]. This property makes, for example, the RCC [12] unsuitable for reasoning in finite structures. In order to remedy these effects, various measures have been proposed, all of which keep an underlying Boolean algebra, e.g. not demanding C5e, changing the basic relations, or employing second order structures [9, 7]. Another possibility is to relax the conditions on the underlying algebraic structure, e.g. not requiring that they be Boolean algebras; this seems sensible if our domain of interest does not include all possible Boolean combinations of regions.

Example 1. Consider the reference set

$$\{\text{North America, Canada,Mexico, Continental USA, } \emptyset\}$$

with a "part–of" relation P. This relation generates the non–distributive lattice L shown in Figure 1. Weak contact in this example is defined by

$$xCy \iff x, y \neq 0 \text{ and } (xPy \text{ or } yPx).$$

Observe that here the overlap relation restricted to non-zero and non-universal regions,

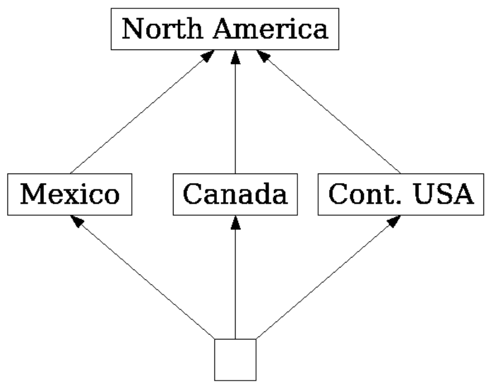


Fig. 1. Continental North America

i.e. regions not equal to the empty region and 'North America', is just the identity.

Example 2. Another example[1] is the (non–distributive) lattice of linear subspaces of an inner product space where contact is given by $UCV \iff \neg(U \perp V)$, i.e., U and V are not orthogonal. We will return to this example in Section 5.

A related approach was put forward by Cohn and Varzi [1]. Motivated by topological considerations, they consider various types $\langle C, Q, \sigma \rangle$ of contact structures, where C is a reflexive and symmetric relation on a family of sets, a "parthood relation" Q is defined from C as

$$xQy \iff (\forall z)[xCz \Rightarrow yCz]. \tag{1.5}$$

and a fusion operator σ, also defined from C. Loosely speaking, the fusion of a family of sets is its union. Q is clearly reflexive and transitive, but it need not be antisymmetric.

[1] We thank A. Urquhart for pointing this out.

It may be worthy to mention that Cohn and Varzi [1] do not, a priori, restrict the domain to algebraic or ordered structures.

In the present paper we investigate weak contact relations C on a lattice L, in particular, the interaction between the axioms C0 – C6 and the algebraic structure of the lattice. Our results are fairly basic, but we hope that they can assist the qualitative spatial reasoning community in choosing the axioms for (weak) contact relations appropriate for the domain under investigation.

2 Additional definitions and notation

If M is an ordered structure with smallest element 0, then $M^+ = \{y \in M : y \neq 0\}$. Furthermore, we shall usually identify a structure with its underlying set.

Throughout, L will denote a bounded lattice. If $a \in L$, the *pseudocomplement* a^* of a is the largest $b \in L$ such that $a \cdot b = 0$. If every $a \in L$ has a pseudocomplement, then L is called *pseudocomplemented*. Observe that the lattice of Figure 3 shows that a pseudocomplemented lattice need not be distributive. On the other hand, it is well known that every complete distributive lattice is pseudocomplemented [10]. $a \in L^+$ is called *dense*, if $a \cdot b \neq 0$ for all $b \in L^+$. It is well known that $a + a^*$ is dense for each $a \in L$, and each dense element can be written in this form [10].

If $a \in L^+$, a pair $\langle b, c\rangle$ of non-zero elements of L is called a *partition of a* if $b + c = a$ and $b \cdot c = 0$.

For a set U, we denote by $\mathrm{Rel}(U)$ the set of all binary relations on U. If $R, S \in \mathrm{Rel}(U)$, then

$$R \,;\, S = \{\langle a, c\rangle : (\exists b)[aRb \text{ and } bRc]\}$$

is the *composition* of R and S, and $R^{\smile} = \{\langle a, b\rangle : bRa\}$ its *converse*. We also define $R(x) = \{y \in L : xRy\}$. The (right) *residual of R with respect to S* is the relation

$$R \diagdown S = -(R^{\smile} \,;\, -S). \tag{2.1}$$

Here, complementation $-$ is taken in $\mathrm{Rel}(U)$. It is well known that

$$x(R \diagdown S)y \iff R^{\smile}(x) \subseteq S^{\smile}(y), \tag{2.2}$$

and it is easy to see that $R \diagdown R$ is a quasi order, i.e. reflexive and transitive. Furthermore, if $R, S, T \in \mathrm{Rel}(U)$, then the *de Morgan equivalences*

$$(R \,;\, S) \cap T = \emptyset \iff (R^{\smile} \,;\, T) \cap S = \emptyset \iff (T \,;\, S^{\smile}) \cap R = \emptyset \tag{2.3}$$

hold in $\mathrm{Rel}(U)$.

3 Mereology on weak algebraic structures

Our first results concerns the algebraic structure of the collection of all contact relations on a lattice L:

Theorem 1. *The collection of weak contact relations is a complete lattice with smallest element O and largest element $L^+ \times L^+$.*

Proof. We only show that O is the smallest weak contact relation on L, and leave the rest to the reader. Clearly, O satisfies C0 – C3. Now, suppose that C is a weak contact relation on L. To show that $O \subseteq C$, let $x, y \in L^+$ such that xOy. By (1.4), there is some $z \neq 0$ with $z \leq x, y$. From C0 we obtain zCz which implies

$$zCz \overset{C3}{\Rightarrow} zCx \overset{C2}{\Rightarrow} xCz \overset{C3}{\Rightarrow} xCy. \qquad \square$$

It is known that for weak contact lattices which are Boolean algebras and satisfy C4, the axioms C5c, C5e, and C5d are equivalent. In our more general setting, we only have

Theorem 2. *C5c $\Rightarrow$ C5e $\Rightarrow$ C5d.*

Proof. The first implication follows immediately from the antisymmetry of $\subseteq$ and $\leq$. Suppose that C satisfies C5e. If xCz for all $z \neq 0$, then $x = 1$ by C5e which shows that C satisfies C5d. $\square$

Let us next consider the case that C satisfies C4:

Theorem 3. *If $C \models C4$, then*

$$C \models C5c \Longleftrightarrow C \models C5e.$$

Proof. "$\Rightarrow$" follows from Theorem 2. For the other direction, let $x, y \in L$ such that $C(x) \subseteq C(y)$; we will show that $x + y = y$: Suppose that $(x + y)Cz$; by C4, we have xCz or yCz, and from $C(x) \subseteq C(y)$, we obtain yCz in any case. Thus, $C(x + y) \subseteq C(y)$; since $C(y) \subseteq C(x + y)$ by C3, it follows that $C(x + y) = C(y)$, and C5e now implies that $x + y = y$, i.e. $x \leq y$. $\square$

The following example shows that C4 and C5c are independent:

Example 3. There are weak contact lattices $\langle L_1, C\rangle, \langle L_2, C\rangle$ such that

1. $\langle L_1, C\rangle$ satisfies C5c and not C4.
2. $\langle L_2, C\rangle$ satisfies C4 and not C5e (and, hence, not C5c).

Indeed, for 1. consider Figure 2, and for 2. consider Figure 3.

Fig. 2. $\langle L_1, O\rangle$

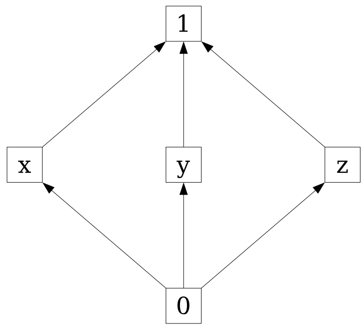
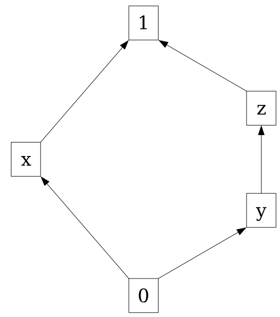


Fig. 3. $\langle L_2, O\rangle$



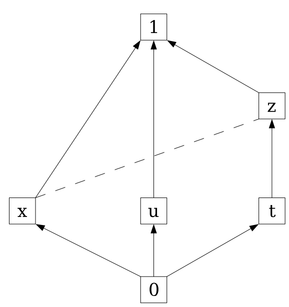
For the general case, we have

Theorem 4. *C5d $\not\Rightarrow$ C5e $\not\Rightarrow$ C5c.*

Proof. The lattice $\langle L_1, C\rangle$ of Figure 3 with $C = O$ satisfies C5d but not C5e: For each non-zero element there is a non-zero element disjoint to it, showing C5d. On the other hand, $O(y) = O(z)$, and $y \neq z$.

The lattice $\langle L_3, C\rangle$ shown in Figure 4 with $C = O \cup \{\langle x, z\rangle, \langle z, x\rangle\}$, satisfies C5e but not C5c: We have $C(x) \subseteq C(z)$, but $x \not\leq z$. $\square$

Fig. 4. $\langle L_3, C\rangle$



This leads to connections between C, $C \diagdown C$, and P of different strengths:

Lemma 1. *1.* $C\ ;\ (C \diagdown C) \subseteq C$.
2. $C \diagdown C$ *is antisymmetric if and only if* $C \models$ C5*e*.
3. $P \subseteq C \diagdown C$.
4. $C \diagdown C \subseteq P$ *if and only if* $C \models$ C5*c*.

Proof. 1. This follows immediately from the residual property of $C \diagdown C$; for completeness, we give a short proof. Assume that $C\ ;\ (C \diagdown C) \nsubseteq C$, i.e. $[C\ ;\ (C \diagdown C)] \cap -C \neq \emptyset$. Then, the de Morgan equivalences (2.3), and the fact that C is symmetric show that

$$\begin{aligned}
[C\ ;\ (C \diagdown C)] \cap -C \neq \emptyset &\iff (C\ ;\ -C) \cap (C \diagdown C) \neq \emptyset \\
&\iff (C\ ;\ -C) \cap -(C\ ;\ -C) \neq \emptyset,
\end{aligned}$$

a contradiction.

We observe that 1. is a compatibility condition such as C3 with respect to $C \diagdown C$. Thus, it is implicitly valid in the setup of [1].

2. This was already observed without proof in [1], see also [4]. Let $C \diagdown C$ be antisymmetric, and $a, b \in L$ such that $C(a) = C(b)$. From (2.2) and the fact that C is symmetric, we obtain that $a(C \diagdown C)b$ and $b(C \diagdown C)a$, and our hypothesis implies that $a = b$. Thus, $C \models C5e$. Conversely, let $a(C \diagdown C)b$ and $b(C \diagdown C)a$. By (2.2), this implies $C(a) = C(b)$, and hence, $a = b$ by C5e.

3. Consider the following:

$$(x \leq y \wedge xCz) \stackrel{C2}{\Rightarrow} (zCx \wedge x \leq y) \stackrel{C3}{\Rightarrow} zCy \stackrel{C2}{\Rightarrow} yCz \Rightarrow C(x) \subseteq C(y) \Rightarrow x(C \diagdown C)y.$$

4. This is just the definition. □

Observe that Figure 4 shows a weak contact lattice where $C \diagdown C$ is a partial order, that strictly contains P; the additional pairs are $\langle x, z\rangle, \langle z, x\rangle$. It may be noted that, even if $C \diagdown C$ is a partial order, it need not be a lattice order.

Corollary 1. *C5c implies C3.*

Proof. Since $C \models C5c$, Lemma 1(3,4) give us $P = C \diagdown C$. C3 now follows from Lemma 1(1). □

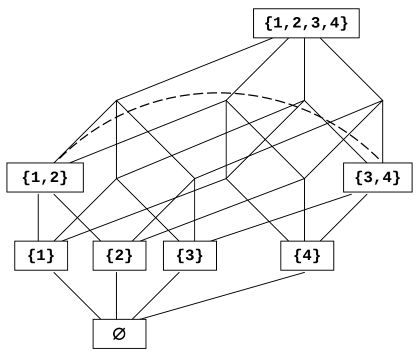
Already our weakest extensionality axiom may have an effect on the algebraic structure of L:

Theorem 5. *If L is a distributive bounded pseudocomplemented lattice and C satisfies C5d, then L is a Boolean algebra.*

Proof. Suppose that $x \neq 0, 1$, and assume that $x + x^* \neq 1$. Then, by C5d, there is some $z \neq 0$ such that $z(-C)(x + x^*)$. Hence, $z \cdot (x + x^*) = 0$, which contradicts the fact that $x + x^*$ is dense; therefore, $x + x^* = 1$, i.e. x^* is a complement of x. Since in distributive lattices complements are unique, it follows that L is a Boolean algebra. □

This strengthens a Theorem of [3].

In [6] we have shown that for a distributive L, $C \models C4$ and $C \models C5e$ imply that $C = O$. Our next example shows that C4 is essential:



$C(\{1\})$	$= \uparrow\{1\}$
$C(\{2\})$	$= \uparrow\{2\}$
$C(\{3\})$	$= \uparrow\{3\}$
$C(\{4\})$	$= \uparrow\{4\}$
$C(\{1,2\})$	$= \uparrow\{3,4\}\cup\uparrow\{1\}\cup\uparrow\{2\}$
$C(\{1,3\})$	$= \uparrow\{1\}\cup\uparrow\{3\}$
$C(\{1,4\})$	$= \uparrow\{1\}\cup\uparrow\{4\}$
$C(\{2,3\})$	$= \uparrow\{2\}\cup\uparrow\{3\}$
$C(\{2,4\})$	$= \uparrow\{2\}\cup\uparrow\{4\}$
$C(\{3,4\})$	$= \uparrow\{1,2\}\cup\uparrow\{3\}\cup\uparrow\{4\}$
$C(\{1,2,3\})$	$= \uparrow\{1\}\cup\uparrow\{2\}\cup\uparrow\{3\}$
$C(\{1,3,4\})$	$= \uparrow\{1\}\cup\uparrow\{3\}\cup\uparrow\{4\}$
$C(\{1,2,4\})$	$= \uparrow\{1\}\cup\uparrow\{2\}\cup\uparrow\{4\}$
$C(\{2,3,4\})$	$= \uparrow\{2\}\cup\uparrow\{3\}\cup\uparrow\{4\}$

Fig. 5. A BA with $C \neq O$ satisfying C5c and not C4

Example 4. Consider the sixteen element Boolean algebra shown in Figure 5. There, C is the smallest contact relation containing O and $\langle\{1,2\},\{3,4\}\rangle$; $\uparrow M$ denotes the upset $\{x : (\exists y \in M) y \leq x\}$ induced by M. Clearly, C is a weak contact relation different from O, and we see from the table that it satisfies C5e, but not C4.

An example of a weak contact relation satisfying all of C0–C5c with an underlying modular and non–distribute lattice is shown in Figure 6.

4 Overlap and the lattice structure

Since $xOy \Longleftrightarrow x \cdot y \neq 0$, we can expect that additional properties of the weak contact relation O are strongly related to the lattice properties. In this section, we will explore this relationship.

First, observe that (1.4) implies that O is the universal relation on L^+ in case 0 is meet irreducible. Thus, in this case O is the only contact relation.

Theorem 6. $C = O \Longleftrightarrow (\forall x)(\forall y)(\forall z)[xC(y \cdot z) \Leftrightarrow (x \cdot y)Cz]$.

Proof. "$\Rightarrow$": This follows immediately from (1.4) and the associativity of $\cdot$.

"$\Leftarrow''$: By Theorem 1 it is sufficient to show $C \subseteq O$. Suppose xCy. Then, $xC(y \cdot 1)$, and hence $(x \cdot y)C1$ using the assumption on C. From C0 we conclude $x \cdot y \neq 0$. $\square$

Lemma 2. *1. If L is distributive then O satisfies C4.*
2. *If O satisfies C4 and C5e then L is distributive.*
3. *If L is a bounded pseudocomplemented lattice, then O satisfies C5c.*

4. *If L is a bounded pseudocomplemented lattice, then*

$$L \text{ is a Boolean algebra} \iff O \text{ satisfies C4 and C5e.}$$

Proof. 1. Suppose that L is distributive, and let $aO(b+c)$; then $a \cdot (b+c) \neq 0$. Since L is distributive, this is equivalent to $a \cdot b + a \cdot c \neq 0$. It follows that $a \cdot b \neq 0$ or $a \cdot c \neq 0$, i.e. aOb or aOc.

2. Let $a,b,c,d \in L$. Then, we have

$$\begin{aligned}
dO[a \cdot (b+c)] &\iff (d \cdot a)O(b+c) && \text{by Theorem 6}\\
&\iff (d \cdot a)Ob \text{ or } (d \cdot a)Oc && \text{by C3 and C4}\\
&\iff dO(b \cdot a) \text{ or } dO(c \cdot a) && \text{by Theorem 6}\\
&\iff dO(a \cdot b + a \cdot c) && \text{by C3 and C4}
\end{aligned}$$

Thus, $O(a \cdot (b+c)) = O(a \cdot b + a \cdot c)$, and C5e implies $a \cdot (b+c) = a \cdot b + a \cdot c$.

3. Suppose $O(x) \subseteq O(y)$. If $x \cdot y^* \neq 0$ we conclude $y^* \in O(x) \subseteq O(y)$ and hence $y \cdot y^* \neq 0$, a contradiction. Therefore, $x \cdot y^* = 0$ which is equivalent to $x \leq y$.

4. "$\Rightarrow$": By 1. we know that O satisfies C4 and by 2. that O satisfies C5c. The latter implies C5e.

"$\Leftarrow''$: By 2. we know that L is distributive. Since C5e implies C5d, we may conclude from Theorem 5 that L is a Boolean algebra . □

It can be seen from Figure 3 that the converse of 1. of Lemma 2 is not true: O satisfies C4, but the lattice is not distributive. Furthermore, we cannot replace O by an arbitrary contact relation C: If L is the eight element Boolean algebra with atoms x,y,z, let C be the smallest contact relation on L such that $xC(y+z)$. Then, $x(-C)y$ and $x(-C)z$, showing that $C \not\models C4$.

The converse of 2. of Lemma 2 does not hold either: In any bounded chain we have $C(x) = C(1)$ for all $x \neq 0$, and thus, C5d is not satisfied.

Observe that in the proof of Lemma 2(3), C4 was only used to establish distributivity. Since every finite distributive bounded lattice is pseudocomplemented, the previous results show that, for finite L, we have to give up C4 or C5e or both not to result in a Boolean algebra, where O is the only contact relation which satisfies C0 – C5e. In other words, if L is a bounded distributive lattice which is not a Boolean algebra, then O does not satisfy C5d, see Lemma 2.

5 Orthogonality

Our notion of orthogonality is motivated by the weak contact relation in an inner product space given in Example 2. There, weak contact of two linear subspaces U and V was defined by $UCV \iff \neg(U \perp V)$, i.e. there are $u \in U$ and $v \in V$ with $\langle u,v \rangle \neq 0$;

here, $\langle u, v\rangle$ is the inner product of u and v. It is easy to verify that this relation indeed satisfies C0-C3.

Example 2 suggests the following definition: A pair of non-zero elements $\langle y, z\rangle$ is called an *orthogonal partition* of an element x if $y(-C)z$ and $y+z=x$. Note, that $y(-C)z$ implies $y \cdot z = 0$, and that the definition is symmetric.

By Theorem 1 we have $O \subseteq C$ so that every orthogonal partition is a partition.

In the example, we are able to switch from a given partition to an orthogonal one using the orthonormalization procedure by Gram and Schmidt; this can be done by keeping one subspace and varying the other one. We use an abstract version of this technique as an additional property of weak contact providing orthogonalization:

N. $(\forall x)(\forall y)(\forall z)[(y,z)$ is a partition of $x \Rightarrow$
$(\exists u)(y,u)$ is an orthogonal partition of $x]$.

The weak contact relation in Example 2 has property N. Notice, that O always satisfies N.

If the underlying lattice is distributive, then N is very restrictive:

Lemma 3. *Let L be distributive.*

1. *If $\langle y, u\rangle$ and $\langle y, v\rangle$ are partitions of x then $u = v$.*
2. *If C satisfies N then $C = O$.*

Proof. 1. By definition we have $y \cdot u = 0 = y \cdot v$ and $y+u = x = y+v$ so that we conclude $u = v$ by the distributivity of L.

2. By Theorem 1 it remains to show that $C \subseteq O$. Suppose $y(-O)z$. If $y = 0$ or $z = 0$ we conclude $y(-C)z$ using C0. Otherwise, $\langle y, z\rangle$ is a partition of $y+z$. By property N there is a $u \in L$ so that $\langle y, u\rangle$ is an orthogonal partition of $y+z$. Since every orthogonal partition is a partition we conclude $z = u$ using 1. This implies $y(-C)z$. □

Corollary 2. *If L is a bounded pseudocomplemented lattice, then*

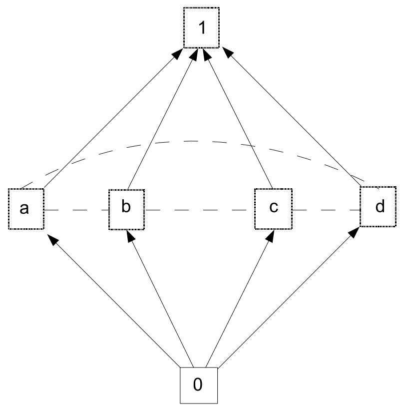
$$L \text{ is a Boolean algebra and } C \text{ satisfies N} \Longleftrightarrow C = O \text{ and } C \text{ satisfies C4 and C5e.}$$

Proof. The claim follows immediately from Lemma 3 and Lemma 2. □

The weak contact structure $\langle L, C\rangle$ given in Figure 6 satisfies C0–C5d as well as N; furthermore, $C \neq O$.

In any RCC model we have $C \neq O$ because of C6. Therefore, the previous theorem shows that there is no RCC model satisfying property N.

Finally, the weak contact structure of Figure 6 exhibits a weak contact relation $C \neq O$ on a non–distributive modular lattice which satisfies C0-C5e and N.



$$
\begin{aligned}
C(0) &= \emptyset \\
C(a) &= \{a,b,d,1\} \\
C(b) &= \{a,b,c,1\} \\
C(c) &= \{b,c,d,1\} \\
C(d) &= \{a,c,d,1\} \\
C(1) &= \{a,b,c,d,1\}
\end{aligned}
$$

Fig. 6. A non–distributive modular lattice with $C \neq O$ satisfying C0–C5d and N

6 Conclusion and outlook

We have looked at the "fine structure" of the interplay between properties of contact relations and those of the underlying algebra. In particular, we have shown that the various forms of extensionality – and their interplay with C4 – do not coincide when our basic structure is not a Boolean algebra. Our results are elementary and not difficult to prove; nevertheless, we hope that they are useful for the axiomatization for application domains where not all possible Boolean combinations of regions are required. We intend to continue our investigations by considering still weaker structures such as semi–lattices, and also topological domains in the spirit of [1]. Additionally, the topic of orthogonal partitions and weak contact relations satisfying N merits further attention.

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