Betweenness and comparability obtained from binary relations

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Abstract. We give a brief overview of the axiomatic development of betweenness relations, and investigate the connections between these and comparability graphs. Furthermore, we characterize betweenness relations induced by reflexive and antisymmetric binary relations, thus generalizing earlier results on partial orders. We conclude with a sketch of the algorithmic aspects of recognizing induced betweenness relations.

1 Introduction

The study of betweenness relations goes back to at least 1917, when Huntington and Kline [10] published "Sets of independent postulates for betweenness." The concept of betweenness can have rather different meanings – we quote from [10]:

- K is the class of points on a line; AXB means that point X lies between the points A and B.
- K is the class of natural numbers, AXB means that number X is the product of the numbers A and B.
- K is the class of human beings; AXB means that X is a descendant of A and an ancestor of B.
- K is the class of points on the circumference of a circle; AXB means that the arc $A-X-B$ is less than 180°.

In the sequel they concentrate on the geometric case. Throughout, B is a ternary relation on a suitable set, and $B(x,y,z)$ is read as "y lies between x and z." Quantifier free axioms are assumed to be universally quantified. The notation $\#M$ means that all elements of M are different.

Their first set of four postulates is concerned with three elements:

⋆ Both authors gratefully acknowledge support from the Natural Sciences and Engineering Research Council of Canada.

HK A. $B(a,b,c) \implies B(c,b,a)$.
HK B. $\#\{a,b,c\} \implies B(b,a,c) \vee B(c,a,b) \vee B(a,b,c) \vee B(c,b,a) \vee B(a,c,b) \vee B(b,c,a)$.
HK C. $\#\{a,b,c\} \implies \neg(B(a,b,c) \wedge B(a,c,b))$.
HK D. $B(a,b,c) \implies \#\{a,b,c\}$.

They proceed by adding another eight universal postulates which describe the configurations with four distinct elements, and state "If we think of *a* and *b* as two given points on a line, the hypotheses of these postulates state all the possible relations in which two other distinct points *x* and *y* of the line can stand in regard to *a* and *b*." For later use, we mention

HK 1. $\#\{b,c\} \wedge B(a,b,c) \wedge B(b,c,d) \implies B(a,b,d)$.
HK 2. $B(a,b,c) \wedge B(b,d,c) \implies B(a,b,d)$.

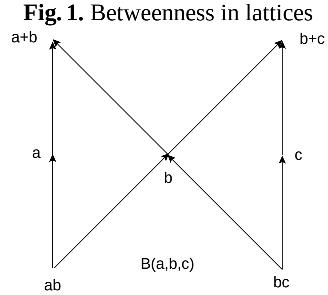
While HK A is widely accepted in many contexts (unless one requires one-way streets), the other postulates make very strong assumptions: HK B says that for any three different elements, one is between the other two, and HK D rules out what one might call degenerate triples. Postulate HK C prohibits "nesting." Their set of postulates completely axiomatizes betweenness if we restrict the domain to linear orders. The postulates HK 1 and HK 2 subsequently became known as "outer transitivity" and "inner transitivity", respectively. Some years later, Huntington [9] proposed a ninth postulate,

H 9. $\#\{a,b,c,x\} \wedge B(a,b,c) \implies B(a,b,x) \vee B(x,b,c)$.

and showed that the axiom system $\{HK\ A - HK\ D, H\ 9\}$ is equivalent to the one given in [10].

Betweenness in metric spaces was investigated by Karl Menger [12], and, in a further development, Pitcher and Smiley [14] direct their interest to betweenness relations in lattices, and define $B(a,b,c) \iff a \cdot b + b \cdot c = b = (a+b) \cdot (b+c)$., see Figure 1. Observe that $a \leq b \leq c$ implies $B(a,b,c)$.

Fig. 1. Betweenness in lattices



The main difference from the system of [9] is the omission of HK B, which is geared to linear orders, and the introduction of "degenerate triples" which contain at most two distinct variables. Thus, their basic system consists only of the symmetry axiom HK A, and

PS β. $B(a,b,c) \wedge B(a,c,b) \Longleftrightarrow b = c.$.

They continue to explore various transitivity conditions and their connection to lattice properties. For example,

Theorem 1 *[14] A lattice L is modular if and only if its betweenness relation satisfies HK 2*

In a parallel development, Tarski proposed an axiom system for first order Euclidean plane geometry based on two relations: equidistance and betweenness. An overview of the system and its history can be found in [16], and axiom numbers below refer to this exposition. His axioms for betweenness are of course very much tailored for the purpose that he had in mind, and many of these are specific to the geometric context. We mention those which are of a more general nature:

Ax 6. $B(a,b,a) \Rightarrow a = b$. (Identity)
Ax 12. $B(a,b,b)$. (Reflexivity)
Ax 13. $a = b \Rightarrow B(a,b,a)$. (Equality)
Ax 14. $B(a,b,c) \Rightarrow B(c,b,a)$. (Symmetry)
Ax 15. $B(a,b,c)$ and $B(b,d,c) \Rightarrow B(a,b,d)$. (Inner transitivity)
Ax 16. $B(a,b,c)$ and $B(b,c,d)$ and $b \neq c \Rightarrow B(a,b,d)$. (Outer transitivity)

In a further generalization, Birkhoff [2] defines a betweenness relation on U by the condition $B(a,b,c) \Longleftrightarrow a \leq b \leq c$ or $c \leq b \leq a$, where $\leq$ is a partial order. The reader is then invited to prove the following:

Birk 0. $B(a,b,c) \Rightarrow B(c,b,a)$.
Birk 1. $B(a,b,c)$ and $B(a,c,b) \Rightarrow b = c$.
Birk 2. $B(a,b,c)$ and $B(a,d,b) \Rightarrow B(a,d,c)$.
Birk 3. $B(a,b,c)$ and $B(b,c,d)$ and $b \neq c \Rightarrow B(a,b,d)$.
Birk 4. $B(a,b,c)$ and $B(a,c,d) \Rightarrow B(b,c,d)$.

It turns out that Birk 1 and Birk 2 follow from Birk 0, Birk 3, and Birk 4. Furthermore, these properties are not sufficient to characterize those betweenness relations which are induced by a partial order. In a fundamental article in 1950, Martin Altwegg [1] obtained an axiom system for such betweenness relations:

Z_1. $B(a,a,a)$.
Z_2. $B(a,b,c) \Rightarrow B(c,b,a)$.
Z_3. $B(a,b,c) \Rightarrow B(a,a,b)$.
Z_4. $B(a,b,a) \Rightarrow a = b$.
Z_5. $B(a,b,c)$ and $B(b,c,d)$ and $b \neq c \Rightarrow B(a,b,d)$.
Z_6. Suppose that $\langle a_0, a_1, \ldots, a_{2n}, a_{2n+1} \rangle$ and $a_0 = a_{2n+1}$. If $B(a_{i-1}, a_{i-1}, a_i)$ for all $0 < i \leq 2n+1$, and not $B(a_{i-1}, a_i, a_{i+1})$ for all $0 < i < 2n+1$, then $B(a_{2n}, a_0, a_1)$.

Shortly after Altwegg's paper, Sholander [15] investigated betweenness for various kind of orderings, and derived Altwegg's characterization as a Corollary. His system, however, is somewhat shorter. In addition to Z_6, he only supposes

Sho B. $B(a,b,a) \iff a = b$.
Sho C. $B(a,b,c) \land B(b,d,e) \implies B(c,b,d) \lor B(e,b,a)$.

Altwegg's work seems to have been largely forgotten – a notable exception being [3] –, and a search on the Science Citation Index reveals only three citations since 1965. A case in point is the widely studied area of comparability graphs that are closely connected to betweenness relations; as far as we know, researchers in this area were not aware of the earlier results. It is one of the aims of this paper to draw attention to Altwegg's work, and point out some connections between betweenness relations and comparability graphs.

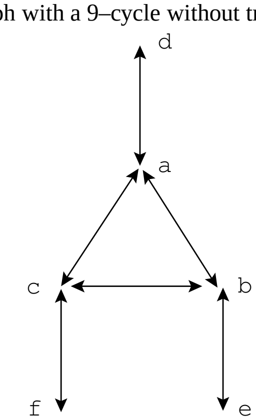
2 Notation

The universe of our relations is a non-empty set U. The identity relation on U is denoted by $1'_U$, or just $1'$, if U is understood. For each $n \in \omega$, $\mathbf{n}$ denotes the set of all $k < n$. For $M \subseteq U$, we abbreviate by $\#M$ the statement that all elements of M are different.

A partial order $\leq$ is called *connected* if there is a path in $\leq \cup \geq$ from a to b for all $a, b \in U$. A *component of* $\leq$ is a maximally connected subset of $<$.

Graphs are assumed to be undirected, without loops or multiple edges. In other words, a graph is just a symmetric irreflexive binary relation on U. A *cycle C in G of length n* is a sequence of elements $a_0, \dots, a_{n-1}$ of U such that $a_0 G a_1 \dots a_{n-2} G a_{n-1} G a_0$; repetitions are allowed. A cycle is sometimes called a *closed path in G*. A cycle is *strict*, if $\#\{a_0, \dots, a_{n-1}\}$, and we denote by C_n the strict cycle of length n. A *triangular chord of the cycle C* is an edge $\{a_i, a_{i+2}\}$ of G; here, addition is modulo n. For example, the graph of Figure 2 contains the cycle $d, a, b, e, b, c, f, c, a, d$ of length 9, which has no triangular chords [5].

Fig. 2. A graph with a 9–cycle without triangular chords



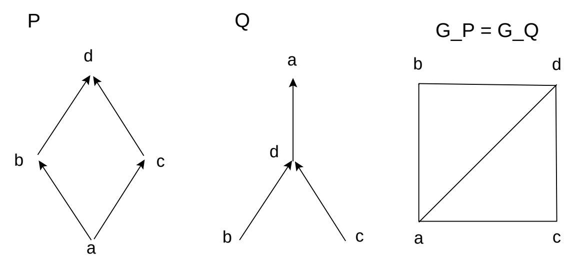
3 Comparability graphs

If P is a partial order on U, its *comparability graph* G_P is the set of all comparable proper pairs, i.e. $G_P = (P \cup P\breve{}) \setminus 1'$; here, $P\breve{}$ is the relational converse of P. A graph G is called a *comparability graph*, if $G = G_P$ for some partial order P. We denote the class of comparability graphs by $\mathbb{G}_{\leq}$.

Two partial orders P, Q on the same set U are called *equivalent* if for all components M_P of P, $P \upharpoonright M_P = Q \upharpoonright M_P$ or $P \upharpoonright M_P = Q\breve{} \upharpoonright M_P$.

Example 1 *Consider the partial orderings P, Q, shown in Figure 3; obviously, these are not equivalent. In both cases, the only non–comparable elements are b and c, so that $G_P = G_Q$.* □

Fig. 3. Non-equivalent partial orders with the same comparability graph



Comparability graphs have been investigated since the early 1960s, and we invite the reader to consult the overview by Kelly [11] for more information. A characterization of comparability graphs is as follows:

Theorem 2 *[4, 5]*

GH. G is a comparability graph if and only if every odd cycle of G contains a triangular chord.

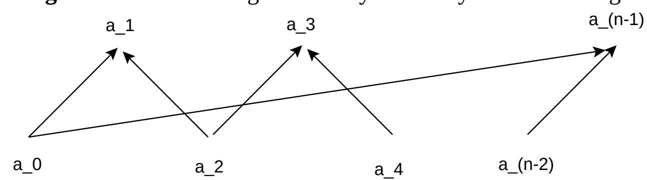
For example, the graph of Figure 2 is not a comparability graph. It is instructive, and will be useful later on, to consider the strict partial orders $<$ obtained from strict cycles of even length n. As these cycles have no triangles, each path in $<$ has length 2, and, consequently, $a_0 < a_1 > a_2 < a_3, > \ldots, a_{n-2} < a_{n-1} > a_0$, or its converse. see Figure 4. Conversely, each crown induces a cycle of even length.

In the sequel, let J be the set of all odd natural numbers greater than 3.

If G is a graph and $n \in J$, let σ_n express that each cycle of G of length n has a triangular chord:

$$\sigma_n : (\forall x_0, \ldots, x_{n-1})[x_0 G x_1 G \ldots G x_{n-1} G x_0 \implies x_0 G x_2 \vee x_1 G x_3 \vee \ldots \vee x_{n-2} G x_0 \vee x_{n-1} G x_1].$$

Fig. 4. A crown ordering induced by a strict cycle of even length



By Theorem 2, G is a comparability graph if and only if it satisfies σ_n for each $n \in J$, and thus, $\mathbb{G}_{\leq}$ has a universal first order axiomatization. Hence, $\mathbb{G}_{\leq}$ is closed with respect to substructures.

The following comes as no surprise:

Theorem 3 *$\mathbb{G}_{\leq}$ is not axiomatizable with a finite number of variables..*

Proof. Assume that Σ is a set of sentences with altogether n variables which axiomatizes $\mathbb{G}_{\leq}$; we can assume w.l.o.g. that $n = 2r \geq 4$. Let $U = \mathbf{n} + \mathbf{1}$, and G be the cycle on U of length $n+1$, say, $0, 1, 2, 3, \ldots n, 0$. Then, since G is an odd cycle without triangular chord, it is not in $\mathbb{G}_{\leq}$.

Suppose that $U' \subseteq U$ with $|U'| = n$, w.l.o.g. $U' = \{0, 1, \ldots, n-1\}$, and H is the restriction of G to U'. Then, $H = G \setminus \{\langle n-1, n\rangle, \langle n, 0\rangle\}$, and H is the comparability graph of the crown of Figure 4 with $\langle a_0, a_{n-1}\rangle$ removed.

Now, since the satisfaction in $\langle U, G\rangle$ of sentences with at most n variables depends only on its satisfaction in the n - generated substructures of $\langle U, G\rangle$, we have $\langle U, G\rangle \models \Sigma$. This contradicts the fact that $G \notin \mathbb{G}_{\leq}$.

4 Betweenness relations

Considering the plethora of proposed axiomatizations of betweenness relations, we need to decide which axioms to use. Our strategy will be to start with those postulates that are most common, present the least restrictions and still let us obtain a sensible theory. We will allow "degenerate triples", as it enables us to go from triples to pairs and vice versa.

A ternary relation on a set U is called a *betweenness relation* if it satisfies

BT 0. $B(a,a,a)$.
BT 1. $B(a,b,c) \Rightarrow B(c,b,a)$.
BT 2. $B(a,b,c) \Rightarrow B(a,a,b)$
BT 3. $B(a,b,c)$ and $B(a,c,b) \Rightarrow b = c$.

We denote the class of all betweenness relations by $\mathbb{B}$. Observe that at this stage we do not include any transitivity conditions. Since $\mathbb{B}$ is a universal Horn class, it is closed under substructures, and thus, under unions of chains, and also under direct products; note that for any set M of triples consistent with the axioms, there is a smallest betweenness relation B containing M, and that B is finite just when M is finite.

With BT 1, BT 2, and BT 3 one can easily prove

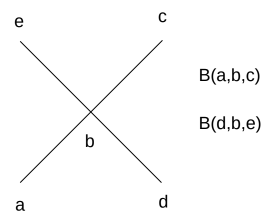


Fig. 5. A betweenness relation not induced by a binary relation

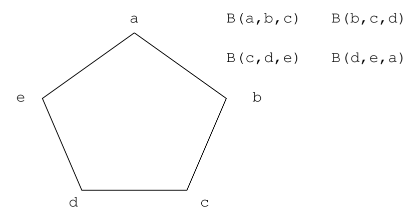


Fig. 6. A betweenness relation based on a pentagon

Lemma 1. *1. $B(a,b,c)$ implies $B(a,b,b)$, $B(b,b,c)$, and $B(b,c,c)$.*
2. $B(a,b,a) \Rightarrow a = b$.

However, in the absence of transitivity axioms, $B(a,a,c)$ does not follow.

A triple $\langle a,b,c\rangle$ is called *proper*, if $\#\{a,b,c\}$. We say that a,b are *comparable*, if $B(a,a,b)$, and let C_B be the set of all comparable pairs. By BT 0, C_B is reflexive, and by Lemma 1 it is symmetric. If aC_Bb, and $a \neq b$ we call a and b *strictly comparable*, and denote the graph of all strictly comparable pairs by C_B^+. Note that C_B^+ does not necessarily determine B, as Example 1 shows.

Conversely, if R is a reflexive antisymmetric binary relation on U we let $B_R = \{\langle a,b,c\rangle : aRbRc \text{ or } cRbRa\}$, and say that B *is induced by* R, if $B = B_R$; it is straightforward to see that $B_R \in \mathbb{B}$. The question arises whether every betweenness relation is induced by a binary relation. The following two examples show that the answer is "no."

Example 2 *Let $U = \{a,\dots,e\}$ and B be the smallest betweenness relation on U containing $\langle a,b,c\rangle$and $\langle d,b,e\rangle$, see Figure 5 ; inspection shows that these are the only nontrivial triples of B.*

Assume that B is induced by the binary relation R. Then, $B(a,b,c)$ implies $aRbRc$ or $cRbRa$, and $B(d,b,e)$ implies $dRbRe$ or $eRbRd$. If, for example, $aRbRc$ and $dRbRe$, then $dRbRc$, implying $B(d,b,c)$, which is not the case. The other cases are similar. $\square$

Another instructive example is the pentagon shown in Figure 6. We let B be the smallest betweenness relation containing $\langle a,a,b\rangle, \langle b,b,c\rangle, \langle c,c,d\rangle, \langle d,d,e\rangle$, and $\langle e,e,a\rangle$; then, C_B is the pentagon, and each triple in B contains at most two different entries..

Assume that $B = B_R$ for some reflexive and antisymmetric relation R. Then, aRb or bRa; suppose w.l.o.g. that aRb. Since B contains no proper triples, we then must have cRb, cRd, eRd, eRa. But then, $B(e,a,b)$, a contradiction. On the other hand, if B is generated by $\{\langle a,b,c\rangle, \langle b,c,d\rangle, \dots, \langle e,a,b\rangle\}$, then B is induced by the reflexive and antisymmetric relation $aRbRcRdReRa$ enhanced by $1'$. This shows that C_B can have cycles of odd length without a triangular chord.

5 Axiomatizing betweenness relations

In this section, we generalize Altwegg's theorem characterizing betweenness relations arising from partially ordered sets. It turns out that the basic ideas of his proof go through even in the absence of transitivity. Our main theorem in this section characterizes the betweenness relations arising from reflexive, antisymmetric orderings, from which we obtain Altwegg's result as a corollary.

The construction $R \longmapsto B_R$, as defined in §4, produces a structure satisfying the axioms of a betweenness relation, defined on the same universe, if R is reflexive and antisymmetric. The main idea of the present section is to show that there is an inverse map $B \longmapsto R_B$; however, the map is not unique, rather it depends on an arbitrary choice of orientation for each component of the strict comparability graph C_B^+. If we produce a new relation R', from a reflexive, antisymmetric relation R by reversing the direction of all pairs in R within a fixed component of its comparability graph, then R and R' generate the same betweenness relation. However, if we regard two such relations as equivalent if they differ from one another only with respect to an arbitrary choice of orientation for a set of components, then the map $B \longmapsto R_B$ determines a relation that is unique up to equivalence. This notion generalizes the concept of equivalence for partially ordered sets defined in §3.

We begin by recalling some terminology from Altwegg's paper [1]. Suppose that B is a betweenness relation on U. A sequence $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ is called a *C–path in B*, if $a_0 C_B a_1 C_B \dots C_B a_n$, i.e. every two consecutive entries are comparable. It is called a *B–path*, if $B(a_i, a_{i+1}, a_{i+2})$ for all $i \leq n-2$. Every C–path can be made into a B–path by doubling a_i for each $0 < i < n$.

Having derived a B–path $a_0, a_1, \dots, a_n$ from a C–path, it can be reduced in the following way:

1. If $a_{i_0} = a_{i_1} = \dots = a_{i_k}$, then remove $a_{i_2}, \dots, a_{i_k}$.
2. If $a_i, a_{i+1}, a_{i+2}, a_{i+3}$, $a_{i+1} = a_{i+2}$, and $B(a_i, a_{i+1}, a_{i+3})$, then remove a_{i+2}.

A completely reduced path is called a *chain*. Note that by the construction of a chain $a_0, a_1, \dots, a_n$ from a B–path, for $0 \leq i, i+1, i+2, i+3 \leq n$,

$$\#\{a_i, a_{i+1}, a_{i+2}\} \Rightarrow B(a_i, a_{i+1}, a_{i+2}), \tag{5.1}$$

$$a_{i+1} = a_{i+2} \Rightarrow \neg B(a_i, a_{i+1}, a_{i+3}). \tag{5.2}$$

A chain is called *simple*, if consecutive entries are different. We also call a, b a simple chain, if $a \neq b$ and $B(a, a, b)$. Clearly, for each $0 \leq k < m \leq n$, $a_k, a_{k+1}, \dots, a_m$ is again a simple chain, and the inverse $a_n, \dots, a_0$ of a (simple) chain $a_0 \dots, a_n$ is also a (simple) chain.

Definition 1. *We define the notion of a B-*walk of size *n by induction on n:*

1. *A simple chain is a B-walk of size 1;*
2. *If $W = a, \dots, p, q$ is a B-walk of size n, and $C = q, r \dots, z$ a simple chain where $\neg B(p, q, r)$, then the sequence $a, \dots, p, q, r, \dots, z$ obtained by identifying the last element of W with the first element of C is a B-walk of size $n+1$.*

In other words, a B-walk consists of a sequence obtained by gluing together simple chains; the gluing consists of identifying their endpoints. If $W = a, b, \dots, c, d$ is a B-walk, then we say that it is a *B-walk from a, b to c, d*. A B-walk is even or odd depending on whether its size is even or odd. The *length* of a B-walk is its length, considered as a sequence, so, for example, the B-walk a,b,a,b has length 4. Note that length and size may differ; indeed, it is the definition of *size* in the various scenarios that causes GH, Z_6, and BT 4 below to look so similar. A *B-cycle* is a B-walk $a_0, a_1 \ldots, a_{n-1}, a_n$, in which the first and last two elements are the same ($a_0 = a_n$), and $\neg B(a_{n-1}, a_0, a_1)$.

Lemma 2. *Let R be a reflexive, antisymmetric relation, and B_R the betweenness relation generated by R. Assume that aRb and that $W = a,b,\ldots,c,d$ is a B_R-walk from a,b to c,d. If W is odd, then cRd, while if W is even, then dRc.*

Proof. The proof is by induction on the length of W. For a B_R-walk of length 2, it holds by assumption. Assuming for walks of length $n > 1$, let $a,b,\ldots,c,d,e$ be an odd B_R-walk of length $n+1$. If $a,b,\ldots,c,d$ is also odd, then $B_R(c,d,e)$, and cRd, hence dRe. On the other hand, if $a,b,\ldots,c,d$ is even, then $\neg B_R(c,d,e)$, and dRc by inductive assumption, showing that dRe. The proof for even walks is symmetrical. □

The main theorem of this section is:

Theorem 4 *The theory $\mathbb{B}_R$ of betweenness relations generated by a reflexive, antisymmetric relation is axiomatized by the following postulates:*

BT 0. $B(a,a,a)$.
BT 1. $B(a,b,c) \Rightarrow B(c,b,a)$.
BT 2. $B(a,b,c) \Rightarrow B(a,a,b)$.
BT 3. $B(a,b,c)$ *and* $B(a,c,b) \Rightarrow b = c$.
BT 4. There are no odd B-cycles.

The fact that BT 4 holds for a betweenness relation B_R generated by a reflexive, antisymmetric relation R follows easily from Lemma 2. Note that Altwegg's postulate Z_6 is a special case of our BT 4. The more general formulation is needed here because the transitivity axioms are not available. To illustrate the axiom BT 4, let us consider two of the betweenness relations from the previous section. In Example 2, the sequence a,b,e,b,d,b,a is an odd B-cycle. The five simple chains making up the cycle are

$$a,b \mid b,e \mid e,b,d \mid d,b \mid b,a.$$

In the next example (the pentagon of Figure 6), the sequence a,b,c,d,e,a is an odd B-cycle. For any betweenness relation that is not generated by a reflexive, antisymmetric relation, there is an odd B-cycle that is a witness to this fact.

Lemma 3. *Let B be a betweenness relation satisfying the axiom BT 4 whose strict comparability graph C_B^+ is connected. If $\{a,b\}$, $\{c,d\}$ are distinct edges in this strict graph, then there is an odd B-walk from a,b to c,d or an odd B-walk from a,b to d,c, but not both.*

Proof. Since the strict comparability graph of B is connected, there is a C-path, and hence a B-path joining the edges $\{a,b\}$ and $\{c,d\}$. This path (or its inverse) must have one of the four forms: $a,b,\ldots,c,d$, $b,a,\ldots,c,d$, $b,a,\ldots d,c$ or $a,b,\ldots d,c$. By successive reductions, we can assume that this B-path is in fact a chain. Simplify this chain by removing immediate repetitions from it. Then the result is a B-walk from a,b to c,d, or from b,a to c,d, or from b,a to d,c, or from a,b to d,c.

For any $e,f,g,h \in U$, there is an even B-walk from e,f to g,h. if and only if there is an odd B-walk from e,f to h,g, since if $e,f,\ldots,g,h$ is an even B-walk from e,f to g,h, then $e,f,\ldots,g,h,g$ is an odd B-walk from e,f to h,g, and conversely. If the B-walk joining the edges $\{a,b\}$ and $\{c,d\}$ starts with a,b, then we are through. If on the other hand, it starts with b,a, then there is a B-walk of opposite parity starting with a,b, by the same argument as above. Hence, there is an odd B-walk from a,b to c,d or an odd B-walk from a,b to d,c.

It remains to be shown that there cannot be odd walks from a,b to both e,f and f,e, for any distinct comparable elements e,f. Suppose that $W_1 = a,b,\ldots,e,f$ and $W_2 = a,b,\ldots f,e$ are both odd B-walks. Then the inverse of W_2, $W_3 = e,f,\ldots b,a$ is odd. Let W_4 be the walk $a,b,\ldots,e,f,\ldots b,a$ resulting from the identification of the last two elements of W_1 and the first two elements of W_3. Then W_4 is an odd B-cycle, contradicting BT 4. $\square$

If U is a fixed universe, and R a reflexive, antisymmetric relation defined on U, then we write $\mathscr{B}(R) = \langle U, B_R \rangle$ for the betweenness relation defined from R. In the next definition, we describe the inverse construction.

Definition 2. *Let B be a betweenness relation defined on the set U, satisfying the axiom BT 4, and whose strict comparability graph C_B^+ is connected. In addition, let $\{a,b\}$ be an edge in C_B^+. Then $\mathscr{R}(B,a,b)$ is the relational structure $\langle U,R \rangle$ defined on U as follows. For $c,d \in U$, cRd holds if and only if $c = d$, or $\{c,d\}$ is an edge in C_B^+, and there is an odd B-walk from a,b to c,d.*

It follows from Lemma 3 that $\mathscr{R}(B,a,b)$ is reflexive and antisymmetric.

Theorem 5 *Let R be a reflexive, antisymmetric relation on U, and let aRb, where $a \neq b$. In addition, assume that the strict comparability graph of $\mathscr{B}(R)$ is connected. Then $\mathscr{R}(\mathscr{B}(R),a,b) = \langle U,R \rangle$.*

Proof. This follows from Lemmas 2 and 3. $\square$

Theorem 6 *Let B be a betweenness relation defined on the set U, satisfying the axiom BT 4, and whose strict comparability graph C_B^+ is connected. In addition, let $\{a,b\}$ be any edge in C_B^+. Then $\mathscr{B}(\mathscr{R}(B,a,b)) = \langle U,B \rangle$*

Proof. Let R be the relation defined from B in $\mathscr{R}(B,a,b)$. We need to show for any c,d,e in U, that $B(c,d,e)$ holds if and only if $B_R(c,d,e)$.

First, let us assume that $B(c,c,d)$, $c \neq d$. Then the edge $\{c,d\}$ belongs to the comparability graph C_B^+, so that cRd or dRc holds by Lemma 3, hence $B_R(c,c,d)$. Second, assume that $B(c,d,e)$ holds, where $\#\{c,d,e\}$. Then $\{c,d\}$ and $\{d,e\}$ are edges in C_B^+, and so cRd or dRc holds, and similarly dRe or eRd. Let us suppose that $\{c,d\}$ and $\{d,e\}$ are not consistently oriented, so that (say) cRd, but eRd. By construction, there are odd B-walks $W_1 = a,b,\ldots,c,d$ and $W_2 = a,b,\ldots,e,d$. Now consider the B-walk

$a, b, \ldots, c, d, e, \ldots, b, a$ obtained by identifying the last element of W_1 with the first element of the inverse of W_2. This walk is an odd B-cycle, contradicting BT 4. It follows that $B_R(c,d,e)$.

For the converse, let us assume that $B_R(c,d,e)$, where $\#(c,d,e)$, but not $B(c,d,e)$. By construction, there are odd B-walks $W_1 = a, b, \ldots, c, d$ and $W_2 = a, b, \ldots, d, e$, hence there is an even B-walk $W_3 = a, b, \ldots, e, d$. The B-cycle $a, b, \ldots, c, d, e, \ldots, b, a$ obtained by identifying the last element of W_1 with the first element of W_3 is odd, contradicting BT 4. Hence, $B(c,d,e)$ must hold, completing the proof. $\square$

We can now use the previous results to prove the main theorem.

Proof of Theorem 4: We have already observed that Lemma 2 implies that betweenness relations generated from reflexive, antisymmetric relations satisfy BT 4. Conversely, let B be a betweenness relation satisfying BT 4. Then $B = \bigcup_{i \in I} B_i$, where each B_i is the restriction of B to one of the connected components C_i of C_B^+. Each such B_i also satisfies BT 4. If C_i contains no edges, then the universe U_i of this component is a unit set, and we can set R_i to be the identity relation on U_i. If C_i contains at least one edge $\{a_i, b_i\}$, then choose an orientation a_i, b_i for this edge. By Theorem 6, $\mathscr{B}_i(\mathscr{R}(B_i, a_i, b_i)) = \langle U_i, B_i \rangle$. Hence, setting $\mathscr{R}(B) = \bigcup_{i \in I} \mathscr{R}(B_i, a_i, b_i)$, $\mathscr{B}(\mathscr{R}(B)) = \langle U, B \rangle$, showing that the class of betweenness structures satisfying BT 4 is identical with those betweenness structures arising from reflexive, antisymmetric relations. $\square$

Theorem 4 is quite powerful, and we can deduce results for restricted classes of relations with its help. The next theorem is equivalent to Altwegg's result of 1950; it shows that it is sufficient to add the outer transitivity axiom to our basic set of postulates.

Theorem 7 *The theory $\mathbb{B}_{\leq}$ of betweenness relations generated by a partial order is axiomatized by the following postulates:*

BT 0. $B(a,a,a)$.
BT 1. $B(a,b,c) \Rightarrow B(c,b,a)$.
BT 2. $B(a,b,c) \Rightarrow B(a,a,b)$
BT 3. $B(a,b,a) \Rightarrow a = b$.
BT 4. There are no odd B-cycles.
BT 5. $B(a,b,c)$ *and* $B(b,c,d)$ *and* $b \neq c \Rightarrow B(a,b,d)$.

Proof. In view of Theorem 6, it is sufficient to prove that if a betweenness relation B satisfies BT 5, that the relation $\mathscr{R}(B)$ is transitive. Suppose that aRb and bRc hold in $\mathscr{R}(B)$, where $\#\{a,b,c\}$. Then by Theorem 6, $B(a,b,c)$ holds. By BT 2 and BT 5, we have $B(a,a,c)$, so that a and c are comparable, hence aRc or cRa. Now if cRa holds, then we have $B(b,c,a)$, so by BT 2, $B(a,b,a)$, a contradiction. Hence, aRc, showing that R is transitive.

To prove Altwegg's theorem, that his axiom system Z_1 to Z_6 also characterizes this set of betweenness relations, it is sufficient to show that BT 4 can be deduced from BT 0, BT 1, BT 2, BT 3 and BT 5, together with Z_6. Now Altwegg's postulate Z_6 asserts, using our earlier terminology, that there is no odd B-cycle in which the simple chains that compose it are of length 2 (that is to say, of minimum length). However, in the presence of the outer transitivity axiom, it is not hard to show that if $a, b, \ldots, c, d, e, \ldots, f, g$ is a simple chain, then so is $a, b, \ldots, c, e, \ldots, f, g$; that is to say, the intermediate elements in a simple chain can be removed, and the result is still a simple chain. Consequently, if we postulate outer transitivity, then Z_6 implies the more general version BT 4. □

The following can be proved using basically the same construction as in Theorem 3:

Theorem 8 *The theories $\mathbb{B}_{\leq}$ and $\mathbb{B}_R$ are not axiomatizable with a finite number of variables.*

6 Algorithmic aspects

In this section, we give a brief sketch of the algorithmic aspects of betweenness relations. In the case of comparability graphs arising from partially ordered sets, very efficient algorithms are known for both the recognition problem and colouring problems. The reader is referred to the work of Golumbic [6–8] for descriptions of these algorithms, and to the article by Möhring [13] for an informative survey of this area.

The characterization given in §5 rests on the fact that if we have assigned orientations to some edges in the comparability graph of a betweenness relation, then other orientations are forced by the betweenness structure. If we use the notation $a \to b$, $a \leftarrow b$ to symbolize the fact that we have assigned the orientation (a,b) (respectively (b,a)) to the unordered edge $\{a,b\}$, then the following implications hold:

Imp 0. $[B(a,b,c) \wedge (a \neq b) \wedge (b \neq c) \wedge (a \to b)] \Rightarrow (b \to c)$;
Imp 1. $[B(a,b,c) \wedge (a \neq b) \wedge (b \neq c) \wedge (a \leftarrow b)] \Rightarrow (b \leftarrow c)$;
Imp 2. $[\neg B(a,b,c) \wedge (a \neq b) \wedge (b \neq c) \wedge (a \to b)] \Rightarrow (b \leftarrow c)$;
Imp 3. $[\neg B(a,b,c) \wedge (a \neq b) \wedge (b \neq c) \wedge (a \leftarrow b)] \Rightarrow (b \to c)$.

Let us say that a set S of ordered pairs (a,b), $a \neq b$, where a,b belong to the universe of a betweenness relation $\langle U,B \rangle$, is *implicationally closed* if it is closed under these implications (interpreting "$a \to b$" as "$(a,b) \in S$" and "$a \leftarrow b$" as "$(b,a) \in S$") and that it is an *implicational class of B* if it is a minimal non-empty implicationally closed subset of $U \times U$. If A is an implicational class, then $A\breve{}$ is the implicational class representing the result of reversing the orientation of all edges in A. Using this terminology, we can give an alternative characterization of betweenness relations arising from reflexive, antisymmetric relations; this is the analogue of a corresponding theorem of Golumbic for comparability graphs [6].

Theorem 9 *A betweenness relation B is induced by a reflexive antisymmetric relation if and only $A \cap A\breve{} = \emptyset$ for all implicational classes of B.*

Proof. If B is a betweenness relation that is *not* generated by such a relation, then by Theorem 4, there must be an odd B-cycle. Choose any edge (a,b) in this cycle, and consider the smallest implicational class A containing it. By the argument of Lemma 2, the oriented edge (b,a) must also belong to this class, showing that $A = A\breve{}$.

For the converse, let us assume that $A \cap A\breve{} \neq \emptyset$ for some implicational class A, and let $(a,b),(b,a) \in A$. Since A is the smallest implicational class containing (a,b), it follows that there is a sequence of elements of U, $a_0,\ldots,a_k$, and a sequence of statements

$S_1,\ldots,S_i,\ldots,S_k$, where each statement S_i is of the form $(a_{i-1} \to a_i)$ or $(a_{i-1} \leftarrow a_i)$, $S_1 = (a_0 \to a_1) = (a \to b)$, $S_k = (b \to a)$, and every statement in the sequence, except for the first, is derived from the preceding statement by one of the implicational rules Imp 0 – Imp 3. In Example 2 of §4, such a sequence of statements is given by: $(a \to b),(b \leftarrow e),(e \to b),(b \to d),(d \leftarrow b),(b \to a)$. Then it is straightforward to check that the sequence of elements $a_0,\ldots,a_k$ is an odd B-cycle. $\square$

Theorem 9 immediately suggests an algorithm to determine whether or not a betweenness relation is generated by a reflexive, antisymmetric relation. The algorithm consists of generating all of the implicational classes generated by directed edges in the comparability graph of the relation, while checking to see whether any overlap ever occurs between an implication class A and its converse $A\breve{}$. If we succeed in generating all such classes without an overlap, then they can be used to orient the edges appropriately, while if an overlap occurs, then Theorem 9 tells us that the betweenness relation cannot be generated by a reflexive, antisymmetric relation.

If $\langle U,B\rangle$ is a betweenness relation, and $b \in U$, then the *betweenness degree* of b is the number of proper triples (a,b,c) in B; the *betweenness degree* $\Delta(B)$ of the relation B is the maximum betweenness degree of any element in U. The *comparability degree* $\delta(B)$ of the relation B is the maximum degree of any vertex of the comparability graph of B.

Theorem 10 *There is an algorithm to determine whether a given betweenness relation B is generated by a reflexive, antisymmetric relation that runs in $O((\Delta(B)+\delta(B))|B|)$ time and $O(|B|)$ space.*

Proof. We provide only a brief sketch of this result. The basic ideas of the algorithm are all to be found in the original paper of Golumbic [6], and the reader can consult this paper for the details of the implementation.

We initialize the data structures for the algorithm by setting up two arrays of linked lists, one for the proper triples in B, the other for the edges in the comparability graph. This takes space $O(|B|)$. Then we start from an arbitrarily selected edge $\{a,b\}$ in the comparability graph, and generate the smallest implicational class A containing (a,b), simultaneously with its converse $A\breve{}$. The time complexity of the algorithm can be estimated through an upper bound on the time taken to look up the appropriate implication, when extending the implication classes. Suppose that (a,b) belongs to our class, and that we wish to see if there is an edge (b,c) or (c,b) that should be added because of some implication. First, we search for such an edge in the array representing the comparability graph; this takes time $O(\delta(B))$. Second, if we have found such an edge, we look for an appropriate proper triple with the middle element b; this takes time $O(\Delta(B))$, assuming that we have indexed such triples by their middle elements. Consequently, the entire procedure takes time $O((\Delta(B)+\delta(B))|B|)$. $\square$

7 Summary and Outlook

We have given an outline of the history of axiomatizations of betweenness relations, and have shown that the class of betweenness relations generated by a reflexive and antisymmetric binary relation is first order axiomatizable, albeit with an infinite number of variables. Furthermore, we have pointed out the connection of betweenness relations to comparability graphs. Such a graph may be generated by essentially different partial orders; in contrast, betweenness relations carry, in some sense, total information: If B is generated by a reflexive and antisymmetric binary relation, then this relation is determined up to taking converse on its components. In further work, we plan to investigate more deeply the relation between comparability graphs and betweenness relations, and also to give characterizations of induced betweenness relations in terms of forbidden substructures.

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