An algebraic approach to preference relations

Ivo Düntsch[1] and Ewa Orłowska[2]

[1] Brock University
St. Catharines, Ontario, Canada, L2S 3A1
duentsch@brocku.ca
[2] National Institute of Telecommunications
Szachowa 1, 04–894 Warsaw, Poland
orlowska@itl.waw.pl

Abstract. We define a class of structures – *preference algebras* – such that properties of preference relations can be expressed with their operations. We prove a discrete duality between preference algebras and preference relational structures.

1 Introduction

The concept of a preference structure appears in a variety of fields such as social choice theory, economics, and game theory [14, 8, 11], fuzzy logic [3, 12], among others. For a recent concise introduction to preference structures we refer the reader to [7], and for a general presentation of relations and their applications the reader is invited to consult [13]. Typically, a preference is viewed as a binary relation, say P on a set of alternatives. The statement xPy is intuitively interpreted as *x is preferred to y*. Together with a preference relation a binary relation of indifference, I, it is often assumed; here, xIy is interpreted as *x is similar to y* or *there is no preference for x or y*. In any particular theory there are various axioms assumed for preference and indifference relations. Reusch [9, 10] proposes the following (minimal) set of requirements on preference and indifference: For all alternatives x and y,

1. If x is preferred to y, then it is not the case that y is preferred to x.
2. If x is preferred to y, then it is not the case that x is indifferent to y.
3. x is indifferent to x.
4. If x is indifferent to y, then y is indifferent to x.

Postulates 1 and 2 may be interpreted as saying that the underlying preference structure is conflict-free.

In this paper we define a class of preference algebras which will be shown to correspond to the class of preference structures in the precise sense of discrete duality.

A preference algebra is a join of two mixed algebras, see [1], such that the properties of the preference and indifference relations can be expressed with their operations. The method of discrete duality employed in the paper enables us to consider logics of preference either with an algebraic semantics determined by preference algebras or with relational semantics determined by preference structures. The representation theorems presented in the paper yield the equivalence of these two semantics as it is shown in [5].

Given a class of preference algebras, Alg, and a class of preference relational structures (preference frames), Frm, a discrete duality between Alg and Frm is a tuple $\langle \mathsf{Alg}, \mathsf{Frm}, \mathfrak{Cf}, \mathfrak{Cm} \rangle$ such that $\mathfrak{Cf}$ is a mapping from Alg to Frm, $\mathfrak{Cm}$ is a mapping from Frm to Alg every A in Alg is embeddable into $\mathfrak{Cm}\,\mathfrak{Cf}(A)$, and every F in Frm is embeddable into $\mathfrak{Cf}\,\mathfrak{Cm}(F)$.

2 Definitions and notation

Suppose that $\langle B, \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra. With some abuse of language we will identify algebras with their base set. If $A \subseteq B$ and $h : B \to B$ a function, then $h[A] \stackrel{\text{df}}{=} \{h(a) : a \in A\}$ is the complex image of A under h. A *modal operator* on B is a function $f : B \to B$ which satisfies $f(0) = 0$, and $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in B$. A *sufficiency operator* on B is a function $g : B \to B$ which satisfies $g(0) = 1$, and $g(a \vee b) = g(a) \wedge g(b)$ for all $a, b \in B$.

A *mixed algebra* (MIA) [1] is a structure $\langle B, f, g \rangle$ where B is a Boolean algebra and $f, g : B \to B$ are functions on B such that

(2.1) f is a modal operator on B,

(2.2) g is a sufficiency operator on B.

and, for all ultrafilters F, G of B,

(2.3) $$F \cap g[G] \neq \emptyset \iff f[G] \subseteq F.$$

The condition 2.3 is easily seen to be equivalent to the one originally given in [1].

A *weak MIA* is complete and atomic Boolean algebra with a modal operator f and a sufficiency operator g such that $f(p) = g(p)$ for all atoms p of B. Each complete and atomic MIA is a weak MIA, and the converse does not necessarily hold:

Suppose that B is a complete and atomic MIA. Then, for all atoms p of B^c, $g^\pi(p) = f^\sigma(p)$, since B is a MIA. Since B can be embedded into B^c, and atoms of B stay atoms in B^c under this embedding, B is a weak MIA. □

Mixed algebras are an extension of Jónsson – Tarski Boolean algebras [4] which add additive and normal operators to the Boolean operations by sufficiency operators which are co–additive and co–normal. Not every modal algebra can be extended to a MIA: If, for example, f is the identity function in a Boolean algebra B, then there is no sufficiency operator such that $\langle B, f, g\rangle$ is a MIA [1].

In terms of canonical structures, a modal operator f talks about R, while its sufficiency operator talks about $-R$. Reflexivity, for example, can be expressed by a modal operator, but to express irreflexivity one needs a sufficiency operator. Also, antisymmetry can be expressed by a mixed modal–sufficiency expression, but not by a modal or sufficiency expression alone [2].

Observe that condition (2.3) is second order, and it is known that the class of mixed algebras is not first order axiomatizable [2].

If $\langle B, f, g\rangle$ is a MIA, define a binary relation R_B on $\mathrm{Ult}(B)$ by $\langle F, G\rangle \in R_B \overset{\mathrm{df}}{\iff} f[G] \subseteq F$. The structure $\mathfrak{Cf}(B) \overset{\mathrm{df}}{=} \langle \mathrm{Ult}(B), R_B\rangle$ is called the *canonical structure of* B.

The set of all binary relations on a set U is denoted by $\mathrm{Rel}(U)$; if $x \in U$, we let $R(x) \overset{\mathrm{df}}{=} \{z \in U : xRz\}$; the relational converse is denoted by $R^{\smile}$. For $R \in \mathrm{Rel}(U)$, we define two operators on 2^U by

$$\langle R\rangle(S) \overset{\mathrm{df}}{=} \{x : (\exists y)[xRy \text{ and } y \in S]\} = \{x : R(x) \cap S \neq \emptyset\}. \tag{2.4}$$

$$[[R]](S) \overset{\mathrm{df}}{=} \{x : (\forall y)[y \in S \Rightarrow xRy]\} = \{x : S \subseteq R(x)\}. \tag{2.5}$$

It is well known that $\langle R\rangle$ is a complete modal operator on the power set algebra of U, and that $[[R]]$ is a complete sufficiency operator, see e.g. [1]. Furthermore, $\langle 2^U, \langle R\rangle, [[R]]\rangle$ is a MIA [1], called the *complex algebra of* $\langle U, R\rangle$.

The following correspondences are well known:

Lemma 1. *1. R is reflexive if and only if $X \subseteq \langle R\rangle(X)$ for all $X \subseteq U$.*
2. R is symmetric if and only if $X \subseteq [[R]]([[R]](X))$ for all $X \subseteq U$.

3 Preference frames and preference algebras

In this section we introduce preference frames and preference algebras. Here, we use as a basis the "most traditional preference model" [7], also known as a $\langle P, I\rangle$ – *structure*: A *preference frame* is a structure $\langle X, P, I\rangle$, where X is a nonempty set and P, I, are binary relations on X which satisfy

F_1. $P \cap P^{\smile} = \emptyset$.
F_2. $P \cap I = \emptyset$.
F_3. I is reflexive.
F_4. I is symmetric.

These axioms reflect the postulates 1 - 4 from Section 1. We shall usually denote a preference frame $\langle X, P, I\rangle$ just by its universe X.

A *preference algebra*

$$\langle B, \vee, \wedge, \neg, 0, 1, f_1, g_1, f_2, g_2\rangle$$

is a structure such that $\langle B, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra, $\langle B, f_1, g_1\rangle$ and $\langle B, f_2, g_2\rangle$ are MIAs, and, for all $a \in B$,

A_1. $a \wedge f_1(g_1(a)) = 0$.
A_2. $a \wedge f_1(g_2(a)) = 0$.
A_3. $a \leq f_2(a)$.
A_4. $a \leq g_2(g_2(a))$.

These axioms correspond to the frame axioms listed above in the precise sense of the modal correspondence theory.

In Section 4 (resp. Section 5) we present an example of a preference algebra (resp. a preference frame). While we are not aware of any universal-algebraic preference structures, a great variety of preference frames can be found in the literature devoted to applications of preference relations.

4 The complex algebra of a preference frame

Let X be a preference frame. The complex algebra B_X of X has as its universe the powerset algebra of X, and the following distinguished operators: If $A \subseteq X$, then

$$f_P(A) \stackrel{\mathrm{df}}{=} \langle P\rangle(A), \qquad g_P(A) \stackrel{\mathrm{df}}{=} [[P]](A),$$
$$f_I(A) \stackrel{\mathrm{df}}{=} \langle I\rangle(A), \qquad g_I(A) \stackrel{\mathrm{df}}{=} [[I]](A).$$

B_X is called the *complex algebra of the preference frame* X, denoted by $\mathfrak{Cm}(X)$.

Theorem 1. *The complex algebra of a preference frame is a weak MIA satisfying* A_1 – A_4.

Proof. We have shown in [1] that both $\langle 2^X, \langle P\rangle, [[P]]\rangle$ and $\langle 2^X, \langle I\rangle, [[I]]\rangle$ are weak MIAs.

A_1: Let $A \subseteq X,\ x \in A$, and assume that $x \in \langle P\rangle[[P]](A)$. Then, there is some y such that xPy and $y \in [[P]](A)$. From the latter we infer that for all s, $s \in A$ implies yPs. Since $x \in A$, by the hypothesis we obtain yPx, which, together with xPy, contradicts the asymmetry F_1 of P.

A_2: Let $A \subseteq X,\ x \in A$, and assume that $x \in \langle P\rangle[[I]](A)$. Similar to A_1 there is some y such that xPy and yIx. Since I is symmetric, we also have xIy, contradicting F_2.

A_3 and A_4 follow directly from Lemma 1.

5 The canonical frame of a preference algebra

Suppose that $\langle B, f_1, g_1, f_2, g_2\rangle$ is a preference algebra, and let $\mathrm{Ult}(B)$ be the set of ultrafilters on the Boolean algebra B. Define binary relations P_B and I_B on $\mathrm{Ult}(B)$ by

$$F P_B G \iff f_1[G] \subseteq F, \tag{5.1}$$

$$F I_B G \iff f_2[G] \subseteq F. \tag{5.2}$$

The structure $\langle \mathrm{Ult}(B), P_B, I_B\rangle$ is called the *canonical frame of the preference algebra* B, denoted by $\mathfrak{Cf}(B)$.

Theorem 2. *The canonical frame of a preference algebra is a preference frame.*

Proof. It is well known that A_3 implies the reflexivity of I, and A_4 implies the symmetry of I, see e.g. [1].

F_1: Assume that P_B is not asymmetric. then, there are ultrafilters F, G such that $f_1[G] \subseteq F$ and $f_1[F] \subseteq G$. By the latter condition and (2.3), there is some $a \in F$ such that $g_1(a) \in G$. From $f_1[G] \subseteq F$ we obtain that $f_1(g_1(a)) \in F$. Since $a \in F$ and F is a proper filter, we have $0 \neq a \wedge f_1(g_1(a))$, contradicting A_1.

F_2: Assume that there are $F, G \in \mathrm{Ult}(B)$ such that $F P_B G$ and $F I_B G$. Since I_B is symmetric, we also obtain $G I_B F$, i.e. $f_1[G] \subseteq F$ and $f_2[F] \subseteq G$. Then, by (2.3), $G \cap g_2[F] \neq \emptyset$, so there is some $a \in B$ such that $a \in F$ and $g_2(a) \in G$. Using $f_1[G] \subseteq F$ we see that $f_1(g_2(a)) \in F$, hence, $a \wedge f_1(g_2(a)) \in F$. Since F is a proper filter, this contradicts A_2.

6 The duality result

Theorem 3. *Suppose that $\langle X, P, I\rangle$ is a preference frame, and $\langle B, f_1, g_1, f_2, g_2\rangle$ a preference algebra.*

1. *The mapping $h : B \to \mathfrak{Cm}\,\mathfrak{Cf}(B)$ defined by $h(a) = \{F \in \mathrm{Ult}(B) : a \in F\}$ is an embedding of preference algebras.*
2. *The mapping $k : X \to \mathfrak{Cf}\,\mathfrak{Cm}(X)$ defined by $k(x) = \{A \in 2^X : x \in A\}$ is an embedding of preference frames.*

Proof. 1. We have shown in [1], Proposition 12, that the mapping h embeds a MIA $\langle B, f, g\rangle$ into the complex algebra of its canonical frame. Following a referee's request, we make this fact explicit.

Since h is a Stone mapping it, is a Boolean embedding, and it is sufficient to show that h preserves the modal and sufficiency operators of preference algebras. Following [6] we show preservation of the operator g_1, that is $h(g_1(a)) = g_{P_{(2^X)}}(h(a))$.

First, observe that

$$F \in g_{P_{(2^X)}}(h(a)) \Longleftrightarrow F \in [[P_B]]h(a) \Longleftrightarrow (\forall G \in \mathrm{Ult}(B))[G \in h(a) \Rightarrow FP_BG],$$

and therefore by (2.3),

$$f_1(G) \in F \Longleftrightarrow (\forall G \in \mathrm{Ult}(B))[a \in G \Rightarrow F \cap g_1(G) \neq \emptyset].$$

"$\subseteq$": Take $G \in \mathrm{Ult}(B)$ such that $a \in G$. Since $g_1(a) \in F, F \cap g_1(G) \neq \emptyset$.

"$\supseteq$": Suppose $g_1(a) \notin F$. Consider set $Z_{g_1} = \{b \in B : g_1^d(b) \notin F\}$, where $g_1^d(b) = \neg g_1(\neg b)$ and $\neg$ is the Boolean complement. Let G' be a filter generated by $Z_{g_1} \cup \{a\}$. G' is a proper filter, for suppose otherwise, then there is $a' \in Z_{g_1}$ such that $a' \wedge a = 0$, which yields $a \leq \neg a'$. Since g_1 is antitone, $g_1(\neg a') \leq g_1(a)$. By definition of Z_{g_1}, $g^d(a') \notin F$ and since F is maximal, $g_1(\neg a') = \neg g^d(a') \in F$. Hence, $g_1(a) \in F$, a contradiction. Thus G' can be extended to a prime filter, say G, containing it. Since $a \in G'$, $a \in G$. Hence, by the assumption, $F \cap g_1(G) \neq \emptyset$. It follows that for some $b \in G$, $g_1(b) \in F$. Then $\neg g_1(b) = g_1^d(\neg b) \notin F$ and hence $\neg b \in Z_{g_1} \subseteq G$, which yields $b \notin G$, a contradiction.

The proof of preservation of modal operators can also be found in [6].

2. Since $k(x)$ is the principal ultrafilter of 2^X generated by $\{x\}$, the mapping k is well defined. Let $x, y \in X$; we need to show that $xPy \Longleftrightarrow k(x)P_{(2^X)}k(y)$ and $xIy \Longleftrightarrow k(x)I_{(2^X)}k(y)$. First, observe that

$$k(x)P_{(2^X)}k(y) \Longleftrightarrow \langle P\rangle[k(y)] \subset k(x) \tag{6.1}$$

$$\Longleftrightarrow (\forall Y \subseteq X)[y \in Y \Rightarrow x \in \langle P\rangle(Y)] \tag{6.2}$$

$$\Longleftrightarrow (\forall Y \subseteq X)[y \in Y \Rightarrow P(x) \cap Y \neq \emptyset]. \tag{6.3}$$

"$\Rightarrow$": Let xPy and $y \in Y$. Then, $P(x) \cap Y \neq \emptyset$, and hence by (6.3) we have $k(x)P_{(2^X)}k(y)$.

"$\Leftarrow$": Suppose that $k(x)P_{(2^X)}k(y)$ for some $x, y \in X$. Setting $Y \stackrel{\mathrm{df}}{=} \{y\}$ and using (6.3) we obtain xPy.

Corollary 1.

1. *Any preference frame can be embedded into the canonical frame of its complex algebra.*
2. *Any preference algebra can be embedded into the complex algebra of its canonical frame.*

7 Conclusion and outlook

In this paper we introduced a class of preference algebras corresponding to the class of traditional $\langle P, I\rangle$ preference frames. We proved representation theorems for these classes, thus obtaining a discrete duality between them.

The preference structures considered in the paper can be extended in a natural way to the structures with multiple pairs of preference and indifference relations associated with agents making choices on a set of alternatives. Algebraic counterparts to these structures will be preference algebras constructed from multiple mixed algebras. Then some axioms reflecting relationships among preferences of various agents may be added. The duality results for those structures will open the way to a study of aggregation of preferences of the agents both in an algebraic and a relational framework.

Other directions for future work may include studying restrictions of the preference and/or indifference relation. In particular, transitivity of preference and/or indifference is of importance. The corresponding algebraic axioms are well known from correspondence theory. In some approaches to preference modeling, a third relation of incomparability, J, is introduced and postulated to be irreflexive and symmetric. In such preference structures, it is usually assumed that the relations P, $P^{\smile}$, I, and J form a partition of the set of all the pairs of alternatives. Further work is planned on discrete dualities for such and similar structures as, for example, interval orders or semi–orders.

Acknowledgment We should like to thank the referees for thoughtful comments and pointers to further literature.

References

1. Düntsch, I., Orłowska, E.: Beyond modalities: Sufficiency and mixed algebras. In: Orłowska, E., Szałas, A. (eds.) Relational Methods in Computer Science Applications. pp. 277–299. Physica Verlag, Heidelberg (2000)
2. Düntsch, I., Orłowska, E.: Boolean algebras arising from information systems. Annals of Pure and Applied Logic 127, 77–98 (2004)
3. Fodor, J., Roubens, M.: Fuzzy Preference Modelling and Multicriteria Decision Support. North Holland, Dordrecht (1994)
4. Jónsson, B., Tarski, A.: Boolean algebras with operators I. American Journal of Mathematics 73, 891–939 (1951)
5. Orlowska, E., Rewitzky, I.: Discrete Duality and Its Applications to Reasoning with Incomplete Information. In: Kryszkiewicz, M., Peters, J.F., Rybinski, H., Skowron, A. (eds.) Rough Sets and Intelligent Systems Paradigms. LNCS, vol. 4585, pp. 51–56. Springer–Verlag, Heidelberg (2007)
6. Orłowska, E., Rewitzky, I., Düntsch, I.: Relational semantics through duality. In: MacCaull, W., Winter, M., Düntsch, I. (eds.) Relational Methods in Computer Science. LNCS, vol. 3929, pp. 17–32. Springer–Verlag, Heidelberg (2006)
7. Öztürk, M., Tsoukiàs, A., Vincke, P.: Preference modelling. In: Ehrgott, M., Greco, S., Figueira, J. (eds.) State of the Art in Multiple Criteria Decision Analysis, pp. 27–72. Springer–Verlag, Heidelberg (2005)
8. Pauly, M.: Formal Methods and the Theory of Social Choice. In: Berghammer, R., Möller, B., Struth, G. (eds.) Relations and Kleene Algebra in Computer Science. LNCS, vol. 4988, pp. 1–2. Springer (2008)

9. Reusch, B.: On the Axiomatics of Rational Preference Structures (2006), `http://lrb.cs.uni-dortmund.de/pdf/On%20the%20Axiomatics%20of%20Rational%20Preference%20Structures2.pdf`, retrieved February 2, 2011
10. Reusch, B.: An axiomatic theory of (rational) preference structures (2008), Manuscript, Technische Universität Dortmund
11. Roubens, M., Vincke, P.: Preference Modelling, Lecture Notes in Economics and Mathematical Sciences, vol. 250. Springer-Verlag, Heidelberg (1985)
12. Saminger-Platz, S.: Basics of preference and fuzzy preference modeling. In: Berghammer, R., Moeller, B., Struth, G. (eds.) Relations and Kleene Algebra in Computer Science, pp. 25–40. Universität Augsburg (2008), PhD Programme at RelMiCS10/AKA5
13. Schmidt, G.: Relational Mathematics, Encyclopedia of Mathematics and its Applications, vol. 132. Cambridge University Press (2010)
14. de Swart, H. (ed.): Logic, Game Theory and Social Choice. Tilburg University Press (1999)

A An first order scenario

Suppose that we weaken the definition of a preference algebra by not assuming that B is a MIA, but that it satisfies only the weaker first order condition

$$a \wedge b \neq 0 \Rightarrow g_i(a) \leq f_i(a) \tag{A.1}$$

for $i = 1, 2$.

A *preference algebra*

$$\langle B, \vee, \wedge, \neg, 0, 1, f_1, g_1, f_2, g_2 \rangle$$

is a structure such that $\langle B, \vee, \wedge, \neg, 0, 1 \rangle$ is a Boolean algebra, and for $i = 1, 2$ f_i is a modal operator, g_i a sufficiency operator, and (A.1) holds. Furthermore, for all $a \in B$,

A$'_1$. $a \wedge f_1(g_1(a)) = 0$.
A$'_2$. $a \wedge f_1(g_2(a)) = 0$.
A$'_3$. $a \leq f_2(a)$.
A$'_4$. $a \leq g_2(g_2(a))$.

The canonical relations are defined by

$$F P_B G \iff F \cap g_1[G] \neq \emptyset.$$
$$F I_B G \iff F \cap g_2[G] \neq \emptyset.$$

Theorem 4. *The canonical frame of a preference algebra is a preference frame.*

Proof. F_1: Assume that P_B is not asymmetric. Then, there are ultrafilters F, G such that $F \cap g_1[G] \neq \emptyset$ and $G \cap g_1[F] \neq \emptyset$. Let $a \in G$ such that $g_1(a) \in F$, and $b \in F$ such that $g_1(b) \in G$. Since G is proper and $a, g_1(b) \in G$, we have $a \wedge g_1(b) \neq 0$, and thus, $g_1(a) \leq f_1(g_1(b))$ by (A.1). Since $g_1(a) \in F$, we have $f_1(g_1(b)) \in F$, and therefore, $b \wedge f_1(g_1(b)) \in F$. Since F is proper, we obtain $0 \neq b \wedge f_1(g_1(b))$, contradicting A_1'.

F_4: Suppose that $F \cap g_2[G] \neq \emptyset$, and let $a \in G$ exemplify this fact. Then, $g_2(g_2(a)) \in g_2[F]$, and $a \in G$ together with A_4' implies $g_2(g_2(a)) \in G$. Hence, $G \cap g_2[F] \neq \emptyset$.

F_2: This is analogous to the proof of F_1: Assume that there are $F, G \in \mathrm{Ult}(B)$ such that $F P_B G$ and $F I_B G$. Since I_B is symmetric, we also obtain $G I_B F$, so that

$$F \cap g_1[G] \neq \emptyset, \tag{A.2}$$

$$G \cap g_2[F] \neq \emptyset. \tag{A.3}$$

Let $a \in G$ such that $g_1(a) \in F$, and $b \in F$ such that $g_2(b) \in G$. Since G is proper and $a, g_2(b) \in G$, we have $a \wedge g_2(b) \neq 0$, and thus, $g_1(a) \leq f_1(g_2(b))$ by (A.1). Since $g_1(a) \in F$, we have $f_1(g_2(b)) \in F$, and therefore, $b \wedge f_1(g_2(b)) \in F$. Since F is proper, we obtain $0 \neq b \wedge f_1(g_2(b))$, contradicting A_2'.

Theorem 5. *The complex algebra of a preference frame is a preference algebra.*

Proof. All we need to show is that $\mathfrak{Cm}(X)$ satisfies (A.1), and this is Lemma 1.5. of the MIA paper. The rest is as in the proof of Theorem 1.

Theorem 6. *Suppose that $\langle X, P, I \rangle$ is a preference frame, and $\langle B, f_1, g_1, f_2, g_2 \rangle$ a preference algebra.*

1. *The mapping $h : B \to \mathfrak{Cm}\,\mathfrak{Cf}(B)$ defined by $h(a) = \{F \in \mathrm{Ult}(B) : a \in F\}$ is an embedding of preference algebras.*
2. *The mapping $k : X \to \mathfrak{Cf}\,\mathfrak{Cm}(X)$ defined by $k(x) = \{A \in 2^X : x \in A\}$ is an embedding of preference frames.*

Proof. 1. By Stone's Theorem, h is an embedding of Boolean algebras, and h preserves modal operators [4]. We show first that h preserves the sufficiency operator g_1, i.e. that $h(g_1(a)) = [[P_B]](h(a))$:

"$\subseteq$": Let $g_1(a) \in F$; we need to show that $F \in [[P_B]](h(a))$, i.e.

$$h(a) \subseteq P_B(F), \text{ where } P_B(F) = \{G : F P_B G\} = \{G : F \cap g_1[G] \neq \emptyset\}.$$

Let $G \in h(a)$, i.e. $a \in G$. Since $g_1(a) \in F$, this shows that $F \cap g_1[G] \neq \emptyset$. Hence, $G \in P_B(F)$.

"$\supseteq$": Suppose that $F \in [[P_B]](h(a))$, i.e. $F \cap g_1[G] \neq \emptyset$ whenever $a \in G$. Assume that $g_1(a) \notin F$, and let $Z = \{b \in B : g_1(\neg b) \in F\}$. If $b \in Z$ and $a \wedge b = 0$, then $a \leq \neg b$, and thus, $g_1(\neg b) \leq g_1(a)$. Since $b \in Z$ we have $g_1(\neg b) \in F$, and thus, $g_1(a) \in F$ contradicting our assumption. Now, $b \wedge a \neq 0$ for all $b \in Z$ implies that there is an ultrafilter G containing $Z \cup \{a\}$. By the hypothesis $F \in [[P_B]](h(a))$ and $a \in G$ it follows that $F \cap g_1[G] \neq \emptyset$. Let $b \in G$ such that $g_1(b) \in F$; then, $\neg b \in Z$, and thus, $\neg b \in G$ since $Z \subseteq G$. This contradicts our choice of b.

Analogously one shows that $h(g_2(a)) = [[I_B]](h(a))$.