An algebraic approach to preference relations

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Abstract. We define a class of structures – preference algebras – such that properties of preference relations can be expressed with their operations. We prove a discrete duality between preference algebras and preference relational structures.

1 Introduction

The concept of a preference structure appears in a variety of fields such as social choice theory, economics, and game theory [14, 8, 11], fuzzy logic [3, 12], among others. For a recent concise introduction to preference structures we refer the reader to [7], and for a general presentation of relations and their applications the reader is invited to consult [13]. Typically, a preference is viewed as a binary relation, say $P$ on a set of alternatives. The statement $xPy$ is intuitively interpreted as $x$ is preferred to $y$. Together with a preference relation a binary relation of indifference, $I$, it is often assumed; here, $xIy$ is interpreted as $x$ is similar to $y$ or there is no preference for $x$ or $y$. In any particular theory there are various axioms assumed for preference and indifference relations. Reusch [9, 10] proposes the following (minimal) set of requirements on preference and indifference: For all alternatives $x$ and $y$,

1. If $x$ is preferred to $y$, then it is not the case that $y$ is preferred to $x$.
2. If $x$ is preferred to $y$, then it is not the case that $x$ is indifferent to $y$.
3. $x$ is indifferent to $x$.
4. If $x$ is indifferent to $y$, then $y$ is indifferent to $x$.

Postulates 1 and 2 may be interpreted as saying that the underlying preference structure is conflict-free.

In this paper we define a class of preference algebras which will be shown to correspond to the class of preference structures in the precise sense of discrete duality.
A preference algebra is a join of two mixed algebras, see [1], such that the properties of the preference and indifference relations can be expressed with their operations. The method of discrete duality employed in the paper enables us to consider logics of preference either with an algebraic semantics determined by preference algebras or with relational semantics determined by preference structures. The representation theorems presented in the paper yield the equivalence of these two semantics as it is shown in [5].

Given a class of preference algebras, \( \text{Alg} \), and a class of preference relational structures (preference frames), \( \text{Frm} \), a discrete duality between \( \text{Alg} \) and \( \text{Frm} \) is a tuple \( \langle \text{Alg}, \text{Frm}, \mathcal{Cf}, \mathcal{Cm} \rangle \) such that \( \mathcal{Cf} \) is a mapping from \( \text{Alg} \) to \( \text{Frm} \), \( \mathcal{Cm} \) is a mapping from \( \text{Frm} \) to \( \text{Alg} \) every \( A \) in \( \text{Alg} \) is embeddable into \( \mathcal{Cm} \mathcal{Cf}(A) \), and every \( F \) in \( \text{Frm} \) is embeddable into \( \mathcal{Cf} \mathcal{Cm}(F) \).

## 2 Definitions and notation

Suppose that \( \langle B, \land, \lor, \neg, 0, 1 \rangle \) is a Boolean algebra. With some abuse of language we will identify algebras with their base set. If \( A \subseteq B \) and \( h : B \to B \) a function, then \( h[A] \overset{\text{df}}{=} \{ h(a) : a \in A \} \) is the complex image of \( A \) under \( h \). A modal operator on \( B \) is a function \( f : B \to B \) which satisfies \( f(0) = 0 \), and \( f(a \lor b) = f(a) \lor f(b) \) for all \( a, b \in B \). A sufficiency operator on \( B \) is a function \( g : B \to B \) which satisfies \( g(0) = 1 \), and \( g(a \lor b) = g(a) \land g(b) \) for all \( a, b \in B \).

A mixed algebra (MIA) [1] is a structure \( \langle B, f, g \rangle \) where \( B \) is a Boolean algebra and \( f, g : B \to B \) are functions on \( B \) such that

\[
(2.1) \quad f \text{ is a modal operator on } B,
\]

\[
(2.2) \quad g \text{ is a sufficiency operator on } B.
\]

and, for all ultrafilters \( F, G \) of \( B \),

\[
(2.3) \quad F \cap g[G] \neq \emptyset \iff f[G] \subseteq F.
\]

The condition 2.3 is easily seen to be equivalent to the one originally given in [1].

A weak MIA is complete and atomic Boolean algebra with a modal operator \( f \) and a sufficiency operator \( g \) such that \( f(p) = g(p) \) for all atoms \( p \) of \( B \). Each complete and atomic MIA is a weak MIA, and the converse does not necessarily hold:

Suppose that \( B \) is a complete and atomic MIA. Then, for all atoms \( p \) of \( B^c \), \( g^*(p) = f^*(p) \), since \( B \) is a MIA. Since \( B \) can be embedded into \( B^c \), and atoms of \( B \) stay atoms in \( B^c \) under this embedding, \( B \) is a weak MIA.

Mixed algebras are an extension of Jónsson – Tarski Boolean algebras [4] which add additive and normal operators to the Boolean operations by sufficiency.
operators which are co-additive and co-normal. Not every modal algebra can be extended to a MIA: If, for example, \( f \) is the identity function in a Boolean algebra \( B \), then there is no sufficiency operator such that \( \langle B, f, g \rangle \) is a MIA [1].

In terms of canonical structures, a modal operator \( f \) talks about \( R \), while its sufficiency operator talks about \( -R \). Reflexivity, for example, can be expressed by a modal operator, but to express irreflexivity one needs a sufficiency operator. Also, antisymmetry can be expressed by a mixed modal–sufficiency expression, but not by a modal or sufficiency expression alone [2].

Observe that condition (2.3) is second order, and it is known that the class of mixed algebras is not first order axiomatizable [2].

If \( \langle B, f, g \rangle \) is a MIA, define a binary relation \( R_B \) on \( \text{Ult}(B) \) by
\[
\langle F, G \rangle \in R_B \iff f[G] \subseteq F.
\]
The structure \( \text{Cf}(B) \) is called the canonical structure of \( B \).

The set of all binary relations on a set \( U \) is denoted by \( \text{Rel}(U) \); if \( x \in U \), we let \( R(x) \) \( \overset{df}{=} \) \{ \( z \in U : xRz \} \); the relational converse is denoted by \( R^\circ \). For \( R \in \text{Rel}(U) \), we define two operators on \( 2^U \) by
\[
\langle R \rangle(S) \overset{df}{=} \{ x : (\exists y)[xRy \text{ and } y \in S] \} = \{ x : R(x) \cap S \neq \emptyset \}.
\]
\[
[[R]](S) \overset{df}{=} \{ x : (\forall y)[y \in S \Rightarrow xRy] \} = \{ x : S \subseteq R(x) \}.
\]
It is well known that \( \langle R \rangle \) is a complete modal operator on the power set algebra of \( U \), and that \( [[R]] \) is a complete sufficiency operator, see e.g. [1]. Furthermore, \( \langle 2^U, \langle R \rangle, [[R]] \rangle \) is a MIA [1], called the complex algebra of \( \langle U, R \rangle \).

The following correspondences are well known:

**Lemma 1.** 1. \( R \) is reflexive if and only if \( X \subseteq \langle R \rangle(X) \) for all \( X \subseteq U \).
2. \( R \) is symmetric if and only if \( X \subseteq [[R]]([[R]](X)) \) for all \( X \subseteq U \).

### 3 Preference frames and preference algebras

In this section we introduce preference frames and preference algebras. Here, we use as a basis the “most traditional preference model” [7], also known as a \( \langle P, I \rangle \) – structure: A preference frame is a structure \( \langle X, P, I \rangle \), where \( X \) is a nonempty set and \( P, I \), are binary relations on \( X \) which satisfy

\begin{align*}
F_1. & \quad P \cap P^\circ = \emptyset. \\
F_2. & \quad P \cap I = \emptyset. \\
F_3. & \quad I \text{ is reflexive.} \\
F_4. & \quad I \text{ is symmetric.}
\end{align*}
These axioms reflect the postulates 1 - 4 from Section 1. We shall usually denote a preference frame \(\langle X, P, I \rangle\) just by its universe \(X\).

A preference algebra
\[
\langle B, \lor, \land, \neg, 0, 1, f_1, f_2, g_2 \rangle
\]
is a structure such that \(\langle B, \lor, \land, \neg, 0, 1 \rangle\) is a Boolean algebra, \(\langle B, f_1, g_1 \rangle\) and \(\langle B, f_2, g_2 \rangle\) are MIAs, and, for all \(a \in B\),

\[
A_1. \ a \land f_1(g_1(a)) = 0.
\]
\[
A_2. \ a \land f_1(g_2(a)) = 0.
\]
\[
A_3. \ a \leq f_2(a).
\]
\[
A_4. \ a \leq g_2(g_2(a)).
\]

These axioms correspond to the frame axioms listed above in the precise sense of the modal correspondence theory.

In Section 4 (resp. Section 5) we present an example of a preference algebra (resp. a preference frame). While we are not aware of any universal-algebraic preference structures, a great variety of preference frames can be found in the literature devoted to applications of preference relations.

4 The complex algebra of a preference frame

Let \(X\) be a preference frame. The complex algebra \(B_X\) of \(X\) has as its universe the powerset algebra of \(X\), and the following distinguished operators: If \(A \subseteq X\), then

\[
f_P(A) \overset{df}{=} (P)(A), \quad g_P(A) \overset{df}{=} [[P]](A),
\]
\[
f_I(A) \overset{df}{=} (I)(A), \quad g_I(A) \overset{df}{=} [[I]](A).
\]

\(B_X\) is called the complex algebra of the preference frame \(X\), denoted by \(\mathfrak{Cm}(X)\).

**Theorem 1.** The complex algebra of a preference frame is a weak MIA satisfying \(A_1 - A_4\).

**Proof.** We have shown in [1] that both \(\langle 2^X, \langle P\rangle, [[P]] \rangle\) and \(\langle 2^X, \langle I\rangle, [[I]] \rangle\) are weak MIAs.

**A1:** Let \(A \subseteq X, \ x \in A\), and assume that \(x \in (P)=[[P]](A)\). Then, there is some \(y\) such that \(xPy\) and \(y \in [[P]](A)\). From the latter we infer that for all \(s, s \in A\) implies \(yPs\). Since \(x \in A\), by the hypothesis we obtain \(yPx\), which, together with \(xPy\), contradicts the asymmetry \(F_1\) of \(P\).

**A2:** Let \(A \subseteq X, \ x \in A\), and assume that \(x \in (P)=[[I]](A)\). Similar to \(A_1\) there is some \(y\) such that \(xPy\) and \(yIx\). Since \(I\) is symmetric, we also have \(xIy\), contradicting \(F_2\).

\(A_3\) and \(A_4\) follow directly from Lemma 1.
5 The canonical frame of a preference algebra

Suppose that \( \langle B, f_1, g_1, f_2, g_2 \rangle \) is a preference algebra, and let \( \text{Ult}(B) \) be the set of ultrafilters on the Boolean algebra \( B \). Define binary relations \( P_B \) and \( I_B \) on \( \text{Ult}(B) \) by

\[
(5.1) \quad FP_B G \iff f_1[G] \subseteq F, \\
(5.2) \quad FI_B G \iff f_2[G] \subseteq F.
\]

The structure \( \langle \text{Ult}(B), P_B, I_B \rangle \) is called the canonical frame of the preference algebra \( B \), denoted by \( \mathcal{Cf}(B) \).

Theorem 2. The canonical frame of a preference algebra is a preference frame.

Proof. It is well known that \( A_3 \) implies the reflexivity of \( I \), and \( A_4 \) implies the symmetry of \( I \), see e.g. [1].

F1: Assume that \( P_B \) is not asymmetric. Then, there are ultrafilters \( F, G \) such that \( f_1[G] \subseteq F \) and \( f_1[F] \subseteq G \). By the latter condition and (2.3), there is some \( a \in F \) such that \( g_1(a) \in G \). From \( f_1[G] \subseteq F \) we obtain that \( f_1(g_1(a)) \in F \). Since \( a \in F \) and \( F \) is a proper filter, we have \( 0 \neq a \land f_1(g_1(a)) \), contradicting \( A_1 \).

F2: Assume that there are \( F, G \in \text{Ult}(B) \) such that \( FP_B G \) and \( FI_B G \). Since \( I_B \) is symmetric, we also obtain \( GI_B F \), i.e. \( f_1[G] \subseteq F \) and \( f_2[F] \subseteq G \). Then, by (2.3), \( G \cap g_2[F] \neq \emptyset \), so there is some \( a \in B \) such that \( a \in F \) and \( g_2(a) \in G \). Using \( f_1[G] \subseteq F \) we see that \( f_1(g_2(a)) \in F \), hence, \( a \land f_1(g_2(a)) \in F \). Since \( F \) is a proper filter, this contradicts \( A_2 \).

6 The duality result

Theorem 3. Suppose that \( \langle X, P, I \rangle \) is a preference frame, and \( \langle B, f_1, g_1, f_2, g_2 \rangle \) a preference algebra.

1. The mapping \( h : B \to \mathcal{Cf}(B) \) defined by \( h(a) = \{ F \in \text{Ult}(B) : a \in F \} \) is an embedding of preference algebras.
2. The mapping \( k : X \to \mathcal{Cf}(\mathcal{Lm}(X)) \) defined by \( k(x) = \{ A \in 2^X : x \in A \} \) is an embedding of preference frames.

Proof. 1. We have shown in [1], Proposition 12, that the mapping \( h \) embeds a MIA \( \langle B, f, g \rangle \) into the complex algebra of its canonical frame. Following a referee’s request, we make this fact explicit.

Since \( h \) is a Stone mapping it is a Boolean embedding, and it is sufficient to show that \( h \) preserves the modal and sufficiency operators of preference algebras. Following [6] we show preservation of the operator \( g_1 \), that is \( h(g_1(a)) = g_{P_{\mathcal{Lm}(X)}}(h(a)) \).
First, observe that
\[ F \in g_{P(x)}(h(a)) \iff F \in [P_B][h(a) \iff (\forall G \in \text{Ult}(B))[G \in h(a) \Rightarrow F \cap G(1) \neq \emptyset], \]

and therefore by (2.3),
\[ f_1(G) \in F \iff (\forall G \in \text{Ult}(B))[a \in G \Rightarrow F \cap g_1(G) \neq \emptyset]. \]

\[ \subseteq: \] Take \( G \in \text{Ult}(B) \) such that \( a \in G \). Since \( g_1(a) \in F \), \( F \cap g_1(G) \neq \emptyset \).

\[ \supseteq: \] Suppose \( g_1(a) \notin F \). Consider set \( Z_{g_1} = \{ b \in B : g_1(b) \notin F \} \), where \( g_1(b) = \neg g_1(\neg b) \). By definition of \( Z_{g_1} \), \( g_1(b) \notin F \) and since \( F \) is maximal, \( g_1(\neg b) = g_1(b') \in F \). Hence, \( g_1(a) \in F \), a contradiction. Thus \( G' \) can be extended to a prime filter, say \( G \), containing it. Since \( a \in G' \), \( a \in G \). Hence, by the assumption, \( F \cap g_1(G) \neq \emptyset \).

It follows that for some \( b \in G \), \( b \notin F \). Then \( \neg b \notin F \) and hence \( b \notin Z_{g_1} \subseteq G \), which yields \( b \notin G \), a contradiction.

The proof of preservation of modal operators can also be found in [6].

2. Since \( k(x) \) is the principal ultrafilter of \( 2^X \) generated by \( \{ x \} \), the mapping \( k \)

is well defined. Let \( x, y \in X \); we need to show that \( xPy \iff k(x) \cap P(x) \cap (x) \iff k(x) \).

\begin{align*}
(6.1) & \quad k(x) \cap P(x) \cap k(y) \iff \langle P \rangle \cap k(x) \in k(x) \\
(6.2) & \quad \iff (\forall Y \subseteq X)[y \in Y \Rightarrow x \in (P \cap Y)] \\
(6.3) & \quad \iff (\forall Y \subseteq X)[y \in Y \Rightarrow P(x) \cap Y \neq \emptyset].
\end{align*}

\[ \Rightarrow: \] Let \( xPy \) and \( y \in Y \). Then, \( P(x) \cap Y \neq \emptyset \), and hence by (6.3) we have \( k(x) \cap P(x) \cap k(y) \).

\[ \subseteq: \] Suppose that \( k(x) \cap P(x) \cap k(y) \) for some \( x, y \in X \). Setting \( Y \) \( \{ y \} \) and using (6.3) we obtain \( xPy \).

**Corollary 1.**

1. Any preference frame can be embedded into the canonical frame of its complex algebra.
2. Any preference algebra can be embedded into the complex algebra of its canonical frame.

**7 Conclusion and outlook**

In this paper we introduced a class of preference algebras corresponding to the class of traditional \( (P, I) \) preference frames. We proved representation theorems for these classes, thus obtaining a discrete duality between them.
The preference structures considered in the paper can be extended in a natural way to the structures with multiple pairs of preference and indifference relations associated with agents making choices on a set of alternatives. Algebraic counterparts to these structures will be preference algebras constructed from multiple mixed algebras. Then some axioms reflecting relationships among preferences of various agents may be added. The duality results for those structures will open the way to a study of aggregation of preferences of the agents both in an algebraic and a relational framework.

Other directions for future work may include studying restrictions of the preference and/or indifference relation. In particular, transitivity of preference and/or indifference is of importance. The corresponding algebraic axioms are well known from correspondence theory. In some approaches to preference modeling, a third relation of incomparability, \( J \), is introduced and postulated to be irreflexive and symmetric. In such preference structures, it is usually assumed that the relations \( P, P^-, I, \) and \( J \) form a partition of the set of all the pairs of alternatives.

Further work is planned on discrete dualities for such and similar structures as, for example, interval orders or semi-orders.

**Acknowledgment** We should like to thank the referees for thoughtful comments and pointers to further literature.

**References**

A An first order scenario

Suppose that we weaken the definition of a preference algebra by not assuming that $B$ is a MIA, but that it satisfies only the weaker first order condition

\[(A.1) \quad a \wedge b \neq 0 \Rightarrow g_i(a) \leq f_i(a)\]

for $i = 1, 2$.

A preference algebra

$\langle B, \vee, \wedge, \neg, 0, 1, f_1, g_1, f_2, g_2 \rangle$

is a structure such that $\langle B, \vee, \wedge, \neg, 0, 1 \rangle$ is a Boolean algebra, and for $i = 1, 2$ $f_i$ is a modal operator, $g_i$ a sufficiency operator, and $(A.1)$ holds. Furthermore, for all $a \in B$,

$A_1'$. \quad a \wedge f_1(g_1(a)) = 0.$
$A_2'$. \quad a \wedge f_1(g_2(a)) = 0.
$A_3'$. \quad a \leq f_2(a).
$A_4'$. \quad a \leq g_2(g_2(a)).

The canonical relations are defined by

$FP_BG \iff F \cap g_1[G] \neq \emptyset.$

$FI_BG \iff F \cap g_2[G] \neq \emptyset.$

**Theorem 4.** The canonical frame of a preference algebra is a preference frame.
Proof. F₁: Assume that $P_B$ is not asymmetric. Then, there are ultrafilters $F, G$ such that $F \cap g_1[G] \neq \emptyset$ and $G \cap g_1[F] \neq \emptyset$. Let $a \in G$ such that $g_1(a) \in F$, and $b \in F$ such that $g_1(b) \in G$. Since $G$ is proper and $a, g_1(b) \in G$, we have $a \land g_1(b) \neq 0$, and thus, $g_1(a) \leq f_1(g_1(b))$ by (A.1). Since $g_1(a) \in F$, we have $f_1(g_1(b)) \in F$, and therefore, $b \land f_1(g_1(b)) \in F$. Since $F$ is proper, we obtain $0 \neq b \land f_1(g_1(b))$, contradicting $A'_1$.

F₄: Suppose that $F \cap g_2[G] \neq \emptyset$, and let $a \in G$ exemplify this fact. Then, $g_2(g_2(a)) \in g_2[F]$, and $a \in G$ together with $A'_4$ implies $g_2(g_2(a)) \in G$. Hence, $G \cap g_2[F] \neq \emptyset$.

F₂: This is analogous to the proof of F₁: Assume that there are $F, G \in \text{Ult}(B)$ such that $FP_B G$ and $FI_B G$. Since $I_B$ is symmetric, we also obtain $GI_B F$, so that

\[(A.2) \quad F \cap g_1[G] \neq \emptyset,
\]

\[(A.3) \quad G \cap g_2[F] \neq \emptyset.
\]

Let $a \in G$ such that $g_1(a) \in F$, and $b \in F$ such that $g_2(b) \in G$. Since $G$ is proper and $a, g_2(b) \in G$, we have $a \land g_2(b) \neq 0$, and thus, $g_1(a) \leq f_1(g_2(b))$ by (A.1). Since $g_1(a) \in F$, we have $f_1(g_2(b)) \in F$, and therefore, $b \land f_1(g_2(b)) \in F$. Since $F$ is proper, we obtain $0 \neq b \land f_1(g_2(b))$, contradicting $A'_2$.

Theorem 5. The complex algebra of a preference frame is a preference algebra.

Proof. All we need to show is that $\mathfrak{Cm}(X)$ satisfies (A.1), and this is Lemma 1.5. of the MIA paper. The rest is as in the proof of Theorem 1.

Theorem 6. Suppose that $⟨X, P, I⟩$ is a preference frame, and $⟨B, f_1, g_1, f_2, g_2⟩$ a preference algebra.

1. The mapping $h : B \rightarrow \mathfrak{Cm} \mathfrak{Cf}(B)$ defined by $h(a) = \{F \in \text{Ult}(B) : a \in F\}$ is an embedding of preference algebras.

2. The mapping $k : X \rightarrow \mathfrak{Cf} \mathfrak{Cm}(X)$ defined by $k(x) = \{A \in 2^X : x \in A\}$ is an embedding of preference frames.

Proof. 1. By Stone’s Theorem, $h$ is an embedding of Boolean algebras, and $h$ preserves modal operators [4]. We show first that $h$ preserves the sufficiency operator $g_1$, i.e. that $h(g_1(a)) = [[P_B]](h(a))$:

“⊆”: Let $g_1(a) \in F$; we need to show that $F \in [[P_B]](h(a))$, i.e.

\[h(a) \subseteq P_B(F), \text{ where } P_B(F) = \{G : FP_B G\} = \{G : F \cap g_1[G] \neq \emptyset\}.
\]

Let $G \in h(a)$, i.e. $a \in G$. Since $g_1(a) \in F$, this shows that $F \cap g_1[G] \neq \emptyset$. Hence, $G \in P_B(F)$.
Denote: Suppose that $F \in [[P_B]](h(a))$, i.e. $F \cap g_1[G] \neq \emptyset$ whenever $a \in G$. Assume that $g_1(a) \notin F$, and let $Z = \{b \in B : g_1(\neg b) \in F\}$. If $b \in Z$ and $a \land b = 0$, then $a \leq \neg b$, and thus, $g_1(\neg b) \leq g_1(a)$. Since $b \in Z$ we have $g_1(\neg b) \in F$, and thus, $g_1(a) \in F$ contradicting our assumption. Now, $b \land a \neq 0$ for all $b \in Z$ implies that there is an ultrafilter $G$ containing $Z \cup \{a\}$. By the hypothesis $F \in [[P_B]](h(a))$ and $a \in G$ it follows that $F \cap g_1[G] \neq \emptyset$. Let $b \in G$ such that $g_1(b) \in F$; then, $\neg b \in Z$, and thus, $\neg b \in G$ since $Z \subseteq G$. This contradicts our choice of $b$.

Analogously one shows that $h(g_2(a)) = [[I_B]](h(a))$. 

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