Discrete dualities for some algebras with relations

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Dedicated to Gunther Schmidt on

the occasion of his 75th birthday.

Abstract. In this paper we present a unifying discrete framework for various representation theorems in the field of spatial reasoning. We also show that the universal and existential quantifiers of restricted scope used in first order languages and represented as binary relations in the syllogistic algebras considered in [12] may be studied in this framework.

1 Introduction

In this paper we present discrete dualities between some classes of Boolean algebras or their reducts, additionally endowed with a binary relation, and the appropriate classes of relational systems (frames). We consider two types of algebras with relations: those studied in connection with spatial reasoning where the relations describe relationships between space regions, and those considered in theories of Aristotelian syllogistic presented in [1] where the relations represent statements of the form "All (resp. some) a are b".

For an overview of recent developments in the region based theory of space and related topics we invite the reader to consult [2] or [3].

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By a *discrete duality* we mean a system $\langle \mathsf{Alg}, \mathsf{Frm}, \mathfrak{Cm}, \mathfrak{Cf} \rangle$ where Alg is a class of algebras, Frm is a class of frames, $\mathfrak{Cm} : \mathsf{Frm} \to \mathsf{Alg}$ is a mapping assigning to every frame X in Frm its *complex algebra* $\mathfrak{Cm}(X)$ in such a way that $\mathfrak{Cm}(X)$ belongs to Alg, $\mathfrak{Cf} : \mathsf{Alg} \to \mathsf{Frm}$ is a mapping assigning to every algebra L in Alg its *canonical frame* $\mathfrak{Cf}(L)$ in such a way that $\mathfrak{Cf}(L)$ belongs to Frm, and the following two representation theorems hold:

1. Representation theorem for algebras: Every L in Alg is embeddable into $\mathfrak{Cm}(\mathfrak{Cf}(L))$.
2. Representation theorem for frames: Every X in Frm is embeddable into $\mathfrak{Cf}(\mathfrak{Cm}(X))$.

In the literature there are some topological representation theorems for algebras of spatial reasoning, see for example [4, 5], but no corresponding abstract frames are considered, only the structures which – in terms of the notions used in the definition of discrete duality (dd-notions) – are counterparts to canonical frames endowed with a topology are introduced. In [6] a discrete representation theorem for Boolean proximity algebras is presented, but there is no a representation theorem for frames. In [1] a necessary and sufficient condition is given for obtaining a representation theorem for syllogistic algebras, saying that every such algebra is embeddable into the algebra which – in terms of dd-notions – is the complex algebra of a frame. In the present paper we develop discrete representations for some of those algebras, introduce the corresponding classes of frames, and prove representation theorems for them.

In Section 3 we present a discrete duality for Boolean algebras endowed with a proximity relation and their corresponding frames. Following [6], the canonical frames are constructed with prime filters – as usual in the Stone representation theorem for Boolean algebras.

In Section 4 a discrete duality for Boolean algebras with a contact relation and the appropriate frames is developed. In this case the canonical frames are constructed with clans as in [5].

In Section 5 a discrete representation theorem is presented for a syllogistic $\forall$-algebra with the relation representing the universal quantifier with a restricted scope as considered in [1].

In Section 6 discrete representation theorems for $\forall\exists$-algebras and $\forall\exists$-frames are developed. The representation structure for frames is built with an algebra with relations representing both the universal and existential quantifiers of restricted scope presented in [12].

2 Notation and first definitions

A frame is a pair $\langle X, R\rangle$, where X is a nonempty set and R is a binary relation on X. Since the frames needed for establishing a discrete duality for Boolean algebras are just nonempty sets, in this paper we also allow for such a 'degenerate' notion of frame. For a frame $\langle X, R\rangle$ we set $R(x) = \{y : xRy\}$, and define two operators on 2^X by

(1) $\langle R\rangle(A) = \{x \in X : (\exists y \in X)[xRy \wedge y \in A]\} \quad = \{x \in X : R(x) \cap A \neq \emptyset\}$,

(2) $[R](A) = \{x \in X : (\forall y \in X)[xRy \Rightarrow y \in A\} \quad = \{x \in X : R(x) \subseteq A\}$.

The following properties of $\langle R\rangle$ and $[R]$ are well known, see [7]:

Lemma 1. *1. $\langle R\rangle$ is a normal operator which distributes over $\cup$.*
2. $\langle R\rangle$ and $[R]$ are dual to each other, i.e. $[R](A) = X \setminus \langle R\rangle(X \setminus A)$.
3. $\langle R\rangle$ is a closure operator if and only if $[R]$ is an interior operator if and only if R is reflexive and transitive.

For the definition of closure and interior operator see [8].

If $\leq$ is a partial order on X, we call $x, y \in X$ *compatible*, if there is some z such that $x \leq z$ and $y \leq z$, otherwise, they are called *incompatible*. We can consider $\langle\leq\rangle$ and $[\leq]$ as topological operators of closure and interior. If $A \subseteq X$, we let $\uparrow A = \{y : (\exists x)[x \in A \text{ and } x \leq y]\}$. If $A = \{x\}$ we often just write $\uparrow x$ instead of $\uparrow \{x\}$. We define $\downarrow A$ analogously.

The *order topology* $\tau_{\leq}$ *on* X *generated by* $\leq$ (also called the *Alexandrov topology*) has the sets of the form $[\leq](A)$ as a basis for the open sets. Since $[\leq]$ is an interior operator, $\tau_{\leq}$ is well defined; indeed, $\tau_{\leq}$ is closed under arbitrary intersections. Since for each $x \in X$, $\uparrow x$ is the smallest open set containing x, we see that the set $\{\uparrow x : x \in X\} \cup \{\emptyset\}$ also is a basis for $\tau_{\leq}$. $\mathrm{RegCl}(X, \tau_{\leq})$, the collection of regular closed sets in this topology, consists of sets of the form $\langle\leq\rangle[\leq](Y)$ for $Y \subseteq X$.

3 A discrete duality of proximity algebras and frames

In this section we show how the representation of proximity algebras of [6] fits into our framework. In this context, we call a structure $\langle X,R\rangle$ a *proximity frame*, if X is a binary relation and R is a binary relation on X.

Suppose that B is a Boolean algebra. A binary relation δ on B is a called a *proximity* if it satisfies the following for all $a,b,c \in B$:

Prox_1. $a\delta b$ implies $a \neq 0$ and $b \neq 0$.
Prox_2. $a\delta(b \vee c)$ if and only if $a\delta b$ or $a\delta c$.
Prox_3. $(a \vee b)\delta c$ if and only if $a\delta c$ or $b\delta c$.

The pair $\langle B,\delta\rangle$ is called a *Boolean proximity algebra* (BPA). These were introduced in [6] as an abstract version of proximity spaces [9]. Among others, the following axiomatic extensions of proximities have been studied:

A proximity δ is called a *contact relation* (or *Čech proximity*) if it satisfies

Prox_4. δ is symmetric.
Prox_5. If $a \wedge b \neq 0$ then $a\delta b$.

If δ is a contact relation on B, then the pair $\langle B,\delta\rangle$ is called a *Boolean contact algebra* (BCA).

The following is well known and easy to show:

Theorem 1. *If B is a subalgebra of the Boolean algebra of regular closed sets* $\mathrm{RegCl}(X)$ *of some topological space X, then the relation δ on B defined by $a\delta b$ if and only if $a \cap b \neq \emptyset$ is a contact relation.*

BCAs of this form are called *standard models*. It is easy to see that the smallest contact relation on B is the *overlap relation* defined by aOb if and only if $a \wedge b \neq 0$.

δ is called *normal* (or a *Efremovič proximity*) if it satisfies

Prox_6. $a(-\delta)b$ implies there is some c such that $a(-\delta)c$ and $-c(-\delta)b$.

The reader is invited to consult the standard text [9] for more information on proximities.

The *canonical frame* $\mathfrak{Cf}(B,\delta)$ of a PBA $\langle B,\delta\rangle$ is the structure $\langle \mathrm{Prim}(B), R_\delta\rangle$, where $\mathrm{Prim}(B)$ is the set of prime filters of B, and, for $F,G \in \mathrm{Prim}(B)$, $FR_\delta G$ if and only if $F \times G \subseteq \delta$.

Theorem 2. *[6] Suppose that $\langle B,\delta\rangle$ is a BPA.*

1. *R_δ is symmetric if and only if δ is symmetric.*
2. *R_δ is reflexive if and only if δ satisfies* Prox_5.
3. *R_δ is transitive if and only if δ satisfies* Prox_6.

Conversely, if R is a binary relation on X, we define a binary relation δ_R on the powerset algebra of X by $A\delta_R B$ if and only if there are $a \in A, b \in B$ such that aRb (see [6, 10]). The pair $\langle 2^X, \delta_R\rangle$ is called the *complex algebra* of $\langle X,R\rangle$, denoted by $\mathfrak{Cm}(X,R)$. It is easy to see that the complex algebra of $\langle X,R\rangle$ is a BPA. However, the full strength of Theorem 2 cannot be kept:

Theorem 3. *[6] Suppose that $\langle X,R\rangle$ is a frame.*

1. *If R is symmetric, then δ_R is symmetric.*
2. *If R is reflexive, then δ_R satisfies* Prox_5.
3. *If R is transitive, then δ_R satisfies* Prox_6.

None of the implications can be reversed.

It was shown in [11] that a binary relation R on set of prime filters of some Boolean algebra B is of the form R_δ for some contact relation δ on B if and only if R is reflexive, symmetric and closed in the product topology of the Stone space of B.

For a BPA $\langle B,\delta\rangle$ let $h : B \to 2^{\mathrm{Prim}(B)}$ be the Stone map defined by $h(a) = \{F : a \in F\}$. The following theorem shows that :

Theorem 4. *[6] (*Discrete representation theorem for BPAs*) Each BPA can be embedded into the complex algebra of its canonical frame.*

Proof. It suffices to show that $a\delta b \iff h(a)\delta_{R_\delta}h(b)$.

For a frame $\langle X,R\rangle$ let $k : X \to \mathfrak{Cf}\mathfrak{Cm}(X,R)$ be defined by $k(x) = F_x$, where F_x is the principal filter of 2^X generated by $\{x\}$. Clearly, k is injective.

Theorem 5. *(*Discrete representation theorem for proximity frames*) Each frame can be embedded into the canonical frame of a BPA.*

Proof. We show that for $x,y \in X$, then xRy if and only if $k(x)R_{\delta_R}k(y)$.

"$\Rightarrow$": Let xRy; then, $\{x\}\delta_R\{y\}$. Suppose that $A \in k(x)$ and $A' \in k(y)$, i.e. $x \in A$ and $y \in A'$. Now, $A\delta_R A'$ if and only if there are $s \in A$ and $t \in A'$ such that sRt, and we may choose $s = x$ and $t = y$.

"$\Leftarrow$": Let $k(x)R_{\delta_R}k(y)$. Then, $F_x \times F_y \subseteq \delta_R$ by definition of R_{δ_R} and thus, $\{x\}\delta_R\{y\}$. The definition of δ_R now implies xRy. $\square$

4 A discrete duality for BCAs based on clans

In this section we exhibit a discrete duality for BCAs following the representation theorem presented in [5] on the basis of clans.

Suppose that $\langle B,\mathscr{C}\rangle$ is a BCA. A non–empty subset Γ of B is called a *clan* if for all $a,b \in L$,

CL1. If $a,b \in \Gamma$ then $a\mathscr{C}b$.
CL2. If $a \vee b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$.
CL3. If $a \in \Gamma$ and $a \leq b$, then $b \in \Gamma$.

Lemma 2. *1. ([5], Fact 3.3) Each prime filter is a clan, and each clan is the union of all prime filters it contains.*
2. ([5], Proposition 3.3.) If $a\mathscr{C}b$, then there is a clan Γ such that $a,b \in \Gamma$.

Indeed, given the canonical frame $\langle \mathrm{Prim}(B),R\rangle$ on prime filters, each union of a clique of R is a clan. Conversely, if Γ is a clan and $F,G \subseteq \Gamma, F,G \in \mathrm{Prim}(B)$, then $F \times G \subseteq \mathscr{C}$ by definition of a clan, and therefore $\langle F,G\rangle \in R$. It follows that

$\{F \in \mathrm{Prim}(B) : F \subseteq \Gamma\}$ is a clique. It may be observed, however, that different cliques of R may result in the same clan.

A topological representation theorem for BCAs such that the representation algebras are built on sets of clans was proved by Dimov and Vakarelov in [5].

We will now develop a representation built on frames. A *BC frame* is a pair $\langle X, \geq \rangle$ where X is a set and $\geq$ is a partial ordering of X. The complex algebra of $\langle X, R \rangle$ is the structure $\langle B_X, \vee_X, \wedge_X, -_X, 0_X, 1_X, \mathscr{C}_X \rangle$, denoted by $\mathfrak{Cm}(X,R)$, where for all $Y, Z \subseteq X$

$$\begin{aligned}
B_X &= \{Y \subseteq X : Y = \langle \geq \rangle [\geq](Y)\}, \\
Y \vee_X Z &= Y \cup Z, \\
Y \wedge_X Z &= \langle \geq \rangle [\geq](Y \cap Z), \\
-_X A &= \langle \geq \rangle [\geq](X \setminus A), \\
0_X &= \emptyset, \\
1_X &= X,
\end{aligned}$$

and, if $Y, Z \in B_X$,

$$Y \mathscr{C}_X Z \iff Y \cap Z \neq \emptyset.$$

Theorem 6. $\mathfrak{Cm}(X,R)$ *is a standard BCA.*

Proof. The elements of B_X are exactly the regular closed sets of the topology $\tau_{\geq}$ on X induced by $\geq$. It is well known that the regular closed sets of any topological space form a complete Boolean algebra under the operations given above, see e.g. [12], Theorem 1.37. □

If $\langle B, \mathscr{C} \rangle$ is a BCA, its canonical frame $\mathfrak{Cf}(B, \mathscr{C})$ is the structure $\langle X_B, \supseteq \rangle$, where $X_B = \mathrm{Clan}(B)$. We can now prove the representation theorem.

Theorem 7. *(*Discrete representation theorem for BCAs*) Each BCA can be embedded into the complex algebra of its canonical frame.*

Proof. Let $h : B \to 2^{X_B}$ be the Stone mapping $h(a) = \{\Gamma \in X_B : a \in \Gamma\}$.

1. h is injective: If $a,b \in B$ and w.l.o.g. $a \not\leq b$, there is some prime filter F such that $a \in F$ and $b \notin F$ by the Prime Ideal Theorem, see e.g. [13]. Since every prime filter is a clan, $F \in h(a)$ and $F \notin h(b)$.
2. $h(0) = \emptyset$ and $h(1) = X_B$: Since no prime filter, hence, no clan, contains 0 we have $h(0) = \emptyset$. Since each clan contains 1, we obtain $h(1) = X_B$.
3. $h(a) = \langle\supseteq\rangle[\supseteq](h(a))$: Note that for every $Y \subseteq X$

   $$\Gamma \in \langle\supseteq\rangle[\supseteq](Y) \iff (\exists\Delta)[\Gamma \supseteq \Delta \text{ and } \underbrace{(\forall\Delta')(\Delta \supseteq \Delta' \Rightarrow \Delta' \in Y)}_{\Delta\in[\supseteq](Y)}]. \tag{3}$$

   "$\subseteq$": Let $a \in \Gamma$. By Lemma 2 there is some prime filter F such that $a \in F$, i.e. $F \in h(a)$, and $F \subseteq \Gamma$, i.e. $F \in \ \supseteq(\{\Gamma\})$. Since F is a minimal clan, $\supseteq(\{F\}) = \{F\} \subseteq h(a)$, and it follows from (2) that $F \in [\supseteq](h(a))$. Altogether, this shows that $\Gamma \in \langle\supseteq\rangle[\supseteq](h(a))$.
   "$\supseteq$": Suppose that $\Gamma \in \langle\supseteq\rangle[\supseteq](h(a))$. By (1), there is some clan Δ such that $\Delta \subseteq \Gamma$ and $\Delta \in [\supseteq](h(a))$. Since $\supseteq$ is reflexive, we have, in particular, $\Delta \in h(a)$. Now, $\Delta \subseteq \Gamma$ implies $a \in \Gamma$.
4. $h(a \vee b) = h(a) \cup h(b)$: This follows immediately from CL2 and CL3.
5. $h(-a) = \langle\supseteq\rangle[\supseteq](X \setminus h(a))$: First, note that $X \setminus h(a) = [\supseteq](X \setminus h(a))$: If $a \notin \Gamma$ and $\Gamma \subseteq \Delta$, then $a \notin \Delta$.
   "$\subseteq$": Let $-a \in \Gamma$. Then, there is some prime filter F such that $-a \in F$ and $F \subseteq \Gamma$. Since F is proper, $a \notin F$, and therefore, $F \in X \setminus h(a)$. Now, $F \subseteq \Gamma$ implies $\Gamma \in \langle\supseteq\rangle(X \setminus h(a))$.
   "$\supseteq$": Let $\Gamma \in \langle\supseteq\rangle(X \setminus h(a))$. Then, there is some $\Delta \in X \setminus h(a)$ such that $\Gamma \supseteq \Delta$. Since $a \notin \Delta$ and $a \vee -a = 1 \in \Delta$, it follows from CL2 that $-a \in \Delta$. Now, $\Gamma \supseteq \Delta$ implies $-a \in \Gamma$.
   Since h preserves $0, 1, -$ and $\vee$, it also preserves $\wedge$.
6. $a\mathscr{C}b$ if and only if $h(a) \cap h(b) \neq \emptyset$:
   "$\Rightarrow$": : Let $a\mathscr{C}b$. By Lemma 2 there is a clan Γ containing a and b. Hence, $\Gamma \in h(a) \cap h(b)$.
   "$\Leftarrow$": If $\Gamma \in h(a) \cap h(b)$, then, $a, b \in \Gamma$ and therefore, $a\mathscr{C}b$ by CL1.

This completes the proof. □

Theorem 8. *(*Discrete representation theorem for contact frames*) Each contact frame can be embedded into the canonical frame of its complex algebra.*

Proof. Define $k : X \to \mathrm{Clan}(B_X)$ by $k(x) = \{Y \in B_X : x \in Y\}$.

1. k is well defined: We need to show that $k(x)$ is a clan of B_X. Clearly, $k(x)$ is closed under $\subseteq$, and thus it satisfies CL3. Next, let $Y, Y' \in B_X$ such that $Y \cup Y' \in k(x)$. Then, $x \in Y$ or $x \in Y'$ and therefore, $Y \in k(x)$ or $Y' \in k(x)$. This shows CL2. Finally, let $Y, Y' \in k(x)$. Then, $x \in Y \cap Y'$, and therefore, $Y \mathscr{C}_{B_X} Y'$. This shows CL1.
2. k is injective: Let $x \in X$. Then, $[\geq](\{x\}) = \{x\}$ since $\geq$ is reflexive, and therefore $\langle\geq\rangle[\geq](\{x\}) = \langle\geq\rangle(\{x\}) = \{z \in X : z \geq x\}$. Suppose that $y \in X$ and $x \neq y$. Since $\geq$ is antisymmetric, we may suppose w.l.o.g. that $y \not\geq x$. Clearly, $\langle\geq\rangle(\{x\}) \in k(x)$. Assume that $\langle\geq\rangle(\{x\}) \in k(y)$. Then, $y \in \langle\geq\rangle(\{x\})$ which implies $y \geq x$, contradicting our hypothesis.
3. k preserves $\geq$: Let $x \geq y$. We need to show that $k(x) \supseteq k(y)$. Therefore, let $Y \in k(Y)$. Then, from
   (a) $Y = \langle\geq\rangle(Y) = \{t \in X : (\exists z \in X)[t \geq z \text{ and } z \in Y\}$,
   (b) $y \in Y$, and
   (c) $x \geq y$,

   we obtain $x \in \langle\geq\rangle(Y) = Y$. Hence, $Y \in k(x)$.

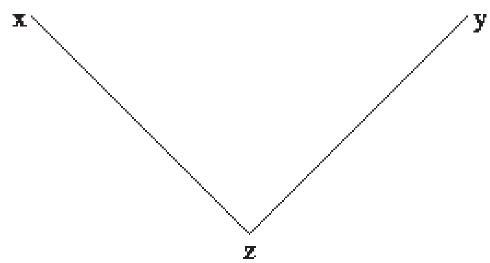
This completes the proof. $\square$

Let us briefly look at the topological properties of the two representations. Suppose that τ is the topology on $\mathrm{Clan}(B)$ of [5] which has the family $\{h(a) : a \in B\}$ as a closed basis, and $\tau_{\geq}$ the topology generated by the partial order $\geq$ in which $Y \subseteq \mathrm{Clan}(B)$ is open if and only if $Y = [\geq](Y)$. If $\Gamma \in \mathrm{Clan}(B)$, then the smallest open set in $\tau_{\geq}$ containing Γ is $\downarrow \Gamma$.

Since each $h(a)$ is (regular) closed in $\tau_{\geq}$, and $h(a)$ is a closed basis of τ we have $\tau \subseteq \tau_{\geq}$. The latter topology may be much larger than τ: Consider the case that B is the finite–cofinite subalgebra of the powerset ω of natural numbers, and that $\mathscr{C}$ is the overlap relation on B. Then, each clan is a prime filter, and the ordering on $\mathrm{Clan}(B)$ is discrete. Therefore, so is the topology $\tau_{\geq}$, and each subset of ω is open in $\tau_{\geq}$. On the other hand, τ is the usual Stone topology on $\mathrm{Prim}(B)$, which, in this case is the one–point compactification of a countable discrete space, and B is isomorphic to $\mathrm{RegCl}(\omega, \tau)$.

While τ is semiregular – i.e. it has an open basis of regular open sets – this is not necessarily true for a topology $\tau_{\geq}$: Consider the ordering on $X = \{x, y, z\}$ shown in Figure 1. There, $\downarrow x = \{x, z\} \subsetneq \{x, y, z\} = \langle \geq \rangle \downarrow x = [\geq]\langle \geq \rangle \downarrow x$.

Fig. 1. An example



5 A discrete representation for a syllogistic ∀-algebra

In [1] an algebraic approach to the Aristotelian syllogistic is presented. The algebras proposed there may be seen as reducts of Boolean algebras endowed with some binary relations relevant for syllogistic. In the present section we consider one of those classes of algebras, namely, $\boldsymbol{\forall}$ – algebras, of the form $\langle L, \neg, \boldsymbol{\forall} \rangle$ with a binary relation which we denote with (a bold version of) the symbol of the universal quantifier for the following reason: In the abstract setting, the $\boldsymbol{\forall}$ – algebras are defined with the set of axioms which say that the relation $\boldsymbol{\forall}$ is transitive, and determine its action on the negated elements. The axioms are chosen so that in case L is a set algebra and $\neg$ is the operator of difference of sets, the relation $\boldsymbol{\forall}$ may be interpreted as set inclusion. From a logical perspective, sets correspond to unary predicates, and set inclusion $A \subseteq B$ is defined with the well known formula $(\forall z)[A(z) \Rightarrow B(z)]$. In this formula, the quantifier $\forall$ is a quantifier with a restricted scope.

Following [1], with some abuse of language we call a structure $\langle L, \neg, \boldsymbol{\forall} \rangle$ an $\forall$-*algebra*, if L is a nonempty set, $\neg$ is a unary operation on L and $\boldsymbol{\forall}$ a binary relation on L such that the following properties are fulfilled for all $a, b, c \in L$:

$\forall_1$. $a \boldsymbol{\forall} \neg\neg a$.

$\forall_2$. $\neg\neg a \forall a$.
$\forall_3$. $a \forall b$ and $b \forall c$ imply $a \forall c$.
$\forall_4$. $a \forall b$ implies $\neg b \forall \neg a$.
$\forall_5$. $a \forall \neg a$ implies $a \forall b$.

Note that $\forall_1$ – $\forall_3$ imply that $\forall$ is reflexive and transitive, and therefore, $[\forall]$ is an interior operator. A primary example of an $\forall$-algebra is a structure $\langle 2^X, \subseteq, \setminus \rangle$, which we call the *complex algebra of* X.

If L is an $\forall$-algebra, $S \subseteq L$ is called $\forall$ – *closed*, if $S = [\forall](S)$. A nonempty subset S of L is called an $\forall$-*set*, if it is $\forall$ – closed and satisfies

$$a, b \in S \Rightarrow a(-\forall)\neg b. \tag{4}$$

For later use we note the following easy fact:

Lemma 3. *The intersection of two $\forall$-sets is an $\forall$-set.*

Let X'_L be the collection of all $\forall$-sets. Observe that L is not an $\forall$-set, since L is nonempty and $\forall_1$ holds. Furthermore, X'_L may be empty, e.g. if $\forall$ is the universal relation on L, in particular, if $|L| = 1$.

Example 1. Suppose that $\langle L, \forall, \neg \rangle$ is an $\forall$-algebra, and $a, a' \notin L, a \neq a'$. Let $L' = L \cup \{a, a'\}$, and extend $\forall$ and $\neg$ by

$$\forall' = \forall \cup (\{a\} \times L') \cup (L' \times \{a'\}), \tag{5}$$

$$\neg' = \neg \cup \{\langle a, a' \rangle, \langle a', a \rangle\}. \tag{6}$$

Then, $\langle L', \forall', \neg' \rangle$ is an $\forall$-algebra:

$\forall_1$: If $b \in L$, then $\neg' b = \neg b$ and $b \forall \neg\neg b$. Otherwise, $a \forall' \neg' \neg' a$, since $a = \neg' \neg' a$ and $a \forall' a$. Similarly, $a' \forall' \neg' \neg' a'$.

$\forall_2$: Similarly.

$\forall_3$: Let $b \forall' c \forall' d$. If $b = a$, then $b \forall' d$ since $\{a\} \times L' \subseteq \forall'$. If $b = a'$, then $c = d = a'$. Thus, let $b \notin \{a, a'\}$; then, $c \neq a$. If $c = a'$, then $a' \forall' d$ implies $d = a'$, and therefore, $b\mathbf{\forall}'d$ since $L' \times \{a'\} \subseteq \mathbf{\forall}'$. Thus, let $\{b,c\} \cap \{a,a'\} = \emptyset$; then, $d \neq a$. If $d = a'$, then, as before, $b\mathbf{\forall}'d$. Otherwise, $b\mathbf{\forall}c\mathbf{\forall}d$, and $b\mathbf{\forall}d$ by $\forall_3$.

$\forall_4$ and $\forall_5$ follow immediately from the definition of $\mathbf{\forall}'$ and $\neg'$.

Observe that $\{a'\}$ is an $\forall$-set, and that no $\forall$-set contains a. $\square$

The following lemma characterizes when an element of an $\forall$-algebra is contained in an $\forall$-set:

Lemma 4. *Let $a \in L$. Then, $\mathbf{\forall}(a)$ is an $\forall$-set if and only if $\{a\} \times L \not\subseteq \mathbf{\forall}$. Furthermore, $\mathbf{\forall}(a)$ is a subset of each $\forall$-set containing a.*

Proof. "$\Rightarrow$": Suppose that $\mathbf{\forall}(a)$ is an $\forall$-set. Then, $a(-\mathbf{\forall})\neg a$ which shows that $\{a\} \times L \not\subseteq \mathbf{\forall}$.

"$\Leftarrow$": Suppose that $\mathbf{\forall}(a)$ is not an $\forall$-set. Since $\mathbf{\forall}(a)$ is $\mathbf{\forall}$ – closed, there are $b,c \in L$ such that $a\mathbf{\forall}b, a\mathbf{\forall}c$, and $b\mathbf{\forall}\neg c$. Then, $a\mathbf{\forall}\neg c$ by transitivity of $\mathbf{\forall}$, and therefore, $\neg\neg c\mathbf{\forall}\neg a$ by $\forall_4$. From $\forall_1$ we obtain $c\mathbf{\forall}\neg\neg c$, and, again by transitivity, we have $c\mathbf{\forall}\neg a$. Using $a\mathbf{\forall}c$ and again transitivity we obtain $a\mathbf{\forall}\neg a$, and therefore, $a\mathbf{\forall}d$ for all $d \in L$ by $\forall_5$.

If S is an $\forall$-set containing a, then the fact that S is $\mathbf{\forall}$ – closed implies that $\mathbf{\forall}(a) \subseteq S$. $\square$

Lemma 5. *$\mathbf{\forall}$ is the universal relation if and only if there is some $a \in L$ such that $(\{a\} \times L) \cup (\{\neg a\} \times L) \subseteq \mathbf{\forall}$.*

Proof. The $\Rightarrow$ direction is obvious. Conversely, let $(\{a\} \times L) \cup (\{\neg a\} \times L) \subseteq \mathbf{\forall}$, and $b,c \in L$. Then, $a\mathbf{\forall}b$ and

$$b\mathbf{\forall}\neg\neg b\mathbf{\forall}\neg a\mathbf{\forall}c,$$

and the claim follows from transitivity. $\square$

If X_L' is nonempty, then, clearly, X_L' is closed under unions of chains, and thus, each element of X_L' is contained in a maximal element. Let X_L be the set of all maximal elements of X_L' which we call the *canonical frame of* L. Our aim is to show under which condition an $\forall$-algebra can be embedded into the complex algebra of its canonical frame. This will follow from a sequence of lemmas.

Lemma 6. *1. If $S \in X_L'$, then there is no $a \in L$ such that $a \in S$ and $\neg a \in S$.*
2. If $S \in X_L$, then $a \in S$ or $\neg a \in S$ for every $a \in L$.

Proof. 1. Assume that $S \in X_L'$ and $a, \neg a \in S$. By (4), $a(-\forall)\neg\neg a$, contradicting $\forall_1$.

2. Let S be a maximal $\forall$-set, and assume there is some $a \in L$ such that $\neg a \notin S$. Let $S' = S \cup \forall(a)$. If we can show that S' is an $\forall$-set, then $S = S'$ because of the maximality of S, and it follows that $a \in S$. Since $[\forall]$ is an interior operator, we have $[\forall](S') \subseteq S'$. Conversely, let $b \in S'$, and $b\forall c$. If $b \in S$, then $c \in S \subseteq S'$ since $[\forall](S) = S$. Otherwise, $a\forall b$, and the transitivity of $\forall$ implies $a\forall c$; hence, $c \in S'$.

Next, let $c, d \in S'$ and assume that $c\forall\neg d$. Then, not both c, d can be in S. Suppose that $c \notin S$; then, $a\forall c$. $\square$

Lemma 7. *Suppose that $h : L \to 2^{X_L}$ is defined by $h(a) = \{S \in X_L : a \in S\}$. Then, $[\subseteq](h(a)) = h(a)$, $a\forall b$ if and only if $h(a) \subseteq h(b)$, and $h(\neg a) = X_L \setminus h(a)$.*

Proof. We first show that $[\subseteq](h(a)) = h(a)$. It is sufficient to consider $\supseteq$, since $\subseteq$ is a partial order: If $S \in h(a)$ and $S \subseteq T$, then clearly $a \in T$.

Next, let $a\forall b$, and $S \in h(a)$, i.e. $a \in S$. Since $S \in X_L$ we see that, in particular, S is $\forall$ – closed, and therefore, $b \in S$ which implies $S \in h(b)$. Conversely, suppose that $a(-\forall)b$. Then, $\{a\} \times L \not\subseteq \forall$, and therefore, $\forall(a)$ is an $\forall$-set by Lemma 4. Since $a(-\forall)b$, it follows from $\forall_4$ that $\neg b(-\forall)\neg a$, and thus, $\forall(\neg b)$ is an $\forall$-set as well. Now, by Lemma 3, $S = \forall(a) \cap \forall(\neg b)$ is an $\forall$-set, and so there is a maximal $\forall$-set S' containing S. By reflexivity of $\forall$ we have $a, \neg b \in S'$. Hence, $S' \in h(a)$, and $b \notin S'$ by Lemma 6.(1). It follows that $h(a) \not\subseteq h(b)$.

Finally, we show that $h(\neg a) = X_L \setminus h(a)$: Suppose that $S \in X_L$.

"$\subseteq$": Let $\neg a \in S$. Then, by Lemma 6.(1), we have $a \notin S$, i.e. $S \notin h(a)$.

"$\supseteq$": Let $a \notin S$. Then, by Lemma 6.(2), we have $\neg a \in S$. $\square$

In general, h is not injective. We have, however,

Theorem 9. *h is injective if and only if $\forall$ is a partial order.*

Proof. "$\Rightarrow$": Suppose that $h(a) = h(b)$ implies $a = b$. All we need to show is that $\forall$ is antisymmetric. Suppose that there are $a, b \in L$ such that $a \forall b$ and $b \forall a$. If $S \in X_L$ and $a \in S$, then $a \forall b$ and the fact that S is $\forall$ – closed imply that $b \in S$. It follows that $h(a) \subseteq h(b)$, and, similarly, $b \forall a$ implies $h(b) \subseteq h(a)$. Injectivity of h now implies $a = b$.

"$\Leftarrow$": Suppose that $\forall$ is antisymmetric and $a \neq b$. Then, $a(-\forall)b$ or $b(-\forall)a$, suppose w.l.o.g. the former. Now, $\forall_3$ implies that $\neg b(-\forall)\neg a$. Thus, $\forall(a)$ and $\forall(\neg b)$ are $\forall$-sets by Lemma 4, and therefore, so is $S = \forall(a) \cap \forall(\neg b)$ by Lemma 3. It follows that any maximal $\forall$-set S' with $S \subseteq S'$ contains a and $\neg b$. By Lemma 6.1, $S' \in h(a)$ and $S' \notin h(b)$. $\square$

The following is now obvious:

Corollary 1. *An $\forall$-algebra $\langle L, \neg, \forall \rangle$ can be embedded into the complex algebra of its canonical frame if and only if $\forall$ is a partial order.*

6 A discrete representation of $\forall\exists$-structures

In this section we consider the structures with two relations $\forall$ and $\exists$ interpreted as universal and existential quantifier, respectively, of restricted scope.

Following [1] by an $\forall\exists$-*algebra* we mean a structure $(L, \forall, \exists)$ such that $L \neq \emptyset$, and $\forall$, $\exists$ are binary relations on L satisfying for all $a, b \in L$

$(\forall\exists_1)$ $\forall$ is reflexive

$(\forall\exists_2)$ $\forall$ is transitive

$(\forall\exists_3)$ $a \forall b$ and $a \exists c$ imply $c \exists b$

$(\forall\exists_4)$ $a \exists b$ implies $a \forall a$

$(\forall\exists_5)$ $a \exists a$ or $a \forall b$.

Note that by $(\forall\exists_1)$ and $(\forall\exists_3)$ relation $\exists$ is symmetric. Although it is not natural to qualify a structure without operations as an algebra, we follow [12] in this respect because it enables us to formulate relationships between $\forall$-algebras and $\forall\exists$-structures.

An $\forall\exists$-*frame* is just a Boolean frame $X \neq \emptyset$.

The complex algebra of X is the structure $(2^X, \forall_X, \exists_X)$ such that for all $A, B \subseteq X$,

$$A \ \forall_X \ B \text{ iff } A \subseteq B$$
$$A \ \exists_X \ B \text{ iff } A \cap B \neq \emptyset.$$

The canonical frame of an $\forall\exists$-algebra L is the set $X_L = \{A \subseteq L : A = [\forall]A$ and $A \times A \subseteq \exists\}$. The elements of X_L are referred to as $\forall\exists$-sets.

$\forall\exists$-algebras and $\forall$-algebras considered in Section 5 are related as presented in the following lemmas based on [12].

Lemma 8. *Given an $\forall$-algebra $(L, \neg, \forall)$, define a binary relation $\exists$ on L by $a \ \exists \ b$ iff $a(-\forall)\neg b$. Then $(L, \forall, \exists)$ is an $\forall\exists$-algebra.*

Proof. If suffices to show that the axioms $(\forall\exists_1), \ldots, (\forall\exists_5)$ are satisfied. $(\forall\exists_3)$ follows from $(\forall_4)$ and $(\forall_5)$. $(\forall\exists_4)$ and $(\forall\exists_5)$ follow from $(\forall_5)$.

□

Conversely, let $(L, \forall, \exists)$ be an $\forall\exists$-algebra. Let $f : L \to L$ be a $1-1$ map of L onto L.

Let $L + f(L)$ be a disjoint union of L and its copy $f(L)$. We define an algebra $\big(L + f(L), \ \neg, \ \forall'\big)$ by

$$\neg a = \begin{cases} f(a) & \text{if} \quad a \in L \\ f^{-1}(a) & \text{if} \quad a \in f(L) \end{cases}$$

$$a \ \forall' \ b \ \text{ iff } \ \begin{cases} a \ \forall \ b & \text{if} \quad a, b \in L \\ f^{-1}(b) \ \forall \ f^{-1}(a) & \text{if} \quad a, b \in f(L) \\ a(-\exists) \ f^{-1}(b) & \text{if} \quad a \in L, \ b \in f(L) \\ false & \text{if} \quad a \in f(L), \ b \in L \end{cases}$$

Lemma 9. *Let $(L, \forall, \exists)$ be an $\forall\exists$-algebra. For all $a, b \in L$ the following conditions are equivalent:*

1. *$a \ \exists \ b$ holds in L*
2. *$a \ \forall' \ \neg \ b$ does not hold in $L + f(L)$.*

Proof. Let $a, b, \in L$. Then $\neg b = f(b) \in f(L)$. By definition of $\forall'$, a $\forall' f(b)$ iff a $(-\exists) f^{-1}\big(f(b)\big)$ and the latter holds if and only if a $(-\exists)\, b$. Hence the required equivalence holds.

$\square$

Lemma 10. *For all $a, b \in L$, $a \exists b$ implies $\forall(a) \times \forall(b) \subseteq \exists$.*

Proof. First, observe that $\forall\,\breve{}\,;\, \exists;\, \forall \subseteq \exists$ because the symmetry of $\exists$ and axiom $\forall\exists_3$ imply that $\forall\,\breve{}\,; \exists \subseteq \exists$, and therefore using $\forall\exists_3$ again we obtain $\forall\,\breve{}\,; \exists; \forall \subseteq \exists; \forall \subseteq \exists$. Now assume $a \exists b$ and let $c \in \forall(a)$ and $d \in \forall(b)$. Thus $a \forall c$ and $b \forall d$. Then $(c, d) \in \forall\breve{}\,;\, \exists;\, \forall \subseteq \exists$.

$\square$

Lemma 11. *Let $a \in L$ and $a \exists a$. Then $\forall(a)$ is the smallest $\forall\exists$-set containing a.*

Proof. Since $\forall$ is reflexive, we have $[\forall]\forall(a) \subseteq \forall(a)$. Conversely, assume that $b \in \forall(a)$ and $c \in \forall(b)$, that is $a \forall b$ and $b \forall c$. By transitivity of $\forall$ we have $a \forall c$ and hence $\forall(b) \subseteq \forall(a)$. It follows that $\forall(a) \subseteq [\forall]\forall(a)$. Next, by $a \exists a$ and Lemma 10, $\forall(a) \times \forall(a) \subseteq \exists$. Thus $\forall(a)$ is an $\forall\exists$-set. Clearly, any $\forall\exists$-set containing a must contain $\forall(a)$ as a subset.

$\square$

Let $h : L \to 2^{X_L}$ be defined by $h(a) = \{A \in X_L : a \in A\}$.

Lemma 12. *(1) $a \forall b$ iff $h(a) \subseteq h(b)$*
(2) $a \exists b$ iff $h(a) \cap h(b) \neq \emptyset$.

Proof. (1) Assume $a \forall b$ and take $A \in X_L$ s.t. $a \in A$. Since $A \times A \subseteq \exists$, $a \exists a$. By Lemma 11 $\forall(a) \subseteq A$ and hence $b \in A$ as required. Conversely, consider the following two cases. If $a\,(-\exists)\,a$, then by $\forall\exists_5$ we have $a \forall b$. If $a \exists a$, then by Lemma 11 we have $\forall(a) \in h(a)$ and hence $\forall(a) \in h(b)$ which implies $a \forall b$.

(2) Assume $a \exists b$. By symmetry of $\exists$ we have $b \exists a$ and by $\forall\exists_4$ we get $a \exists a$ and $b \exists b$. By Lemma 11 $\forall(a), \forall(b) \in X_L$. Consider $A = \forall(a) \cup \forall(b)$. We show that $A \in X_L$, that is $[\forall]A = A$ and $A \times A \subseteq \exists$. $[A]A \subseteq A$ by reflexivity of $\forall$. Conversely, $\forall(a) \cup \forall(b) = [\forall]\forall(a) \cup [\forall]\forall(b) \subseteq [\forall](\forall(a) \cup \forall(b))$. Now, from $a \exists a$, $b \exists b$,

and Lemma 10 we have $\forall(a) \times \forall(a) \subseteq \exists$ and $\forall(b) \times \forall(b) \subseteq \exists$. From $a \exists b$ and Lemma 10, $\forall(a) \times \forall(b) \subseteq \exists$. By symmetry of $\exists$, $\forall(b) \times \forall(a) \subseteq \exists$.

□

Lemma 13. *The mapping h is injective if and only if $\forall$ is a partial order.*

Proof. Assume that h is injective and let $a \forall b$ and $b \forall a$. Then $\forall(a) = \forall(b)$. If $A \in X_L$ and $a \in A$, then $\forall(a) \subseteq A$ and hence $\forall(b) \subseteq A$. Since $\forall$ is reflexive, $b \in A$. Thus $h(a) \subseteq h(b)$. Similarly, we obtain $h(b) \subseteq h(a)$. By injectivity of h, $a = b$.

Conversely, assume that $\forall$ is antisymmetric and let $a, b \in L$ with $a \neq b$. We may assume without loss of generality that $a\,(-\forall)\,b$. Then $b \notin \forall(a)$, so $\forall(a) \notin h(b)$. Clearly, $\forall(a) \in h(a)$, and hence $h(a) \neq h(b)$.

□

Theorem 10. *$\forall\exists$-algebra $(L, \forall, \exists)$ is embeddable into the complex algebra of its canonical frame if and only if relation $\forall$ is a partial order.*

Theorem 11. *(*Discrete representation theorem for $\forall\exists$-frames*).*

Every $\forall\exists$-frame is embeddable into the canonical frame of its complex algebra.

Proof. Consider $k : X \to X_{2^X}$ defined as $k(x) = \{A \subseteq X : x \in A\}$. We show that k is well defined, that is $k(x) = [\forall_X]\,k(x)$ and $k(x) \times k(x) \subseteq \exists_X$. Indeed, if $x \in A$, then $x \in B$ for every $B \supseteq A$. Similarly, if $x \in A$ and $x \in B$ then clearly, $A \cap B \neq \emptyset$. Next, note that k is injective because if for every $A \subseteq X$, $x \in A$ iff $y \in A$, then considering $A = \{x\}$ we get $x = y$.

□

7 Conclusion and outlook

In the first part of the paper we presented a discrete duality between Boolean algebras endowed with a proximity (respectively contact) relation and their corresponding frames. These developments extended some of the existing representation theorems for algebras of spatial reasoning to representation theorems for both algebras and frames such that the topology is not needed for the construction of representation structures into which the given structures are embeddable.

In the second part of the paper we considered some of the algebras related to Aristotelian syllogistic presented in [12]. We presented a discrete representation theorem for $\forall$-algebras with an antisymmetric relation $\forall$ and discrete representation theorems for $\forall\exists$-algebras with an antisymmetric $\mathbf{\forall}$ and for $\forall\exists$-frames. The remaining parts of discrete duality are open problems.

There are various interesting issues for further work. The fact that in the complex algebras of $\forall\exists$-algebras the relation $\forall$ is the intersection of sets and the relation $\exists$ is the overlap relation inspires a systematic study of relationships between syllogistic algebras and algebras of spatial reasoning. In the present paper we considered only two out of six classes of syllogistic algebras discussed in [12]. Discrete dualities for the remaining classes are also worth a study.

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