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# On Deriving Conditional Diagnosability of Interconnection Networks

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## Abstract

We provide general techniques to estimate an upper bound of the conditional diagnosability of a graph  $G$ , and to prove that such a bound is also tight when a certain connectivity result is available for  $G$ . As an example, we derive the exact value of the conditional diagnosability for the  $(n, k)$ -star graph.

*Keywords:* fault diagnosis, self-diagnosable system, comparison diagnosis model, conditional diagnosability,  $(n, k)$ -star graph

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## 1. Introduction

Thanks to constant technological progress, multiprocessor systems with ever increasing number of interconnected computing nodes are becoming a reality. To address the reliability concern of such a system, it is ideal, and technically feasible, to have a self-diagnosable system where the computing nodes are able to detect faulty ones by themselves. One major approach in this regard is called the comparison model [6, 7], where each node performs a diagnosis by sending the same input to each pair of its distinct neighbors and then compares their responses. Based on such comparison results made by all the processors, the faulty status of the system can be decided. The number of detectable faulty nodes in such a multiprocessor system certainly depends

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on the topology of its associated interprocessor structure, as well as the modeling assumptions, and the maximum number of detectable faulty nodes in such a network is called its *diagnosability*. Such a measurement directly characterizes the fault-tolerance ability of an interconnection network and is thus of great interest [4, 5, 8, 11].

When all the neighbors of some processor  $v$  in a network are faulty simultaneously, it is impossible to determine whether or not  $v$  is fault-free. Hence, the unrestricted diagnosability of a network, when represented with a graph  $G$ , is limited by the minimum degree of  $G$ , often too small thus unsatisfying. On the other hand, with the often made statistical assumption of independent and identical distribution (*i.i.d.*) of failures among processors, it is simply unlikely that all the neighbors of a certain processor will fail at the same time, hence the notion of *conditional diagnosability* was introduced in [4] which assumes that no *conditional faulty set* contains all the neighbors of any processor. This more realistic notion leads to an improved characterization of its fault-tolerance property.

In this paper, we suggest general techniques to derive an upper bound of the conditional diagnosability of a general graph and to prove that this upper bound is also tight when a certain connectivity result is available. In particular, we identify the exact value of the conditional diagnosability for the  $(n, k)$ -star graph, which generalizes a result recently reported in this journal [11].

The rest of this paper proceeds as follows: After providing an overview of some related concepts and results in the next section, we describe a derivation technique for the upper bound of the conditional diagnosability of a general network, and derive such an upper bound for the  $(n, k)$ -star graph in Section 3, and then turn to issues related to the lower bound in Section 4, where we provide a lower bound result for a general network, and use it to establish the conditional diagnosability of the  $(n, k)$ -star graph. We conclude this paper in Section 5.

## 2. Related concepts and results

Let  $G$  be a graph,  $T \subseteq S \subset V(G)$ . We use  $N_G(S|T)$  to refer to the neighbors of all the vertices of  $S$  in  $G$ , excluding those in  $T$ . When a path  $p$  occurs in place of  $S$  ( $T$ ), we mean the set of vertices that occur in  $p$ .

According to the comparison model, a vertex  $w \in G$ , called a *faulty comparator*, sends the same input to two of its neighbors  $v$  and  $x$  in  $G$ , and generates a result  $r((v, x)_w)$ , which equals 0 if and only if  $w$  is fault-free and both  $v$  and  $x$  sent back the same response. If  $r((v, x)_w) = 1$ , then at least one of the three vertices is faulty. A collection of all such results is called a *syndrome* of the diagnosis. A subset  $F \subset V(G)$  is said to be *compatible with a syndrome*  $r$  if  $r$  is generated when all vertices in  $F$  are faulty and all vertices in  $V(G) \setminus F$  are fault-free. Finally, a graph  $G$  is called *diagnosable* if, for every syndrome  $r$ , there is a unique  $F \subset V(G)$  compatible with  $r$ .

As observed in [6, 7], two faulty sets may be compatible with the same syndrome, which leads to the notion of a *t-diagnosable graph* [8]: a graph is *t-diagnosable* if the system is diagnosable as long as the number of faulty vertices is no more than  $t$ . In this context, the *diagnosability of a graph*  $G$ , denoted by  $t(G)$ , is defined to be the maximum number of faulty vertices that  $G$  can guarantee to identify, and the *conditional diagnosability of  $G$* , denoted by  $t_c(G)$ , is defined to be the maximum number of faulty vertices that  $G$  can guarantee to identify, when no conditional faulty set includes all the neighbors of any vertex in  $G$ .

Since a faulty comparator can lead to unreliable results, a set of faulty vertices may also produce different syndromes. Two distinct faulty sets  $F_1$  and  $F_2$  are *indistinguishable* if and only if they are compatible with at least one syndrome, *distinguishable* otherwise. Hence,  $t(G)$  equals the maximum number  $c$  such that for all distinct pairs of faulty sets,  $(F_1, F_2)$ ,  $|F_1| \leq c, |F_2| \leq c$ ,  $F_1$  and  $F_2$  are distinguishable. We have a similar definition for  $t_c(G)$ .

The following result [8] tests if a pair of faulty sets is distinguishable, where  $F_1 \oplus F_2$

stands for  $(F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ , the symmetric difference of  $F_1$  and  $F_2$ :

**Theorem 2.1.** *Let  $G$  be a graph. For any two distinct subsets  $F_1$  and  $F_2$  of  $V(G)$ ,  $F_1$  and  $F_2$  are distinguishable if and only if at least one of the following conditions is satisfied,*

1. *there are two distinct vertices  $v$  and  $w$  in  $V(G) \setminus (F_1 \cup F_2)$  and there is a vertex  $x$  in  $F_1 \oplus F_2$  such that  $(v, w, x)$  is a path in  $G$ ;*
2. *there are two distinct vertices  $v$  and  $x$  in  $F_1 \setminus F_2$  and there is a vertex  $w$  in  $V(G) \setminus (F_1 \cup F_2)$  such that  $(v, w, x)$  is a path in  $G$ ; or*
3. *there are two distinct vertices  $v$  and  $x$  in  $F_2 \setminus F_1$  and there is a vertex  $w$  in  $V(G) \setminus (F_1 \cup F_2)$  such that  $(v, w, x)$  is a path in  $G$ .*

The following will be made use of in the proof of a later result.

**Corollary 2.1.** *[5] Let  $G$  be a graph with  $\delta(G)$ , the minimum degree of vertices in  $G$ , being at least 2, and let  $F_1$  and  $F_2$  be two distinct conditional faulty subsets of  $V(G)$ ,  $F_1 \subset F_2$ . Then  $(F_1, F_2)$  is a distinguishable conditional pair under the comparison model.*

### 3. A general technique to derive an upper bound

Let  $p_2 = (v, w, x)$  be a length 2 path in a graph  $G$ ,  $(v, x)$  may or may not be in  $E$ , clearly,  $N_G(p_2|p_2) = \{u \in V(G) \setminus \{v, w, x\} \mid \min(d(u, v), d(u, w), d(u, x)) = 1\}$ .

The following straightforward and insightful result first appeared in [9].

**Lemma 3.1.** *Let  $G$  be a graph, and let  $p_2$  be a length 2 path in  $G$ . Then,  $t_c(G) \leq |N_G(p_2|p_2)|$ .*

Let  $u$  be a vertex in  $N_G(p_2|p_2)$ . By the inclusion-exclusion principle:

$$|N_G(p_2|p_2)| = T_1(v, w, x) - T_2(v, w, x) + T_3(v, w, x), \quad (1)$$

where  $T_i(v, w, x)$ ,  $i = 1, 2$ , or  $3$ , refers to the number of vertices adjacent to  $i$  vertices among  $v, w$ , and  $x$ , excluding these three vertices. The following result is then immediate by Lemma 3.1 and Eq. 1.

**Theorem 3.1.** *Let  $G$  be a graph. Then,*

$$t_c(G) \leq \min_{p_2=(v,w,x) \text{ in } G} \{T_1(v, w, x) - T_2(v, w, x) + T_3(v, w, x)\}.$$

We now apply this general technique to obtain an upper bound of the conditional diagnosability for the  $(n, k)$ -star graph, denoted by  $S_{n,k}$ . This structure is proposed in [2] to address the scalability issue associated with the popular star graph, denoted by  $S_n$ . Let  $n \geq 3, k \in [1, n)$ , the vertex set of  $S_{n,k}$  is simply the collection of all the  $k$ -permutations,  $u_1 u_2 \cdots u_k$ , on  $\langle n \rangle = \{1, 2, \dots, n\}$ , where, for two such  $k$ -permutations,  $u$  and  $v$ ,  $(u, v)$  is an edge in  $S_{n,k}$  if and only if  $v$  can be obtained from  $u$  by either 1) applying a transposition  $(1, i)$  to  $u$ ,  $i \in [2, k]$  (called an  $i$ -edge); or 2) for some  $x \in \langle n \rangle - \{u_i | i \in [1, k]\}$ , replacing  $u_1$  with  $x$  in  $u$  (called a 1-edge). Clearly  $S_{n,k}$  is  $(n - 1)$ -regular. To avoid trivial small examples, we assume  $n \geq 4$ .

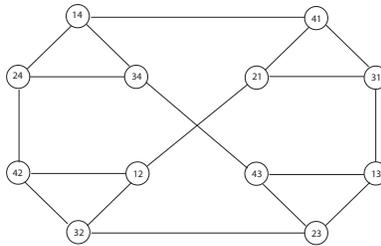


Figure 1:  $S_{4,2}$

Although  $S_{n,k}$  is vertex symmetric, it is not edge symmetric. It turns out that we have to consider three kinds of length 2 paths, depending on if  $(v, w)$  and  $(w, x)$  are either both 1-edges (referred to as a  $p_2^1$  path), a 1-edge and an  $i$ -edge,  $i \in [2, k]$  (referred to as a  $p_2^2$  path), or an  $i$ -edge and a  $j$ -edge,  $j \neq i, i, j \in [2, k]$ , (referred to as a  $p_2^3$  path).

We note that  $p_2^1$  path only exists if  $k \neq n - 1$ ,  $p_2^2$  path only exists if  $k \geq 2$ , and  $p_2^3$  path only exists if  $k \geq 3$ . For  $k \in [1, n - 2]$ , let  $p_2^1 = (v, w, x)$ , where  $w = w_1 w_2 \cdots w_k, v = m_1 w_2 \cdots w_k$ , and  $x = m_2 w_2 \cdots w_k$ , where  $m_1, m_2 \neq w_i, i \in [1, k]$ . Then  $(v, x)$  is also an edge of  $S_{n,k}$ . Since each of these vertices is adjacent to the other two,  $T_1 = 3[(n - 1) - 2] = 3n - 9$ .

Let  $u$  be adjacent to both  $v$  and  $w$ . If  $(u, w)$  is an  $i$ -edge in  $S_{n,k}$ , i.e., for some  $i \in (1, k]$ ,  $u = w_i w_2 \cdots w_1 \cdots w_k$ , then  $u$  is not adjacent to  $v$ . Thus,  $(u, w)$  has to be a 1-edge. Indeed, for  $m' \notin \{w_1, \dots, w_k, m_1, m_2\}$ ,  $u = m' w_2 \cdots w_k, u \neq x$ , is adjacent to

both  $w$  and  $v$ . Thus, there are exactly  $n - k - 2$  vertices that are adjacent to both  $v$  and  $w$ , excluding  $x$ . There are also exactly  $n - k - 2$  vertices adjacent to both  $x$  and  $w$ , excluding  $v$ , and exactly  $n - k - 2$  vertices adjacent to both  $v$  and  $x$ , excluding  $w$ . Thus,  $T_2 = 3(n - k - 2)$ . Finally, consider a vertex  $u$ , adjacent to  $w, v$ , and  $x$ . A similar argument shows that there are exactly  $T_3 = n - k - 2$  such vertices, as well.

By Eq. 1,  $|N_{S_{n,k}}(p_2^1|p_2^1)| = (3n - 9) - 3(n - k - 2) + (n - k - 2) = n + 2k - 5$ . For example, there are exactly three vertices of distance 1 from the triangle  $(42, 12, 32)$ , i.e., 21, 23, and 24 in  $S_{4,2}$ , as shown in Figure 1. By a similar analysis, for  $k \in [2, n - 1]$ ,  $|N_{S_{n,k}}(p_2^2|p_2^2)| = 2n + k - 6$ , and, for  $n \geq 4, k \in [3, n - 1]$ ,  $|N_{S_{n,k}}(p_2^3|p_2^3)| = 3n - 7$ .

By Theorem 3.1, we have the following result:

**Theorem 3.2.** *If  $n \geq 4, k \in [1, n)$ , then  $t_c(S_{n,k}) \leq n + 2k - 5$ .*

As  $S_{n,1}(\equiv K_n)$  is not desirable as interconnection networks, we henceforth assume  $k \geq 2$ .

#### 4. A standard procedure to prove a lower bound

We start with the following result, which generalizes Lemma 5 in [5, 11].

**Lemma 4.1.** *Let  $G$  be a graph such that for any  $T \subset V, |T| \leq c, G - T$  contains a large component and smaller components which contain at most two vertices in total. If  $(F_1, F_2)$  is a pair of two distinct conditional faulty sets of  $V(G)$ ,  $|F_1| \leq c + 1, |F_2| \leq c + 1$ , and  $H$  is a maximum component of  $G - (F_1 \cap F_2)$ , then  $F_1 \oplus F_2 \subset V(H)$ .*

**Proof:** Without loss of generality, let  $u \in F_1 \setminus F_2$ . Since  $F_2$  is a conditional faulty set, there exists a  $v \in V \setminus (F_2 \cup \{u\}), (u, v) \in E(G)$ . If  $u \notin V(H)$ , then  $v \notin V(H)$  either; i.e.,  $(u, v)$  is in a small component of  $G - (F_1 \cap F_2)$ . If  $|F_1 \cap F_2| \geq c + 1$ , then  $|F_1 \setminus F_2| = |F_1| - |F_1 \cap F_2| \leq 0$ , contradicting the assumption that  $F_1 \neq F_2$ . Hence,  $|F_1 \cap F_2| \leq c$ . We thus conclude that  $(u, v)$  forms a  $K_2$  in  $G - (F_1 \cap F_2)$ , i.e.,  $u$  is the only neighbor of  $v$  in  $G - (F_1 \cap F_2)$ , while all the other neighbors of  $v$ , if they do exist, must be in  $(F_1 \cap F_2) \subseteq F_1$ . Hence,  $N_G(v) \subset F_1$ , contradicting the assumption of  $F_1$  being conditional faulty.  $\square$

The following result generalizes Theorem 7 in [5] and Theorem 1 in [11].

**Theorem 4.1.** *Let  $G$  be a graph such that 1)  $\delta(G) \geq 3$ ; 2) for any  $T \subset V(G)$ ,  $|T| \leq c$ ,  $G - T$  contains a large component and smaller components which contain at most two vertices in total; and 3)  $l(G) = |V(G)| - [(\Delta(G) + 2)c + 4] > 0$ , where  $\Delta(G)$  refers to the maximum degree of vertices in  $G$ . Then  $t_c(G) \geq c + 1$ .*

**Proof:** Let  $(F_1, F_2)$  be a pair of two distinct conditional faulty sets of  $V(G)$ ,  $|F_1| \leq c + 1$ ,  $|F_2| \leq c + 1$ . If  $F_1 \subseteq F_2$  or  $F_2 \subseteq F_1$ , by Corollary 2.1,  $(F_1, F_2)$  is indistinguishable. We thus have  $|F_1 \setminus F_2| \geq 1$  and  $|F_2 \setminus F_1| \geq 1$ , then  $|F_1 \cap F_2| \leq c$ . Let  $H$  be a maximum component of  $G - (F_1 \cap F_2)$ , by assumption 2) and Lemma 4.1,  $\emptyset \neq F_1 \oplus F_2 \subset V(H)$ .

By assumption 2), there are at most two vertices not in  $H$ ; for any vertex  $v \in F_1 \cap F_2$ , at most  $\Delta(G)$  neighbors of  $v$  belong to  $H$ ; and  $|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq 2c + 2$ , since  $|F_1 \cap F_2| \geq 0$ . Thus, the number of vertices in  $H \setminus (F_1 \cup F_2)$ , also not adjacent to any vertex in  $F_1 \cap F_2$ , is at least the following:  $l(G) = |V(G)| - 2 - |F_1 \cap F_2| \Delta(G) - |F_1 \cup F_2| \geq |V(G)| - 4 - (\Delta(G) + 2)c$ , which is greater than 0 by assumption 3). Thus, there is at least one such vertex  $v$ .

If  $v$  is not adjacent to any vertex in  $F_1 \oplus F_2$ , since  $v \in H$ , there exists a path  $(v_p, v_{p-1}, \dots, v_0 (= v))$ ,  $p \geq 2$ , where  $v_p \in F_1 \oplus F_2$ , while  $v_{p-1} \notin F_1 \oplus F_2$ ,  $v_{p-1} \neq v$ , and  $v_{p-2}$  might be equal to  $v$ . Then,  $(F_1, F_2)$  is distinguishable by Case 1 of Theorem 2.1, because of the path  $(v_p, v_{p-1}, v_{p-2})$ . Otherwise, by assumption 1),  $v$  either has at least two neighbors in  $F_1 \setminus F_2$ , two neighbors in  $F_2 \setminus F_1$ , or, besides one neighbor in  $F_1 \setminus F_2$  and/or  $F_2 \setminus F_1$ , it has at least another neighbor outside  $F_1 \cup F_2$ , since it is not adjacent to  $F_1 \cap F_2$ . By Theorem 2.1,  $(F_1, F_2)$  is distinguishable when  $v$  falls into any of the above cases.

Since such an  $(F_1, F_2)$  pair is always distinguishable, the proof is completed.  $\square$

The key to Theorem 4.1 is assumption 2), which has to be achieved through a thorough structural analysis. Several such connectivity results have been proved for such graphs as the Cayley graphs generated through transposition trees, including the star graph and the bubble-sort graph [5], and for the alternating group network [11]. If such a result is achieved with  $c$  being a polynomial function in terms of some

network parameters, assumption 3 is likely to follow, since the number of vertices in an interconnection network is often exponential, or factorial, in terms of such network parameters. Indeed, for the aforementioned graphs, the associated  $c$  values are both linear functions in terms of a network parameter  $n$ , while the size of all these networks is  $\Theta(n!)$ . In both cases,  $c + 1$  is proved to be tight by following the process as summarized in Theorem 4.1. The following result is recently proved in [10] for the  $(n, k)$ -star graph.

**Theorem 4.2.** *Let  $T$  be a subset of the vertices of  $S_{n,k}$ ,  $n > k \geq 2$  such that  $|T| \leq n + 2k - 6$ . Then  $S_{n,k} - T$  is either connected or has a large component and smaller components with at most two vertices in total.*

We now derive the exact value of  $t_c(S_{n,k})$ .

**Corollary 4.1.** *For all  $n \geq 4, k \in [3, n], t_c(S_{n,k}) = n + 2k - 5$ .<sup>1</sup>*

**Proof:** By Theorem 3.2, for  $n \geq 4, k \in [1, n], t_c(S_{n,k}) \leq n + 2k - 5$ . To show that  $t_c(S_{n,k}) \geq n + 2k - 5$ , since  $S_{n,k}$ , containing  $n!/(n-k)!$  vertices, is  $(n-1)$ -regular, and  $n-1 \geq 3$ , for all  $n \geq 4$ , by Theorems 4.1 and 4.2, what is left to show is  $l(S_{n,k}) = \frac{n!}{(n-k)!} - [(n+1)(n+2k-6) + 4] > 0$ .

One can check that  $t_c(S_{4,3}) = 5$ . Suppose that  $n \geq 5$ . If  $k = 3$ ,  $l(S_{n,k}) = n(n-1)(n-2) - [(n+1)n + 4] = (n^2 + 1)(n-4) > 0$ . If  $k \in [4, n-1]$ , then  $n!/(n-k)! \geq n(n-1)(n-2)(n-k+1)$  and  $(n+5)(n+2k-6) > (n+1)(n+2k-6) + 4$ . To prove that  $l(S_{n,k}) > 0$  for this general case, it is enough to show that  $n(n-1)(n-2)(n-k+1) > (n+5)(n+2k-6)$ . Clearly  $n(n-1) > n+5$  as  $n \geq 5$ . Now  $(n-2)(n-k+1) \geq n+2k-6$  if and only if  $n^2 - n(k+2) + 4 \geq 0$ , which is also true as  $n \geq k+2$ . Finally, if  $k = n-1$ ,  $l(S_{n,n-1}) = n! - [(n+1)(3n-8) + 4] > 0$ , since  $n \geq 5$ . The proof is completed.  $\square$

Last but not least, it is recently proved [1] that for  $n \geq 3, S_{n,n-2} \equiv AN_n$ , the alternating group network [3]. When taking  $k = n-2$ , we have that, for all  $n \geq 5$ ,  $t_c(S_{n,n-2}) = t_c(AN_n) = 3n - 9$ , the main result given in [11].

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<sup>1</sup>We note that the proof given here is not applicable to  $k = 2$ . Since  $S_{n,2}$  does not have enough vertices with respect to the degree, it is not a good candidate as interconnection networks. So we forgo the discussion on it. Nevertheless, it can be shown that the formula does hold by other means.

## 5. Conclusion

We proposed techniques to estimate an upper bound of the conditional diagnosability of a general interconnection network  $G$  and to show that such an upper bound is also tight if a certain connectivity result is available for  $G$ . As an example, we proved the exact value of the conditional diagnosability for the  $(n, k)$ -star graph.

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