Question 1 (10 marks): Let $\mathbb{C}$ be a category with coproducts. Show that $a + b \cong b + a$ for each pair of objects $a, b$ from $\mathbb{C}$.

Solution: Consider the following diagram:

\[
\begin{array}{ccc}
  & a + b & \\
  i_1 & \downarrow & i_2 \\
 a & \left[ i_2, i_1 \right] & \left[ j_2, j_1 \right] \downarrow \\
  \downarrow & \downarrow & \downarrow \\
 b + a & i_1 & \left[ j_2, j_1 \right] \\
 j_2 & \downarrow & \downarrow \\
 & b & \\
\end{array}
\]

We have

\[
\begin{align*}
\left[ i_2, i_1 \right] \circ \left[ j_2, j_1 \right] \circ i_1 &= \left[ i_2, i_1 \right] \circ j_2 \\
&= i_1 \\
&= \text{id}_{a+b} \circ i_1,
\end{align*}
\]

\[
\begin{align*}
\left[ i_2, i_1 \right] \circ \left[ j_2, j_1 \right] \circ i_2 &= \left[ i_2, i_1 \right] \circ j_1 \\
&= i_2 \\
&= \text{id}_{a+b} \circ i_2
\end{align*}
\]

so that from the uniqueness of the coproduct morphism $\left[ i_2, i_1 \right] \circ \left[ j_2, j_1 \right] = \text{id}_{a+b}$ follows. The equation $\left[ j_2, j_1 \right] \circ \left[ i_2, i_1 \right] = \text{id}_{b+a}$ can be shown analogously.
Question 2 (10 marks): Let $C_1$ and $C_2$ be categories. Show that one can define a category $C_1 \times C_2$ whose objects are pairs $(a, b)$ of objects $a$ from $C_1$ and $b$ from $C_2$ and whose morphisms are pairs of morphisms from $C_1$ and $C_2$, that is $F \in C_1 \times C_2[(a, b), (c, d)]$ iff $F = (f, g)$ with $f \in C_1[a, c]$ and $g \in C_2[b, d]$.

Solution: Suppose $F = (f, g)$ and $G = (h, k)$ with $f \in C_1[a, c], g \in C_2[b, d], h \in C_1[c, e]$ and $k \in C_2[d, f]$. Then define $G \circ F = (h \circ f, k \circ g)$. Then we have

$$F \circ (\text{id}_a, \text{id}_b) = (f \circ \text{id}_a, g \circ \text{id}_b)$$
$$= (f, g)$$
$$= F,$$

$$(\text{id}_c, \text{id}_d) \circ F = (\text{id}_c \circ f, \text{id} \circ g)$$
$$= (f, g)$$
$$= F,$$

i.e., $(\text{id}_a, \text{id}_b)$ is the identity on $(a, b)$. Associativity is shown as follows

$$(H \circ G) \circ F = ((l, m) \circ (h, k)) \circ (f, g)$$
$$= (l \circ h, m \circ k) \circ (f, g)$$
$$= (l \circ h \circ g, m \circ k \circ g),$$

$$H \circ (G \circ F) = (l, m) \circ ((h, k) \circ (f, g))$$
$$= (l, m) \circ (h \circ f, k \circ g)$$
$$= (l \circ h \circ g, m \circ k \circ g).$$

The last lines of each computation above is correct because composition in $C_1$ and $C_2$ is associative so that no brackets are needed.