COSC 5P05 - Introduction to Lambda-Calculus

Term Test 1

Question 1 (5 marks): Perform the following substitutions:

1. $[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\lambda \texttt{x}:\texttt{A}\times\texttt{A}\to\texttt{B}.(\texttt{x}\ \langle \texttt{y},\texttt{y}\rangle)$,
2. $[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\lambda \texttt{y}:\texttt{A}.(\texttt{x}\ \langle \texttt{y},\texttt{y}\rangle)$,
3. $[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\lambda \texttt{z}:\texttt{A}.(\texttt{x}\ \langle \texttt{z},\texttt{z}\rangle)$,
4. $[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\langle\lambda \texttt{z}:\texttt{A}\to\texttt{B}.\langle \texttt{x},\texttt{z}\rangle,(\lambda \texttt{x}:\texttt{A}\times\texttt{A}\to\texttt{B}.\texttt{x})\ \texttt{x}\rangle$.

Solution:

$[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\lambda \texttt{x}:\texttt{A}\times\texttt{A}\to\texttt{B}.(\texttt{x}\ \langle \texttt{y},\texttt{y}\rangle)$
$= \lambda \texttt{x}:\texttt{A}\times\texttt{A}\to\texttt{B}.(\texttt{x}\ \langle \texttt{y},\texttt{y}\rangle)$,
$[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\lambda \texttt{y}:\texttt{A}.(\texttt{x}\ \langle \texttt{y},\texttt{y}\rangle)$
$= \lambda \texttt{y}:\texttt{A}.(\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))\ \langle \texttt{y},\texttt{y}\rangle)$,
$[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\lambda \texttt{z}:\texttt{A}.(\texttt{x}\ \langle \texttt{z},\texttt{z}\rangle)$
$= [\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\lambda \texttt{y}:\texttt{A}.(\texttt{x}\ \langle \texttt{y},\texttt{y}\rangle)$
$= \lambda \texttt{y}:\texttt{A}.(\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))\ \langle \texttt{y},\texttt{y}\rangle)$,
$[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\langle\lambda \texttt{z}:\texttt{A}\to\texttt{B}.\langle \texttt{x},\texttt{z}\rangle,(\lambda \texttt{x}:\texttt{A}\times\texttt{A}\to\texttt{B}.\texttt{x})\ \texttt{x}\rangle$
$= \langle[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}]\lambda \texttt{z}:\texttt{A}\to\texttt{B}.\langle \texttt{x},\texttt{z}\rangle,[\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p}))/\texttt{x}](\lambda \texttt{x}:\texttt{A}\times\texttt{A}\to\texttt{B}.\texttt{x})\ \texttt{x}\rangle$
$= \langle\lambda \texttt{y}:\texttt{A}\to\texttt{B}.\langle\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p})),\texttt{y}\rangle,(\lambda \texttt{x}:\texttt{A}\times\texttt{A}\to\texttt{B}.\texttt{x})\ (\lambda \texttt{p}:\texttt{A}\times\texttt{A}.(\texttt{z}\ \texttt{fst}(\texttt{p})))\rangle$.

Question 2 (5 marks): Find the normal form of the following λ-terms (show intermediate steps):

1. $\langle \mathtt{fst(p)}, \lambda \mathtt{x}:\mathtt{A}.\mathtt{x}\rangle$,
2. $(\lambda \mathtt{p}:\mathtt{A}\times\mathtt{B}.((\lambda \mathtt{x}:\mathtt{A}.\mathtt{x})\ \mathtt{fst(p)}))\ \langle \mathtt{y}, \mathtt{z}\rangle$,
3. $(\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ ((\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ \mathtt{f})$.

Remark: In the reduction of the third term you will apply an η-rule twice.

Solution:
The first term is already in normal form and the others reduce as follows:

$$
\begin{aligned}
&(\lambda \mathtt{p}:\mathtt{A}\times\mathtt{B}.((\lambda \mathtt{x}:\mathtt{A}.\mathtt{x})\ \mathtt{fst(p)}))\ \langle \mathtt{y}, \mathtt{z}\rangle\\
&\to (\lambda \mathtt{x}:\mathtt{A}.\mathtt{x})\ \mathtt{fst}(\langle \mathtt{y}, \mathtt{z}\rangle)\\
&\to (\lambda \mathtt{x}:\mathtt{A}.\mathtt{x})\ \mathtt{y},\\
&\to \mathtt{y},\\
&(\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ ((\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ \mathtt{f})\\
&\to (\lambda \mathtt{p}:\mathtt{A}\to\mathtt{A}\to\mathtt{A}.\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x}))\ (\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{f}\ \mathtt{y})\ \mathtt{x}))\\
&\to \lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.(((\lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{f}\ \mathtt{y})\ \mathtt{x}))\ \mathtt{y})\ \mathtt{x}))\\
&\to \lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\lambda \mathtt{z}:\mathtt{A}.((\mathtt{f}\ \mathtt{z})\ \mathtt{y}))\ \mathtt{x})\\
&\to \lambda \mathtt{x}:\mathtt{A}.\lambda \mathtt{y}:\mathtt{A}.((\mathtt{f}\ \mathtt{x})\ \mathtt{y})\\
&\to \lambda \mathtt{x}:\mathtt{A}.(\mathtt{f}\ \mathtt{x})\\
&\to \mathtt{f}.
\end{aligned}
$$

Question 3 (10 marks): Write a λ-term

$$\mathtt{twist}_{\mathtt{A,B}} : (\mathtt{A}\times\mathtt{B})\to(\mathtt{B}\times\mathtt{A}),$$

so that $\mathtt{twist}_{\mathtt{B,A}}\ (\mathtt{twist}_{\mathtt{A,B}}\ \mathtt{p}) \to \mathtt{p}$ for all $\mathtt{p}:\mathtt{A}\times\mathtt{B}$. Compute the previous property explicitly (show intermediate steps).
Remark: In the reduction mentioned above you will apply an η-rule.

Solution:
Define

$$\mathtt{twist}_{\mathtt{A,B}} \equiv \lambda \mathtt{p}:\ \mathtt{A}\times\mathtt{B}.\langle \mathtt{fst(p)}, \mathtt{snd(p)}\rangle.$$

Then we have

$$
\begin{aligned}
&\mathtt{twist}_{\mathtt{B,A}}\ (\mathtt{twist}_{\mathtt{A,B}}\ \mathtt{p})\\
&\rightarrow (\lambda \mathtt{p} \colon \mathtt{B}\times\mathtt{A}.\langle \mathtt{fst}(\mathtt{p}), \mathtt{snd}(\mathtt{p})\rangle)\ ((\lambda \mathtt{p} \colon \mathtt{A}\times\mathtt{B}.\langle \mathtt{fst}(\mathtt{p}), \mathtt{snd}(\mathtt{p})\rangle)\ \mathtt{p})\\
&\rightarrow (\lambda \mathtt{p} \colon \mathtt{B}\times\mathtt{A}.\langle \mathtt{fst}(\mathtt{p}), \mathtt{snd}(\mathtt{p})\rangle)\ \langle \mathtt{fst}(\mathtt{p}), \mathtt{snd}(\mathtt{p})\rangle\\
&\rightarrow \langle \mathtt{fst}(\langle \mathtt{fst}(\mathtt{p}), \mathtt{snd}(\mathtt{p})\rangle), \mathtt{snd}(\langle \mathtt{fst}(\mathtt{p}), \mathtt{snd}(\mathtt{p})\rangle)\rangle\\
&\rightarrow \langle \mathtt{fst}(\mathtt{p}), \mathtt{snd}(\mathtt{p})\rangle\\
&\rightarrow \mathtt{p}.
\end{aligned}
$$