COSC 5P05 - Introduction to Lambda-Calculus

Term Test 1

Question 1 (5 marks): Perform the following substitutions:

1. $[(\texttt{f z})/\texttt{x}]\lambda\texttt{x}^{\texttt{A}}.(\texttt{y}\ \langle\texttt{x},\texttt{x}\rangle)$,
2. $[(\texttt{f z})/\texttt{x}]\lambda\texttt{y}^{(\texttt{A}\times\texttt{A})\to\texttt{B}}.(\texttt{y}\ \langle\texttt{x},\texttt{x}\rangle)$,
3. $[(\texttt{f z})/\texttt{x}]\lambda\texttt{z}^{(\texttt{A}\times\texttt{A})\to\texttt{B}}.(\texttt{z}\ \langle\texttt{x},\texttt{x}\rangle)$,
4. $[\langle\lambda\texttt{z}^{\texttt{A}}.\texttt{z},\texttt{z}\rangle/\texttt{x}]\langle(\lambda\texttt{x}^{(\texttt{A}\to\texttt{A})\times\texttt{A}}.\texttt{x x}),\lambda\texttt{z}^{\texttt{A}}.(\texttt{fst}(\texttt{x})\ \texttt{z})\rangle$.

Solution:

$$[(\texttt{f z})/\texttt{x}]\lambda\texttt{x}^{\texttt{A}}.(\texttt{y}\ \langle\texttt{x},\texttt{x}\rangle)$$
$$= \lambda\texttt{x}^{\texttt{A}}.(\texttt{y}\ \langle\texttt{x},\texttt{x}\rangle),$$
$$[(\texttt{f z})/\texttt{x}]\lambda\texttt{y}^{(\texttt{A}\times\texttt{A})\to\texttt{B}}.(\texttt{y}\ \langle\texttt{x},\texttt{x}\rangle)$$
$$= \lambda\texttt{y}^{(\texttt{A}\times\texttt{A})\to\texttt{B}}.(\texttt{y}\ \langle(\texttt{f z}),(\texttt{f z})\rangle),$$
$$[(\texttt{f z})/\texttt{x}]\lambda\texttt{z}^{(\texttt{A}\times\texttt{A})\to\texttt{B}}.(\texttt{z}\ \langle\texttt{x},\texttt{x}\rangle)$$
$$= [(\texttt{f z})/\texttt{x}]\lambda\texttt{y}^{(\texttt{A}\times\texttt{A})\to\texttt{B}}.(\texttt{y}\ \langle\texttt{x},\texttt{x}\rangle)$$
$$= \lambda\texttt{y}^{(\texttt{A}\times\texttt{A})\to\texttt{B}}.(\texttt{y}\ \langle(\texttt{f z}),(\texttt{f z})\rangle),$$
$$[\langle\lambda\texttt{z}^{\texttt{A}}.\texttt{z},\texttt{z}\rangle/\texttt{x}]\langle(\lambda\texttt{x}^{(\texttt{A}\to\texttt{A})\times\texttt{A}}.\texttt{x x}),\lambda\texttt{z}^{\texttt{A}}.(\texttt{fst}(\texttt{x})\ \texttt{z})\rangle$$
$$= \langle(\lambda\texttt{x}^{(\texttt{A}\to\texttt{A})\times\texttt{A}}.\texttt{x}\ \langle\lambda\texttt{z}^{\texttt{A}}.\texttt{z},\texttt{z}\rangle),[\langle\lambda\texttt{z}^{\texttt{A}}.\texttt{z},\texttt{z}\rangle/\texttt{x}]\lambda\texttt{y}^{\texttt{A}}.(\texttt{fst}(\texttt{x})\ \texttt{y})\rangle$$
$$= \langle(\lambda\texttt{x}^{(\texttt{A}\to\texttt{A})\times\texttt{A}}.\texttt{x}\ \langle\lambda\texttt{z}^{\texttt{A}}.\texttt{z},\texttt{z}\rangle),\lambda\texttt{y}^{\texttt{A}}.(\texttt{fst}(\langle\lambda\texttt{z}^{\texttt{A}}.\texttt{z},\texttt{z}\rangle)\ \texttt{y})\rangle.$$

Question 2 (5 marks): Find the normal form of the following λ-terms (show intermediate steps):

1. $\lambda \mathtt{x}^{\mathtt{A}}.(\mathtt{y}\ \langle \mathtt{x}, \mathtt{x} \rangle)$,
2. $(\lambda \mathtt{x}^{\mathtt{A}\times\mathtt{A}}.(\mathtt{y}\ \mathtt{fst}(\mathtt{x}))\ \langle \mathtt{z}, \mathtt{z} \rangle)$,
3. $(\lambda \mathtt{p}^{\mathtt{A}\to\mathtt{A}\to\mathtt{A}}.\lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x})\ (\lambda \mathtt{p}^{\mathtt{A}\to\mathtt{A}\to\mathtt{A}}.\lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x})\ \mathtt{f}))$.

Remark: In the reduction of the third term you will apply an η-rule twice.

Solution:
The first term is already in normal form and the others reduce as follows:

$$
\begin{aligned}
&(\lambda \mathtt{x}^{\mathtt{A}\times\mathtt{A}}.(\mathtt{y}\ \mathtt{fst}(\mathtt{x}))\ \langle \mathtt{z}, \mathtt{z} \rangle)\\
&\to (\mathtt{y}\ \mathtt{fst}(\langle \mathtt{z}, \mathtt{z} \rangle))\\
&\to (\mathtt{y}\ \mathtt{z}),\\
&(\lambda \mathtt{p}^{\mathtt{A}\to\mathtt{A}\to\mathtt{A}}.\lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x})\ (\lambda \mathtt{p}^{\mathtt{A}\to\mathtt{A}\to\mathtt{A}}.\lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x})\ \mathtt{f}))\\
&\to (\lambda \mathtt{p}^{\mathtt{A}\to\mathtt{A}\to\mathtt{A}}.\lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.((\mathtt{p}\ \mathtt{y})\ \mathtt{x})\ \lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.((\mathtt{f}\ \mathtt{y})\ \mathtt{x}))\\
&\to \lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.((\lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.((\mathtt{f}\ \mathtt{y})\ \mathtt{x})\ \mathtt{y})\ \mathtt{x})\\
&\to \lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.(\lambda \mathtt{z}^{\mathtt{A}}.((\mathtt{f}\ \mathtt{z})\ \mathtt{y})\ \mathtt{x})\\
&\to \lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{A}}.((\mathtt{f}\ \mathtt{x})\ \mathtt{y})\\
&\to \lambda \mathtt{x}^{\mathtt{A}}.(\mathtt{f}\ \mathtt{x})\\
&\to \mathtt{f}.
\end{aligned}
$$

Question 3 (10 marks): Write two λ-terms

$$
\begin{aligned}
&\mathtt{curry} : ((\mathtt{A} \times \mathtt{B}) \to \mathtt{C}) \to \mathtt{A} \to \mathtt{B} \to \mathtt{C},\\
&\mathtt{uncurry} : (\mathtt{A} \to \mathtt{B} \to \mathtt{C}) \to (\mathtt{A} \times \mathtt{B}) \to \mathtt{C}
\end{aligned}
$$

implementing the curry and uncurry operation. Show that

$$(\mathtt{curry}\ (\mathtt{uncurry}\ \mathtt{f})) \to \mathtt{f}.$$

Solution:
Define

$$\begin{aligned}\texttt{curry} &\equiv \lambda \mathtt{f}^{(\mathtt{A}\times\mathtt{B})\to\mathtt{C}}.\lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{B}}.(\mathtt{f}\ \langle \mathtt{x}, \mathtt{y}\rangle),\\ \texttt{uncurry} &\equiv \lambda \mathtt{g}^{\mathtt{A}\to\mathtt{B}\to\mathtt{C}}.\lambda \mathtt{p}^{\mathtt{A}\times\mathtt{B}}.((\mathtt{g}\ \texttt{fst}(\mathtt{p}))\ \texttt{snd}(\mathtt{p})).\end{aligned}$$

Then we have

$$\begin{aligned}(\texttt{curry}\ (\texttt{uncurry}\ \mathtt{f})) &\to (\texttt{curry}\ \lambda \mathtt{p}^{\mathtt{A}\times\mathtt{B}}.((\mathtt{f}\ \texttt{fst}(\mathtt{p}))\ \texttt{snd}(\mathtt{p})))\\ &\to \lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{B}}.(\lambda \mathtt{p}^{\mathtt{A}\times\mathtt{B}}.((\mathtt{f}\ \texttt{fst}(\mathtt{p}))\ \texttt{snd}(\mathtt{p}))\ \langle \mathtt{x}, \mathtt{y}\rangle)\\ &\to \lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{B}}.((\mathtt{f}\ \texttt{fst}(\langle \mathtt{x}, \mathtt{y}\rangle))\ \texttt{snd}(\langle \mathtt{x}, \mathtt{y}\rangle))\\ &\to \lambda \mathtt{x}^{\mathtt{A}}.\lambda \mathtt{y}^{\mathtt{B}}.((\mathtt{f}\ \mathtt{x})\ \mathtt{y})\\ &\to \lambda \mathtt{x}^{\mathtt{A}}.(\mathtt{f}\ \mathtt{x})\\ &\to \mathtt{f}.\end{aligned}$$