COSC 5P02 - Logic in Computer Science

Term Test 3

Question 1:

(a) Find a derivation \(\Box p, \Box(p \rightarrow \Box p), \Box(\Box p \rightarrow q) \vdash \Box q\) in natural deduction. Justify the application of rules whenever necessary (box conditions) (5 marks).

(b) Show that the formula \((\Diamond p \land \neg \Box q) \rightarrow \Diamond(p \land \neg q)\) is not valid (5 marks).

Hint: A universe with three elements is sufficient. The propositional variables \(p\) and \(q\) may have the same interpretation.

Solution:

(a) \[
\begin{array}{c}
\Box(\Box p \rightarrow q) \quad \Box(p \rightarrow \Box p) \quad \Box E \\
\Box p \rightarrow q \\
\hline
\Box p \rightarrow \Box q \\
\hline
\Box q \quad \Box I
\end{array}
\]

(b) Let \(|\mathcal{M}| := \{a, b, c\}\), \(R = \{(a, b), (a, c)\}\), and \(v(p) = v(q) = \{b\}\). This model is visualized by

\[
\begin{array}{c}
\text{\(c\)} \\
\rightarrow
\end{array}
\]

\(p = q\)

Then \(\mathcal{M}, a \models \Diamond p \land \neg \Box q\), but we have \(\mathcal{M}, a \not\models \Diamond(p \land \neg q)\).
Question 2:

(a) Is the following derivation correct, i.e., a valid derivation in the calculus of natural deduction - why or why not (5 marks)?

\[
\begin{align*}
\frac{\square \neg p}{\neg p} \quad \square E \\
\frac{p}{\neg \neg p} \quad \neg E
\end{align*}
\]

\[
\frac{\neg \neg p}{\neg \square \neg p} \quad \neg I^2 \\
\frac{\square p \rightarrow \neg \square \neg p}{\rightarrow I^1}
\]

(b) Is the formula \( \square p \rightarrow \Diamond p \) valid (5 marks)?

*Hint: Use a universe with 1 element.*

Solution:

(a) The derivation is not correct since it contains an open dashed box (introduced by the two applications of \( \square \) elimination that is not closed by an \( \square \) introduction.

(b) Let \( |\mathcal{M}| := \{a\}, \ R = \emptyset, \ v(p) = \emptyset \). In this model we have \( \mathcal{M}, a \models \square p \) since \( a \) does not have any successor state. Consequently, we also get \( \mathcal{M}, a \not\models \Diamond p. \)