COSC 5P02 - Logic in Computer Science

Term Test 2

Question:

(a) Find a derivation $\forall x{:}\neg p(x) \to \neg\forall x{:}p(x)$. Justify the application of a rule whenever necessary (variable conditions) (8 marks).

(b) Is the following derivation of the converse implication $\neg\forall x{:}p(x) \to \forall x{:}\neg p(x)$ correct, i.e., a valid derivation in the calculus of natural deduction - why, or why not (6 marks)?

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\forall x{:}p(x)]^2 \qquad \dfrac{[p(x)]^1}{\forall x{:}p(x)}\,\forall\mathrm{I}}{\bot}\,\neg\mathrm{E}}{\neg p(x)}\,\neg\mathrm{I}^1}{\forall x{:}\neg p(x)}\,\forall\mathrm{I}}{\neg\forall x{:}p(x) \to \forall x{:}\neg p(x)}\,\to\mathrm{I}^2$$

(c) Show that the converse implication $\neg\forall x{:}p(x) \to \forall x{:}\neg p(x)$ is not valid, i.e., provide a model in which $\neg\forall x{:}p(x)$ is true but $\forall x{:}\neg p(x)$ is false (6 marks).
Hint: A universe with two elements is sufficient.

Solution:

(a)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x{:}\neg p(x)]^2}{\neg p(x)}\ \forall\text{E} \qquad \dfrac{[\forall x{:}p(x)]^1}{p(x)}\ \forall\text{E}}{\bot}\ \neg\text{E}}{\neg\forall x{:}p(x)}\ \neg\text{I}^1}{\forall x{:}\neg p(x) \to \neg\forall x{:}p(x)}\ {\to}\text{I}^2$$

No rule has a variable condition.

(b) The derivation is not correct. The variable condition of the rule $\forall$I applied to the formula $p(x)$ (upper right leaf) requires that x does not occur free in the assumption $p(x)$ of that subtree. This is obviously not the case.

(c) Let $|\mathcal{M}| := \{a, b\}$, and $p^{\mathcal{M}} := \{a\}$. Then $\neg\forall x{:}p(x)$ is true because $b \notin p^{\mathcal{M}}$. On the other hand $\forall x{:}\neg p(x)$ is false because $a \in p^{\mathcal{M}}$.