COSC 5P02 - Logic in Computer Science

Term Test 2

Question 1:

(a) Find a derivation $\vdash \exists x{:}\forall y{:}p(x,y) \rightarrow \forall y{:}\exists x{:}p(x,y)$. Justify the application of a rule whenever necessary (variable conditions) (5 marks).

(b) Show that the converse implication $\forall y{:}\exists x{:}p(x,y) \rightarrow \exists x{:}\forall y{:}p(x,y)$ is not valid, i.e., provide a model so that the formula is not true (5 marks). *Hint: A universe with two elements is sufficient.*

Solution:

(a)

$$\dfrac{\dfrac{\dfrac{[\exists x{:}\forall y{:}p(x,y)]^2 \qquad \dfrac{\dfrac{[\forall y{:}p(x,y)]^1}{p(x,y)}\,\forall\mathrm{E}}{\exists x{:}p(x,y)}\,\exists\mathrm{I}}{\exists x{:}p(x,y)}\,\exists\mathrm{E}^{1,a}}{\forall y{:}\exists x{:}p(x,y)}\,\forall\mathrm{I}^b}{\exists x{:}\forall y{:}p(x,y) \rightarrow \forall y{:}\exists x{:}p(x,y)}\,\rightarrow\mathrm{I}^2$$

a) x does not occur free in $\exists x{:}p(x,y)$ (there are no premises except $\forall y{:}p(x,y)$).
b) y does not occur free in $\exists x{:}\forall y{:}p(x,y)$.

(b) Let $|\mathcal{M}| := \{a,b\}$, and $p^{\mathcal{M}} := \{(a,a),(b,b)\}$, i.e., $p(x,y)$ iff $x = y$. Then $\forall y{:}\exists x{:}p(x,y)$ is true since each element is equal to itself. $\exists x{:}\forall y{:}p(x,y)$ is not valid because neither a nor b is equal to all elements.

Question 2:

(a) Is the following derivation correct, i.e., a valid derivation in the calculus of natural deduction - why, why not or under what assumption (5 marks)?
Hint: An assumption on the free variables of ψ is needed.

$$\dfrac{\exists x{:}(\varphi \land \psi) \qquad \dfrac{\dfrac{\dfrac{[\varphi \land \psi]^1}{\varphi}\,\land\mathrm{E1}}{\exists x{:}\varphi}\,\exists\mathrm{I} \qquad \dfrac{[\varphi \land \psi]^1}{\psi}\,\land\mathrm{E2}}{\exists x{:}\varphi \land \psi}\,\land\mathrm{I}}{\exists x{:}\varphi \land \psi}\,\exists\mathrm{E}^1$$

(b) Is the formula $\exists x{:}(\varphi \land \psi) \to \exists x{:}\varphi \land \psi$ valid if x occurs free in ψ? Prove your claim (5 marks).
Hint: Choose $\varphi = \psi = p(x)$ for a unary predicate symbol p and consider a universe with two elements.

Solution:

(a) The derivation is correct iff x does not occur free in ψ (variable condition for $\exists$E).

(b) The formula is not valid. Let $\varphi = \psi = p(x)$, $|\mathcal{M}| := \{a, b\}$, and $p^{\mathcal{M}} = \{a\}$, i.e., $p(x)$ iff $x = a$. Furthermore, let σ be an environment with $\sigma(x) = b$. Then we have $\models_{\mathcal{M}} \exists x{:}(p(x) \land p(x))[\sigma]$. But we have $\not\models_{\mathcal{M}} \exists x{:}p(x) \land p(x)[\sigma]$ since $\not\models_{\mathcal{M}} p(x)[\sigma]$ by the definition of $p^{\mathcal{M}}$ and σ.