

A logic for rough sets

Ivo Düntsch

School of Information and Software Engineering, University of Ulster,
Newtownabbey, BT 37 0QB, N.Ireland

I.Duentsch@ulst.ac.uk

Abstract

The collection of all subsets of a set forms a Boolean algebra under the usual set theoretic operations, while the collection of rough sets of an approximation space is a regular double Stone algebra [24]. The appropriate class of algebras for classical propositional logic are Boolean algebras, and it is reasonable to assume that regular double Stone algebras are a class of algebras appropriate for a logic of rough sets. Using the representation theorem for these algebras by Katriňák [16], we present such a logic for rough sets and its algebraic semantics in the spirit of Andréka et al [2].

keywords: Rough sets, algebraic semantics, regular double Stone algebras

1 Introduction

Rough set data analysis has been developed by Pawlak and his co-workers since the early 1980s as a method of dealing with coarse information. We invite the reader to consult the monographs by Pawlak [23] and Słowiński [26] for an in depth exposition of the theory and its applications.

The building blocks of rough set analysis are *approximation spaces* $\langle U, \theta \rangle$, where U is a set, and θ is an equivalence relation on U . The intuition behind this is that objects in U can only be distinguished up to their equivalence class, and that objects within one class are indistinguishable with the information at hand. With each approximation space $\langle U, \theta \rangle$ two operators on $\mathfrak{P}(U)$ are associated: If $X \subseteq U$, then

$$\overline{X} := \bigcup \{ \theta x : \theta x \cap X \neq \emptyset \}$$

is the *upper approximation* of X , and

$$\underline{X} := \bigcup \{ \theta x : \theta x \subseteq X \}$$

is its *lower approximation*; for each $X \subseteq U$, a *rough set* is a pair $\langle \underline{X}, \overline{X} \rangle$.

We note in passing that the upper (lower) approximation in $\langle U, \theta \rangle$ is the closure (interior) operator of the topology on U whose non empty open sets are the union of equivalence classes. This topology is sometimes called *Pawlak topology* e.g. in [17]. The terminology seems somewhat unfortunate: It has been known for some time that on a finite set U there are natural correspondences between

- The set of all equivalence relations on U ,
- The set of topologies on U in which each closed set is open,

- The set of all regular (not necessarily T_1) topologies on U ,

see for example [15] and the references therein.

It is easily seen that the two approximation operators can be regarded as modal operators \diamond and \square induced by the frame $\langle U, \theta \rangle$. Thus, one can associate a modal **S5** logic with this concept; this is the approach taken by Orłowska [19].

Another approach is to use the algebraic structure of rough sets in analogy to the correspondence between Boolean algebras and classical propositional logic. It was shown by J. Pomykala and J. A. Pomykala [24] that the collection $\mathfrak{R}_\theta(U)$ of rough sets of $\langle U, \theta \rangle$ can be made into a Stone algebra $\langle \mathfrak{R}_\theta(U), +, \cdot, *, \langle \emptyset, \emptyset \rangle, \langle U, U \rangle \rangle$ by defining

$$\begin{aligned}\langle \underline{X}, \overline{X} \rangle + \langle \underline{Y}, \overline{Y} \rangle &= \langle \underline{X} \cup \underline{Y}, \overline{X} \cup \overline{Y} \rangle, \\ \langle \underline{X}, \overline{X} \rangle \cdot \langle \underline{Y}, \overline{Y} \rangle &= \langle \underline{X} \cap \underline{Y}, \overline{X} \cap \overline{Y} \rangle, \\ \langle \underline{X}, \overline{X} \rangle^* &= \langle -\overline{X}, -\underline{X} \rangle,\end{aligned}$$

where for $Z \subseteq U$, the complement of Z in U is denoted by $-Z$.

This was improved by Comer [7] who noticed that $\mathfrak{R}_\theta(U)$ is in fact a regular double Stone algebra when one defines the dual pseudocomplement $^+$ by

$$\langle \underline{X}, \overline{X} \rangle^+ = \langle -\underline{X}, -\overline{X} \rangle.$$

These algebras of rough sets have the special form of what we call below a *Katriňák algebra*.

We use these facts to present a propositional logic for rough sets with an algebraic semantics and exhibit some of its properties using the algebraic tools of the general approach to logic, where the semantics of formulas is determined by a general “meaning” function, and not only by a truth value assignment

$$v : \text{Formulas} \rightarrow \text{Truth values.}$$

This framework for logic is described in somewhat simplified form in Henkin–Monk–Tarski [14], pp 255ff, where it is credited to a series of papers by Andréka–Németi–Sain ([1, 25, 18, 4]); for a recent exposition we refer the reader to [2]. A related view has later been put forward – for a different purpose – by Epstein [13] as “set assignment semantics”.

There are several closely related, resp. equivalent, constructions in algebraic logic: Using the correspondence of regular double Stone algebras to three valued Łukasiewicz algebras (and thus to Moisil’s algebras) which was first observed by Varlet [28], our results can also be considered a semantic approach to three valued Łukasiewicz logic. Furthermore, since these algebras are equivalent to semi-simple Nelson algebras, the rough set logic is also equivalent to “classical logic with strong negation”, see [27].

The interested reader is invited to consult [22] for an exposition of the algebraic connections between rough set algebras, semi-simple Nelson algebras, three valued Łukasiewicz algebras, and related structures, as well as [11] for an overview of the algebraic properties of rough sets.

A logic for information systems has been presented by Orłowska [20], and Comer [6] provides a class of algebras of information systems which are closely related to cylindric algebras.

2 Definitions and notation

We assume familiarity with the basic concepts of lattice theory, universal algebra, and logic. For definitions not explained here we refer the reader to Balbes and Dwinger [5] for lattice theory and

universal algebra, and to Andr eka et al [2] for logic. Here, we just define regular double Stone algebras which may not be widely known:

A double Stone algebra $\langle L, +, \cdot, *, ^+, 0, 1 \rangle$ is an algebra of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ such that

1. $\langle L, +, \cdot, 0, 1 \rangle$ is a bounded distributive lattice,
2. x^* is the pseudocomplement of x , i.e.

$$y \leq x^* \iff y \cdot x = 0,$$

3. x^+ is the dual pseudocomplement of x , i.e.

$$y \geq x^+ \iff y + x = 1,$$

4. $x^* + x^{**} = 1, x^+ \cdot x^{++} = 0.$

Conditions 2 and 3 are equivalent to the equations

$$\begin{array}{ll} x \cdot (x \cdot y)^* = x \cdot y^*, & x + (x + y)^+ = x + y^+ \\ x \cdot 0^* = x, & x + 1^+ = x \\ 0^{**} = 0, & 1^{++} = 1 \end{array}$$

so that the double Stone algebras form an equational class [29]. L is called *regular*, if it additionally satisfies the equation

$$x \cdot x^+ \leq x + x^*.$$

This is equivalent to

$$x^+ = y^+ \text{ and } x^* = y^* \text{ imply } x = y.$$

Let B be a Boolean algebra, F be a filter on B , and

$$\langle B, F \rangle := \{ \langle a, b \rangle : a, b \in B, a \leq b, a + (-b) \in F \}.$$

If $B = F$, then we shall usually write $B^{[2]}$ for $\langle B, F \rangle$. We now define the following operations on $\langle B, F \rangle$:

$$\begin{array}{l} \langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle, \\ \langle a, b \rangle \cdot \langle c, d \rangle = \langle a \cdot c, b \cdot d \rangle, \\ \langle a, b \rangle^* = \langle -b, -b \rangle, \\ \langle a, b \rangle^+ = \langle -a, -a \rangle. \end{array}$$

The operations on the right hand side are the operations on B . We shall call algebras of this form *Katri ak algebras*. This is motivated by the following representation theorem:

Theorem 2.1. [16] *Each Katri ak algebra is a regular double Stone algebra. Conversely, each regular double Stone algebra is isomorphic to a Katri ak algebra.*

If $\mathbb{S} = \langle B, F \rangle$ is a Katriňák algebra, we can identify B with $\{\langle a, a \rangle : a \in B\}$, and F with $\{\langle a, 1 \rangle : \langle a, 1 \rangle \in \mathbb{S}\}$.

We denote the category of Katriňák algebras by \mathbf{K} , and the variety generated by \mathbf{K} by $\mathbf{V}_{\mathbf{K}}$. Theorem 2.1 shows that $\mathbf{V}_{\mathbf{K}}$ is the variety of regular double Stone algebras.

The connection to the rough sets can now be described as follows:

If $\langle U, \theta \rangle$ is an approximation space, we can view the classes of θ as atoms of a complete subalgebra of the Boolean algebra $\mathfrak{P}(U)$. Conversely, any atomic complete subalgebra B of $\mathfrak{P}(U)$ gives rise to an equivalence relation θ on U via

$$x\theta y : \iff x \text{ and } y \text{ are contained in the same atom of } B,$$

and this correspondence is bijective. If $\{a\} \in B$, then, for every $X \subseteq U$ we have

$$\text{If } a \in \underline{X}, \text{ then } a \in X,$$

and the rough sets of the corresponding approximation space are the elements of the regular double Stone algebra $\langle B, F \rangle$, where F is the filter of B which is generated by the union of singleton elements of B

Homomorphisms of Katriňák algebras were described in [9]:

Theorem 2.2. *Let $\mathbb{L} = \langle B, F \rangle$ and $\mathbb{M} = \langle C, G \rangle$ be Katriňák algebras, and $f : \mathbb{L} \rightarrow \mathbb{M}$ a homomorphism. Then, $f \upharpoonright B$ is a Boolean homomorphism for which $f[F] \subseteq G$. Conversely, if $h : B \rightarrow C$ is a Boolean homomorphism with $h[F] \subseteq G$, then h can be uniquely extended to a \mathbf{K} -homomorphism $\mathbb{L} \rightarrow \mathbb{M}$*

An *epimorphism* is a morphism f such that for all morphisms g, h

$$g \circ f = h \circ f \text{ implies } g = h,$$

see [5]. The following will be used below to describe a property of our logic:

Corollary 2.3. *\mathbf{K} contains non-surjective epimorphisms.*

Proof. Let $\mathbb{S} = \langle B, F \rangle$, $B \neq F$, $\mathbb{T} = \langle B, B \rangle \in \mathbf{K}$, and $f : B \rightarrow B$ be the identity. Since $f[F] = F \subseteq B$, f can be uniquely extended to a \mathbf{K} -morphism $\bar{f} : \mathbb{S} \rightarrow \mathbb{T}$, and it follows from $F \neq B$ that \bar{f} is not surjective. Let $\bar{g}, \bar{h} : \mathbb{L} \rightarrow \mathbb{M} = \langle C, G \rangle$ be \mathbf{K} -morphisms with $\bar{g} \circ \bar{f} = \bar{h} \circ \bar{f}$, and g, h be the respective restrictions to B . By Theorem 2.2 it suffices to show that $g = h$:

$$\begin{aligned} \bar{g} \circ \bar{f} = \bar{h} \circ \bar{f} &\Rightarrow g \circ f = h \circ f, \\ &\Rightarrow g(f(x)) = h(f(x)) \text{ for all } x \in B, \\ &\Rightarrow g(x) = h(x), \end{aligned}$$

which proves our claim. □

3 Rough set logic

The language of rough set logic \mathcal{L} consists of a nonempty set P of propositional variables, two binary connectives \wedge, \vee , two unary connectives $*, +$ which represent two forms of negation, as well as the

constant \top which represents truth. Formulas are built from the propositional variables in the usual recursive way, so that the set Fml of \mathcal{L} – formulas with these operations becomes an absolutely free algebra of type $\langle 2, 2, 1, 1, 0 \rangle$, generated by the elements of P .

A model of \mathcal{L} is a pair $\langle W, v \rangle$, where W is a set, and $v : P \rightarrow \mathfrak{P}(W) \times \mathfrak{P}(W)$ a mapping – called the *valuation function* – for which for all $p \in P$,

$$\text{If } v(p) = \langle A, B \rangle, \text{ then } A \subseteq B.$$

The equation $v(p) = \langle A, B \rangle$ expresses that

p holds at all states of A , and does not hold at any state outside B .

The following characterisation of *valuation* demonstrates the relationship to three-valued Łukasiewicz logic:

For each $p \in P$ let $v_p : W \rightarrow \mathbf{3} = \{0, \frac{1}{2}, 1\}$ be a mapping. Then $v : P \rightarrow \mathfrak{P}(W)^{[2]}$ defined by

$$v(p) = \langle \{w \in W : v_p(w) = 1\}, \{w \in W : v_p(w) \neq 0\} \rangle$$

is a valuation.

Conversely, let v be a valuation and for each $p \in P$, let $v_p : W \rightarrow \mathbf{3}$ be defined as follows: If $v(p) = \langle A, B \rangle$, then

$$v_p(w) = \begin{cases} 1, & \text{if } w \in A, \\ \frac{1}{2}, & \text{if } w \in B \setminus A, \\ 0, & \text{otherwise.} \end{cases}$$

If $t : P \rightarrow \mathfrak{P}(W)^{[2]}$ is defined by

$$t(p) = \langle \{w \in W : v_p(w) = 1\}, \{w \in W : v_p(w) \neq 0\} \rangle,$$

then it is easily seen that $v = t$.

Given a model $\mathfrak{M} = \langle W, v \rangle$, we define its meaning function $\text{mng} : \text{Fml} \rightarrow \mathfrak{P}(W) \times \mathfrak{P}(W)$ as an extension of the valuation v as follows:

$$(1) \quad \text{mng}(\top) = \langle W, W \rangle.$$

For each $p \in P$,

$$(2) \quad \text{mng}(p) = v(p).$$

If $\text{mng}(\varphi) = \langle A, B \rangle$ and $\text{mng}(\psi) = \langle C, D \rangle$, then

$$(3) \quad \text{mng}(\varphi \wedge \psi) = \langle A \cap C, B \cap D \rangle,$$

$$(4) \quad \text{mng}(\varphi \vee \psi) = \langle A \cup C, B \cup D \rangle,$$

$$(5) \quad \text{mng}(\varphi^*) = \langle -B, -B \rangle,$$

$$(6) \quad \text{mng}(\varphi^+) = \langle -A, -A \rangle.$$

Here, $-A$ is the complement of A in $\mathfrak{P}(W)$.

Let $\text{ran}(\text{mng}) := \{\text{mng}(\varphi) : \varphi \in \text{Fml}\}$. We define operations on $\text{ran}(\text{mng})$ in the obvious way:

$$\begin{aligned}\text{mng}(\varphi) \cdot \text{mng}(\psi) &= \text{mng}(\varphi \wedge \psi), \\ \text{mng}(\varphi) + \text{mng}(\psi) &= \text{mng}(\varphi \vee \psi), \\ \text{mng}(\varphi)^* &= \text{mng}(\varphi^*), \\ \text{mng}(\varphi)^+ &= \text{mng}(\varphi^+).\end{aligned}$$

Strictly speaking we should write $\text{mng}_{\mathfrak{M}}$ instead of mng , since it is dependent on \mathfrak{M} . However, we shall omit the subscript if the choice of \mathfrak{M} is clear.

Theorem 3.1. *With these operations, $\text{ran}(\text{mng})$ is a Katriňák algebra, and mng is a homomorphism.*

Proof. Set $\mathbb{S} = \text{ran}(\text{mng})$. It is enough to find a Boolean algebra $B \leq \mathfrak{P}(W)$ and a filter F of B , such that $\mathbb{S} = \langle B, F \rangle$. Let $B := \{A : \langle A, A \rangle \in \mathbb{S}\}$. Then, $W \in B$ by $\text{mng}(\top) = \langle W, W \rangle$, and $\emptyset \in B$ by $\text{mng}(\top)^* = \langle \emptyset, \emptyset \rangle$. Clearly, \mathbb{S} is a 0–1 sublattice of $\mathfrak{P}(W) \times \mathfrak{P}(W)$, and thus, B is closed under \cap and \cup . If $\text{mng}(\varphi) = \langle A, A \rangle \in B$, then $\text{mng}(\varphi^+) = \text{mng}(\varphi)^+ = \langle -A, -A \rangle$, and thus, B is closed under complementation.

Now, let $F := \{C : \langle C, W \rangle \in \mathbb{S}\}$. If $\langle C, W \rangle, \langle D, W \rangle \in \mathbb{S}$, then $\langle C \cap D, W \rangle \in \mathbb{S}$, and thus, F is closed under \cap . Finally, let $C \in F$, $D \in B$, and $C \subseteq D$. Then, there are $\varphi, \psi \in \text{Fml}$ such that $\text{mng}(\varphi) = \langle C, W \rangle$ and $\text{mng}(\psi) = \langle D, D \rangle$. Now, $\text{mng}(\varphi \vee \psi) = \langle C \cup D, W \cup D \rangle = \langle D, W \rangle$ shows that $D \in F$.

The fact that mng is a homomorphism follows immediately from the definitions. □

Observe that the homomorphism condition says that \mathcal{L} is truth–functional, i.e. it satisfies Frege’s compositionality principle.

The class of all models of \mathcal{L} is denoted by Mod . A formula φ holds in a model $\mathfrak{M} = \langle W, v \rangle$, written as $\mathfrak{M} \models \varphi$, if $\text{mng}(\varphi) = \langle W, W \rangle$. A set Σ of sentences entails a formula φ , if every model of Σ is a model of φ .

We define additional operations on Fml by

$$\begin{aligned}\varphi \rightarrow \psi &:= \varphi^* \vee \psi \vee (\varphi^+ \wedge \psi^{**}), \\ \varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).\end{aligned}$$

Theorem 3.2. *If $\mathfrak{M} = \langle W, v \rangle \in \text{Mod}$ and $\varphi, \psi \in \text{Fml}$, then*

1. $\mathfrak{M} \models \varphi \leftrightarrow \psi$ if and only if $\text{mng}(\varphi) = \text{mng}(\psi)$.
2. $\mathfrak{M} \models \top \leftrightarrow \varphi$ if and only if $\mathfrak{M} \models \varphi$.
3. If h is a homomorphism of Fml and $\mathfrak{M} = \langle W, v \rangle \in \text{Mod}$, then there exists some $\mathfrak{N} \in \text{Mod}$ such that $\text{mng}_{\mathfrak{M}} = \text{mng}_{\mathfrak{N}} \circ h$.

Proof. (i). Let $\text{mng}(\varphi) = \langle A, B \rangle$ and $\text{mng}(\psi) = \langle C, D \rangle$. Then,

$$\begin{aligned}\text{mng}(\varphi \rightarrow \psi) &= \text{mng}(\varphi)^* + \text{mng}(\psi) + \text{mng}(\varphi)^+ \cdot \text{mng}(\psi)^{**}, \\ &= \langle -B, -B \rangle + \langle C, D \rangle + \langle -A, -A \rangle \cdot \langle D, D \rangle, \\ &= \langle -B \cup C \cup (D \cap -A), -B \cup D \rangle\end{aligned}$$

“ \Rightarrow ”: If $\text{mng}(\varphi \leftrightarrow \psi) = \langle W, W \rangle$, then

$$-B \cup C \cup (D \cap -A) = W$$

and, using $A \subseteq B$,

$$C \cap A = W \cap A = A.$$

Furthermore, $-B \cup D = W$ implies $B \subseteq D$, so that $\text{mng}(\varphi) \leq \text{mng}(\psi)$. The proof of the other direction is analogous.

“ \Leftarrow ”: Suppose that $A = C$ and $B = D$; then

$$\begin{aligned} \text{mng}(\varphi \rightarrow \psi) &= \langle -B \cup C \cup (D \cap -A), -B \cup D \rangle, \\ &= \langle -B \cup A \cup (B \cap -A), -B \cup B \rangle, \\ &= \langle W, W \rangle, \end{aligned}$$

again using $A \subseteq B$. The proof of the other direction is analogous.

(ii) This follows immediately from (i) and the definition of mng . The condition says that we have one designated truth value, namely, \top .

(iii) Define $m : P \rightarrow \mathfrak{P}(W)^{[2]}$ by $m = \text{mng}_{\mathfrak{M}} \circ h$. Since h and mng are homomorphisms, $\langle W, m \rangle$ is the desired model. \square

Define the *algebraic counterpart* of \mathcal{L} as

$$(7) \quad \text{Alg}_m(\mathcal{L}) := \{\text{ran}(\text{mng}_{\mathfrak{M}}) : \mathfrak{M} \in \text{Mod}\}.$$

Theorem 3.3. *The variety generated by $\text{Alg}_m(\mathcal{L})$ is the variety of regular double Stone algebras.*

Proof. Let \mathbf{V} be the variety generated by $\text{Alg}_m(\mathcal{L})$. Each algebra of \mathbf{V} is a regular double Stone algebra by Theorem 3.1.

Conversely, let $W = \mathbf{2}$, and $\mathfrak{S} = \{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 1 \rangle\}$. Define $v : P \rightarrow \mathfrak{S}$ by $v(p) = \langle 0, 1 \rangle$ for all $p \in P$. Then, $\text{ran}(\text{mng}) = \mathfrak{S} \cong \mathbf{3}$. It is well known that $\mathbf{3}$ generates the variety of regular double Stone algebras, which proves our claim. \square

It follows that \mathcal{L} is a strongly nice general logic in the sense of Andr eka et al [2], 3.1. and 3.2. Strictly speaking, we should differentiate between rough set logics with sets of propositional variables of different size κ . However, Theorem 3.3 shows that this is not necessary since the varieties are the same regardless of κ .

Finally, using the translations

$$\text{Logic} \rightarrow \text{Algebra} \rightarrow \text{Logic}$$

developed in [2], we mention some properties of \mathcal{L} :

Theorem 3.4. 1. \mathcal{L} has a finitely complete and strongly sound Hilbert style axiom system.

2. \mathcal{L} has a compactness theorem.

3. \mathcal{L} does not have the Beth definability property.

Proof. (i) and (ii) follow from the fact that \mathbf{K} generates a finitely axiomatizable variety and 3.11 and 3.20 of [2].

(iii) follows from Corollary 2.3 and 2.24 of [3]. \square

4 Outlook

The approach taken in the previous section can be generalized in various ways. If we keep the definition of rough set as above, but add additional structure to the object set U in such a way that our objects are binary relations on some set, we arrive at algebras of rough relations, introduced by Comer [7] and further studied by Düntsch [10]. It would be interesting to know whether the Katriňák representation of algebras of rough relations can serve as an adequate algebraic counterpart to a rough generalization of arrow logics and their relatives as described in [2]. A practical application of these concepts can be found in preference modeling, investigations into which we are currently undertaking in [12].

In another direction we add “granularity to uncertainty” in the following sense:

Let δ be a complete linear order type, and $\mathfrak{A} = \langle W, A, +, \cdot, m_i, 0, 1 \rangle_{i \in \delta}$ be an algebra with the following properties:

1. Elements of A have the form $\langle X_i \rangle_{i \in \delta}$, where $X_i \subseteq W$, and $i < j < \delta$ implies $X_i \subseteq X_j$.
2. $\langle A, +, \cdot, 0, 1 \rangle$ is a 0–1 sublattice of ${}^\delta\mathfrak{P}(W)$.
3. For each $\alpha \in \delta$ and each $\langle X_i \rangle_{i \in \delta}$, we have $m_\alpha(\langle X_i \rangle_{i \in \delta}) = \langle Y_i \rangle_{i \in \delta}$, where $Y_i = -X_\alpha$ for all $i \in \delta$.

In analogy to rough sets we call the elements of \mathfrak{A} δ -rough sets. These are interpreted in such a way that δ measures the relative degree of uncertainty. It may be interesting to explore these algebras and their relationship to δ -valued Łukasiewicz algebras and fuzzy sets.

Acknowledgements

I should like to thank the referees for their constructive remarks and the pointers to earlier literature, Hajnal Andréka for providing the information regarding the history of the algebraic semantics approach, and Piero Pagliani for pointing out the details of the connection between $\langle U, \theta \rangle$ and $\langle B, F \rangle$.

References

- [1] Hajnal Andréka, *Universal algebraic logic*, PhD dissertation, Hungarian Academy of Sciences, Budapest, 1975.
- [2] Hajnal Andréka and István Németi, *General algebraic logic: A perspective on ‘What is logic’*, What is a Logical System? (Dov M. Gabbay, ed.), Studies in Logic and Computation, vol. 4, Clarendon Press, Oxford, 1994, pp. 393–443.
- [3] Hajnal Andréka, István Németi, and Ildikó Sain, *Abstract model theoretic approach to algebraic logic*, CCSOM Working Paper 92–92, Department of Statistics and Methodology, University of Amsterdam, 1992.
- [4] Hajnal Andréka and Ildikó Sain, *Connections between algebraic logic and initial algebra semantics of CF languages*, In Dömölki and Gergely [8], pp. 25–83.
- [5] Raymond Balbes and Philipp Dwinger, *Distributive lattices*, University of Missouri Press, Columbia, 1974.

- [6] Steve Comer, *An algebraic approach to the approximation of information*, Fund. Inform. **14** (1991), 492–502.
- [7] ———, *On connections between information systems, rough sets, and algebraic logic*, Algebraic Methods in Logic and Computer Science, Banach Center Publications 28, 1993.
- [8] B. Dömölki and T. Gergely (eds.), *Mathematical logic in computer science (Proc. Coll. Salgótarján, 1978)*, Colloq. Math. Soc. J. Bolyai, vol. 26, Amsterdam, 1981.
- [9] Ivo Düntsch, *On free or projective Stone algebras*, Houston J. Math. **9** (1983), 455–463.
- [10] ———, *Rough relation algebras*, Fund. Inform. **21** (1994), 321–331.
- [11] ———, *Rough sets and algebras of relations*, In Orłowska [21].
- [12] Ivo Düntsch and Günther Gediga, *Entailment of relations*, Work in progress, 1996.
- [13] Richard L. Epstein, *The semantic foundations of logic – propositional logics*, Nijhoff International Philosophy Series, vol. 35, Kluwer, Dordrecht, 1990.
- [14] Leon Henkin, J. Donald Monk, and Alfred Tarski, *Cylindric algebras, Part II*, North Holland, Amsterdam, 1985.
- [15] Jeanette Huebener, *Complementation in the lattice of regular topologies*, Pacific J. Math. **41** (1972), 139–149.
- [16] Tibor Katriňák, *Construction of regular double p -algebras*, Bull. Soc. Roy. Sci. Liège **43** (1974), 294–301.
- [17] T. Y. Lin, *Topological and fuzzy rough sets*, In *Intelligent decision support: Handbook of applications and advances of rough set theory* [26], pp. 287–304.
- [18] István Németi, *Connections between cylindric algebras and initial algebra semantics of CF languages*, In Dömölki and Gergely [8], pp. 561–605.
- [19] Ewa Orłowska, *Modal logics in the theory of information systems*, Z. Math. Logik Grundlag. Math. **30** (1984), 213–222.
- [20] ———, *Logic for reasoning about knowledge*, Z. Math. Logik Grundlag. Math. **35** (1989), 559–572.
- [21] Ewa Orłowska (ed.), *Modeling incomplete information – fundamentals and application*, to appear, Springer, 1997.
- [22] Piero Pagliani, *Rough sets theory and logic-algebraic structures*, In Orłowska [21].
- [23] Zdzisław Pawlak, *Rough sets: Theoretical aspects of reasoning about data*, Kluwer, Dordrecht, 1991.
- [24] J. Pomykala and J. A. Pomykala, *The Stone algebra of rough sets*, Bull. Polish Acad. Sci. Math. **36** (1988), 495–508.
- [25] Ildikó Sain, *There are general rules for specifying semantics: Observations on abstract model theory*, Computational Linguistics and Computer Languages **13** (1979), 195–250.

- [26] Roman Słowiński, *Intelligent decision support: Handbook of applications and advances of rough set theory*, Kluwer, Dordrecht, 1992.
- [27] D. Vakarelov, *Notes on N-lattices and constructive logic with strong negation*, *Studia Logica* **36** (1977), 109–125.
- [28] J. C. Varlet, *Algèbres des Łukasiewicz trivalentes*, *Bull. Soc. Roy. Sci. Liège* **36** (1968), 399–408.
- [29] ———, *A regular variety of type $\langle 2, 2, 1, 1, 0, 0 \rangle$* , *Algebra Universalis* **2** (1972), 218–223.