# A tutorial on relation algebras and their application in spatial reasoning 

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## 1 Introduction

It is fair to say that the development of algebraic logic started in the middle of last century with the work of George Boole [16]. The formalisation of what we know as classical propositional logic (which goes back to at least Aristotle) became immensely successful. At about the same time, the shortcomings of propositional (Aristotelian ) logic, which were aptly summarised in de Morgan's aphorism
"All the logic of Aristotle does not permit us, from the fact that a horse is an animal, to conclude that the head of a horse is the head of an animal". [cited in 83]
caused the investigation into the theory of relations as a foundation for mathematical logic. After the initial efforts of de Morgan [25], it was in particular the work of Peirce [73] and Schröder [79] who pioneered the study of the (equational) calculus of relations. It can be said that
"algebraic logic was mathematical logic, or was, at any rate, the late-nineteenth century state-of-the-art version of mathematical logic". [11]

With the advent of Frege-Peano-Russell style of "quantificational logic" and the appearance of the Principia in which the equational theory was subsumed under the quantification formalism, algebraic logic lay more or less dormant for over forty years. It was K. Twardowski, a Polish philosopher and a student of Franz Brentano in Vienna, who became interested in Schröder's work. He, together with Leśniewski, Łukasiewicz, and Leśniewski’s sole doctoral student, Tarski, formed the core of the Lwów - Warsaw school of Logic and Philosophy, which
" ... in the $20 \mathrm{~s}-30$ s of this century made the University of Warsaw perhaps the most important research centre in the world for formal logic". [13]

In the seminal paper "On the calculus of relations" [83], Tarski picked up where Schröder had left off forty five years earlier. He gave two axiomatisations of a theory of binary relations, one in the style of Hilbert and Ackermann, and one as an equational formalism. At the end of this paper, Tarski raises some questions in the solution of which he would be engaged for the rest of his life:

1. Is every model of the axiom system of the calculus of relations isomorphic to an algebra of binary relations?
2. What is the expressive power of the calculus of relations? To what extent can this calculus provide a framework for the development of first order logic or, indeed, Mathematics?
3. Is there a decision procedure for expressible first order sentences?

Tarski had proved in the late 1940s that set theory and number theory could be formulated in the calculus of relation algebras:
"It has even been shown that every statement from a given set of axioms can be reduced to the problem of whether an equation is identically satisfied in every relation algebra. One could thus say that, in principle, the whole of mathematical research can be carried out by studying identities in the arithmetic of relation algebras". [19]

A full account of this appeared for the first time in 1987 after Tarski's death [86]. The theory of cylindric algebras led to an algebraisation of first order logic [43, 44], just as Boolean algebras were an algebraisation of the propositional calculus.

Further references for relation algebras are [19, 49, 50]. For a brief overview of the history of algebraic logic with a large bibliography, we invite the reader to consult [11] and also [62]; a survey of the current state of algebraisation of quantifier logics is [67].

In our days, the calculus of relations has found many applications in Logic and Computer Science, references to many of which can be found in Németi's survey [67], and also in the publications of the International Seminar on Relational Methods in Computer Science (RELMICS) [18, 71]. As a general introduction to algebraic logic I recommend Andréka et al. [10] ${ }^{1}$.

Why would relation algebras be interesting to researchers of spatial reasoning? A large part of (no pun intended) contemporary spatial reasoning is based on the investigations of the behaviour of "part of" relations and their extensions to "contact relations" in various domains [see e.g. 21, 36, 87], going back, among others, to $[24,58,68]$. Also, the consistency of topological relations can be checked by the techniques of relation algebras. The relational calculus tells us which relations must exist, given several basic operations, such as Boolean operations on relations, relational composition and converse. Each equation in the calculus corresponds to a theorem, and, for a situation where there are only finitely many relations, one can construct a composition table (defined below) which can serve as a look up table for the relations involved. Furthermore, since the calculus handles relations, no knowledge about the concrete geometrical objects is necessary. Relation algebras were introduced into spatial reasoning in [38] with additional results published in [35, 39], and I would like to refer the reader to these papers for additional motivation.

We will see below, that relational reasoning in general corresponds to a fragment of first order logic. On some domains, however, the relations definable by equations are those which are definable by full first order logic. In these cases, the calculus is sufficient to express all first order properties of the relations in question.

The tutorial is structured as follows: In Section 2 I will present basic facts on binary relations and their algebras. This will be followed by an introduction to abstract relation algebras. I will keep this

[^0]Section brief, since we will be more concerned with concrete relations. Expressiveness and powers of definability of the calculus of binary relations will be explored in Section 4. As a gentle introduction to relation algebras occurring in reasoning about time and space, we will recall Allen's interval algebra and the algebra of closed circles in the Euclidean plane. Contact relations and some small relation algebras generated by them are introduced in Section 6. The smallest relation algebras on an atomless Boolean algebra generated by a contact relation whose associated order is the Boolean order will be presented in Section 7. In the next Section, I will introduce the Region Connection Calculus of [77] (RCC), and will explore which relations must be present in any model of the RCC, in particular, in any standard topological model whose base consists of regular open sets. I will also interpret some of these relations topologically in the Euclidean plane. Section 9 presents a sound and complete proof system for relation algebras generated by a contact relation, and, finally, I will propose a frame for reasoning about regions with imperfect information, which is based on the data model of rough sets.

I should like to finish this introduction with the closing sentences of Tarski's 1941 paper, which express a feeling for Mathematics which often is lost in our days where commercial exploitability is everything and recognition (and funding) is given by the criterion of immediate applicability, but which, at least for me, is still a major motivation for engaging in the pursuit of mathematical knowledge:
> "Aside from the fact that the concepts occurring in this calculus possess an objective importance and are in these times almost indispensable in any scientific discussion, the calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it." [83]

## 2 Binary relations and their algebras

A binary relation $R$ on a set $U$ is a subset of $U \times U$, i.e. a set of ordered pairs $\langle x, y\rangle$ where $x, y \in U$. I shall usually just speak of $R$ as a relation, and instead of $\langle x, y\rangle \in R$, we shall usually write $x R y$. The collection of all binary relations on $U$ is denoted by $\operatorname{Rel}(U)$. The smallest relation on $U$ is the empty relation, and the largest one the universal relation $U \times U$, which we will denote by $V$.

A pictorial representation of the fact that $x R y$ can be given by drawing an arrow from $x$ to $y$, which is labelled $R$ :


Let $R$ be a relation on $U$.

1. $R$ is reflexive if $x R x$ for all $x \in U$.
2. $R$ is irreflexive if $x R x$ for no $x \in U$.
3. $R$ is antisymmetric if for all $x, y \in U, x R y$ and $y R x$ implies $x=y$.
4. $R$ is asymmetric if for all $x, y \in U, x R y$ implies $\neg y R x R$.
5. $R$ is transitive if for all $x, y, z \in U, x R y$ and $y R z$ implies $x R z$.
6. $R$ is functional, if for all $x, y, z \in U, x R y$ and $x R z$ imply $y=z$.

A function $f: \operatorname{Rel}(U)^{n} \rightarrow \operatorname{Rel}(U)$ is called an $n$-ary relational operator. Since relations on $U$ are sets, we have the binary operators $\cup, \cap$. We also have the unary operator of set theoretic complementation $V \backslash R$, which we just denote by - if $V$ is understood. Our first observation now is

Proposition 2.1. $\langle\operatorname{Rel}(U), \cup, \cap,-, \emptyset, V\rangle$ is a Boolean algebra.

We are going to introduce two more standard operators on relations: The composition or relative multiplication $R \circ S$ of two relations is defined as

$$
\begin{equation*}
R \circ S=\{\langle x, y\rangle:(\exists z)[x R z \text { and } z S y]\} \tag{2.1}
\end{equation*}
$$



The fact that $x(R \circ S) y$ implies the existence of some $z$ with $x R z$ and $z S y$. This is sometimes called existential import [12]. Let us denote the identity relation $\langle x, x\rangle: x \in U$ by $1^{\prime}$, and its complement by $0^{\prime}$. Then,

Proposition 2.2. $\left\langle U, \circ, 1^{\prime}\right\rangle$ is a monoid, i.e.

1. $\circ$ is associative.
2. $1^{\prime} \circ R=R \circ 1^{\prime}=R$ for all $R \in \operatorname{Rel}(U)$.

Another distinguished unary operator is relational converse or just converse:

$$
\begin{equation*}
R^{\hookrightarrow}=\{\langle y, x\rangle: x R y\} \tag{2.2}
\end{equation*}
$$

The interplay between $\circ$ and ${ }^{`}$ is given by
Proposition 2.3. ${ }^{\checkmark}$ is an involution on the semigroup $\langle\operatorname{Rel}(U), \circ\rangle$, i.e.

1. ${ }^{\smile}$ is bijective and of order two, i.e. $x^{\hookleftarrow}=x$.
2. $(R \circ S)^{\cup}=S^{\sim} \circ R^{\llcorner }$for all $R, S \in \operatorname{Rel}(U)$.

We observe that $1^{\prime}, \circ,{ }^{`}$ are first order definable. There are, of course, widely used relational operators which are not first order definable, an example in point being the transitive closure of a relation.

An algebra of binary relations (BRA) is a structure $\left\langle A, \cap, \cup,-, \emptyset, E, \circ,{ }^{\prime}, 1_{E}^{\prime}\right\rangle$, where $E$ is an equivalence relation on some set $U, 1_{E}^{\prime}=E \cap 1^{\prime}$, and $A \subseteq 2^{E}$ is closed under the operations listed and contains the constants. If $E=U \times U$ and $A=2^{U \times U}$, then the algebra is called the full BRA on $U$, and I denote it simply by $\operatorname{Rel}(U)$. The subalgebras of some $\operatorname{Rel}(U)$ are called BRAs on $U$. In the sequel I will mean by a BRA always a BRA on some U (i.e. with $V=U^{2}$ ), unless stated otherwise.

I shall usually identify algebras with their base set, and, with some abuse of notation, I will also denote classes of algebras by the abbreviation of their type, e.g. BRA is also the class of all algebras of binary relations. If $A$ is an algebra and $B$ is a subalgebra of $A$, I denote this by writing $B \leq A$.

For $\mathcal{R} \subseteq \operatorname{Rel}(U)$, let $[\mathcal{R}]$ be the subalgebra of $\operatorname{Rel}(U)$ generated by $\mathcal{R}$. In other words, $[\mathcal{R}]$ is the smallest subset of $U$ which is closed under the Boolean and relational operators, and contains the constants.

If $A$ is a finite BRA, then, as a Boolean algebra, it is atomic; hence the actions of the Boolean operators are uniquely determined. To determine the structure of $A$ it is therefore enough to specify the relative multiplication and the converse operation. This is usually done in a composition table, where rows and columns are labelled by the atoms, and a cell contains all atoms below the (result of) the relative multiplication. If $1^{\prime}$ is an atom, it is usually omitted from the table. We observe that the converse of an atom is again an atom, and that each atom either is contained in $1^{\prime}$ or disjoint from it; thus, in a relational representation of a BRA, we can obtain the converse of $R$ by looking for the unique element $Q$ of the table for which $(R \circ Q) \cap 1^{\prime} \neq \emptyset$.

I need to mention another form of relational composition which has appeared in the literature [77], and which I will call weak composition to distinguish it from the usual relational composition: Suppose that $\mathcal{R}$ is a set of relations on $U$, and $R, S \in \mathcal{R}$. Then,

$$
\begin{equation*}
R \circ_{w} S=\bigcup\{T \in \mathcal{R}:(R \circ S) \cap T \neq \emptyset\} . \tag{2.3}
\end{equation*}
$$

Necessary and sufficient conditions for a composition table to be the composition table of a relation algebra are given in Proposition 3.1.

As our first example of a composition table, let $S_{1}$ be the disjoint union of a $K_{3}$ and a $K_{4}$ on a seven element set $U$ (see Fig. 1). $S_{1}$ generates a relation algebra $A$ on $U$ with atoms $S_{1}, T_{1}, 1^{\prime}$ and the composition shown in Table 1. I use lower case letters in the table to emphasise that the table can be used for various situations.

If $S_{2}$ is the relation shown in Figure 2, then the table of the BRA generated by $S_{2}$ will also be given by 1 . This shows that different BRAs can have the same algebraic structure, and that, in general, the algebraic structure of a BRA is too weak to determine the size of the base set or what the relations look like. Nevertheless, the expressive power of BRA can be surprisingly strong. We shall return to this theme in Section 4.

Figure 1: The relation $S_{1}$


Table 1: The BRA $\mathcal{S}$

| 0 | $s$ | $t$ |
| :---: | :---: | :---: |
| $s$ | $-t$ | $t$ |
| $t$ | $t$ | $-t$ |

Figure 2: The relation $S_{2}$


At the other end of the spectrum is the BRA generated by the relation shown in Figure 3. The

Figure 3: Pentagon and pentagram


Table 2: The pentagon algebra

| $\circ$ | $R$ | $S$ |
| :---: | :---: | :---: |
| $R$ | $1^{\prime}, S$ | $0^{\prime}$ |
| $S$ | $0^{\prime}$ | $1^{\prime}, R$ |

relation $R$ is the pentagon, $S$ is the pentagram, and they generate a BRA whose composition is given in Table 2. It can be shown that every BRA with such a table must be defined on a base set with five elements, and consist of the relations given in Figure 3 [7].

A finite BRA need not act on a finite set: Let $U$ be the set $\mathbb{Q}$ of rational numbers, and let $A \leq \operatorname{Rel}(U)$ be generated by the natural strict ordering $P P$ on $\mathbb{Q}$. The resulting relation algebra has the three atoms $P P, P P^{\checkmark}, 1^{\prime}$ and composition as in Table 3. In fact, any representation of $A$ must be on an infinite

Table 3: The dense order algebra

| $\circ$ | $P P$ | $P P^{\checkmark}$ |
| :---: | :---: | :---: |
| $P P$ | $P P$ | $V$ |
| $P P^{\checkmark}$ | $V$ | $P P^{\checkmark}$ |

set: Since $P P \cap P P^{\checkmark}=\emptyset$, we see that $P P$ is asymmetric; in particular, $P P \cap 1^{\prime}=\emptyset$. The fact that $P P \circ P P=P P$ tells us two things:

1. $P P$ is transitive, since $P P \circ P P \subseteq P P$.
2. $P P$ is dense, since $P P \subseteq P P \circ P P$, i.e. between two different elements of $U$ there is a third one:

$$
(\forall x, y)[x P P y \Rightarrow(\exists z)(x P P z \text { and } z P P y)]
$$

This implies that $U$ is infinite.
As a further example consider the algebra shown in Table 4. Its domain is the set of all open disks in the plane, and we set

$$
\begin{align*}
x P P y & \Longleftrightarrow x \subsetneq y,  \tag{2.4}\\
x P P & \Longleftrightarrow x \supsetneq y,  \tag{2.5}\\
x P O y & \Longleftrightarrow x \cap y \neq \emptyset, x \nsubseteq y, x \nsupseteq y,  \tag{2.6}\\
x D C y & \Longleftrightarrow x \cap y=\emptyset . \tag{2.7}
\end{align*}
$$

This algebra is also known as the containment algebra [56].

Table 4: Open disk algebra $\mathcal{D}_{o}$

| $\circ$ | $P P$ | $P P^{\triangleleft}$ | $P O$ | $D C$ |
| :---: | :---: | :---: | :---: | :---: |
| $P P$ | $P P$ | $V$ | $-P^{\checkmark}$ | $D C$ |
| $P P^{\triangleleft}$ | $-D C$ | $P P^{\triangleleft}$ | $P P^{\triangleleft}, P O$ | $-P$ |
| $P O$ | $P P, P O$ | $-P$ | $V$ | $-P$ |
| $D C$ | $-P^{\checkmark}$ | $D C$ | $-P^{\triangleleft}$ | $V$ |

## 3 Abstract relation algebras

One of Tarski's aims was to give a formal axiomatisation of the calculus of relatives. This led to the definition of the class of relation algebras.

A relation algebra (RA)

$$
\left\langle A,+, \cdot,-, 0,1, \circ,,^{\smile}, 1^{\prime}\right\rangle
$$

is a structure of type $\langle 2,2,1,0,0,2,1,0\rangle$ which satisfies
(R0). $\langle A,+, \cdot,-, 0,1\rangle$ is a Boolean algebra.
(R1). $x \circ(y \circ z)=(x \circ y) \circ z$.
(R2). $(x+y) \circ z=(x \circ z)+(y \circ z)$.
(R3). $x \circ 1^{\prime}=x$.
(R4). $x^{\omega}=x$.
(R5). $(x+y)^{\breve{ }}=x^{\breve{ }}+y^{\breve{ }}$.
(R6). $(x \circ y)^{\llcorner }=y^{\breve{ }} \circ x^{\llcorner }$.
(R7). $\left(x^{\leftrightharpoons} \circ-(x \circ y)\right) \leq-y$.

This axiom system is the one given in [44]. With some abuse of language, I will denote the class of relation algebras also by RA.

A decisive property of RA is the following cycle law, which is de Morgan's Theorem K [25]:

The following conditions are equivalent:

In concrete relations, (3.1) expresses the fact that if one of the directed triangles in Figure 4 exists, then so do the others. It is not hard to see that each BRA is an RA, and an RA is called representable

Figure 4: Condition (3.1) for binary relations

(RRA) if it is isomorphic to a BRA.
The following Proposition makes precise when a composition table is indeed the composition table of a relation algebra:

Proposition 3.1. [47] Let $B$ be a complete and atomic Boolean algebra with $1^{\prime} \in B$ a distinguished element, $\smile$ a unary operation on $B$, and $\circ$ a binary operation, both of which are completely distributive over + , and for which $0^{\llcorner }=0$ and $(0 \circ x)=(x \circ 0)=0$. Let $A t(B)$ be the set of atoms of $B$. Then, $\left\langle B, \circ,{ }^{\lrcorner}, 1^{\prime}\right\rangle$ is an $R A$ if and only if the following conditions hold for all $x, y, z \in A t(B)$ :

$$
\begin{aligned}
& x^{\smile} \in A t(B) \\
& x \circ(y \circ z) \leq(x \circ y) \circ z \\
& x \circ 1^{\prime}=1^{\prime} \\
& x \leq y \circ z \text { implies } x^{\breve{ } \leq z^{\breve{ }} \circ y^{\breve{ }} \text { and } y \leq x \circ z .} \text {. }
\end{aligned}
$$

For things to come, I will introduce some more concepts at this stage. In analogy to rings, an RA $A$ is called integral, if for all $x, y \in A$,

$$
\begin{equation*}
x \circ y=0 \text { implies } x=0 \text { or } y=0 \text {. } \tag{3.2}
\end{equation*}
$$

$A$ is called simple, if every onto homomorphism with domain $A$ is an isomorphism. We now have the following properties [47, 52]:

Proposition 3.2. 1. A is integral if and only if $1^{\prime}$ is an atom.
2. If $A$ is an integral $R R A$, then it has a representation as a subalgebra of some Rel $(U)$.
3. $A$ is simple if and only if $x \circ 1 \circ x=1$ for all $x \neq 0$.
4. To every open formula $\varphi$ in the language of relation algebras there is a term $\varphi^{*}$ in the same language such that for every simple $R A A$,

$$
A \models \varphi \Longleftrightarrow \varphi^{*}=1
$$

Another concept we require is that of residuation. Since $\left\langle A, \circ, 1^{\prime}\right\rangle$ is in general not a group, the equation $a \circ x=b$ does not necessarily have a solution. However, it can be shown that the inequality

$$
a \circ x \leq b
$$

always has a greatest solution, called the (right) residual of b by $a$, written as $a \backslash b$. The concept of residuation is intimately related to Axiom (3.1) of RAs, cf. [62, 76] and also [15].

The residual can be expressed as an RA term in $a$ and $b$ by

$$
\begin{equation*}
a \backslash b=-(a \frown-b) \tag{3.3}
\end{equation*}
$$

If $R, S \in \operatorname{Rel}(U)$, then the residual is given by the condition

$$
\begin{equation*}
x(R \backslash S) y \Longleftrightarrow R^{\lrcorner} x \subseteq S^{\sim} y \tag{3.4}
\end{equation*}
$$

Some properties of the residual are given in
Proposition 3.3. Suppose that $A$ is an $R A$ and $a \in A$.

1. [75] $a \backslash a$ is reflexive and transitive.
2. [34] If $a$ is reflexive and symmetric, then $(a \backslash a)^{\smile} \circ(a \backslash a) \leq a$.

One of the first questions which arose was whether the system (R0) - (R7) captures RRA, i.e. whether each RA is representable. Unfortunately, this is not the case; the first non-representable RA was found by Lyndon [59]. It had 56 atoms and was constructed using projective planes; other examples were subsequently given by [46] and [60]. A non-representable RA $A$ of smallest size was found by McKenzie [64]. It is integral, has four atoms, and is 1-generated; its composition is listed in Table 5. It is not hard to show that $A$ is not representable: Assume that $a, b, c \in \operatorname{Rel}(U)$ for some $U$; since $A$ is integral, we can assume that $1=U \times U$ by Proposition 3.2. Now,

Table 5: A non-representable RA

| $\circ$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: |
| $b$ | $b$ | 1 | $b+d$ |
| $c$ | 1 | $c$ | $c+d$ |
| $d$ | $b+d$ | $c+d$ | $b+c+1^{\prime}$ |

1. $1^{\prime} \leq b \circ c$ implies that $c=b^{\breve{ }}$.
2. $1^{\prime} \cdot b=0$, and $b \circ b=b$ imply that $b$ is a strict dense partial order. I will sometimes write $\leq$ for $b$, and $\leq$ for $b+1^{\prime}$.
3. $b \circ c=c \circ b=1$ imply that for each pair $\langle x, y\rangle$, there are $p, q \in U$ such that $p \leq x, y$ and $x, y \leq q$.
4. $d \circ d=b+b^{\breve{ }}+1^{\prime}$ implies that $x, y$ are comparable if and only if they are incomparable to a third element.

These conditions cannot live together: Suppose that $x, y \in U$ are incomparable, and that $p \leq x, y \leq q$ as provided by 3 . above. By 4. there is an $s \in U$ such that $s$ is incomparable to $p$ and $x$. If $s$ were incomparable to $y$, then, by the other direction of 4 ., $x$ would be comparable to $y$, which is not the case. Hence, $s$ is comparable to $y$; furthermore, $s \leq y$, since otherwise, $p \leq y \leq s$. Similarly, there is some $t \in U$, such that $t \lesseqgtr x$, and $t$ is incomparable to $p$ and $y$. Since $p$ is incomparable to both $s$ and $t$, 4. implies that $s$ and $t$ are comparable. If $s \leq t$, then $s \leq x$, and if $t \leq s$, then $t \leq y$, a contradiction in both cases.

The following proposition summarises the structural properties of RRA:

## Proposition 3.4. 1. RRA is an equational class [84].

2. The equational theory of RRA is undecidable [see 86, Section 8.7. for references].
3. RRA is not finitely axiomatisable [65].
4. RRA is not axiomatisable with finitely many variables [49].

As already noted by Tarski, at times the property of associativity of the relational composition is too strong, and weaker properties are considered [61]. A structure similar to RAs is called a

1. non-associative $R A$ (NA), if it satisfies (R0) and (R2) - (R7).
2. weakly associative $R A(\mathrm{WA})$, if it satisfies (R0) and (R2) - (R7) and

$$
\begin{equation*}
\left(\left(1^{\prime} \cdot x\right) \circ 1\right) \circ 1=\left(1^{\prime} \cdot x\right) \circ(1 \circ 1) \tag{3.5}
\end{equation*}
$$

3. semi-associative $R A(\mathrm{SA})$, if it satisfies (R0) and (R2) - (R7) and

$$
\begin{equation*}
(x \circ 1) \circ 1=x \circ(1 \circ 1) . \tag{3.6}
\end{equation*}
$$

It was shown by Maddux [61] that

$$
R A \subseteq S A \subseteq W A \subseteq N A
$$

An alternative axiomatisation of NA consists of (R0), (R2), (R4), (R5), the identities

$$
1^{\prime} \circ x=x \circ 1^{\prime}=x,
$$

and the cycle law (3.1) [61].
The equational theory of WA is decidable [66]. Moreover, each WA is isomorphic to a subalgebra of an algebra $\left\langle 2^{W}, \cup, \cap,-W, \emptyset, W, \circ_{W},{ }^{\smile}, 1^{\prime}\right\rangle$, where $W$ is a reflexive and symmetric binary relation on a set U , and $x \circ_{W} y=(x \circ y) \cap W$.

For many decidability results for various classes of relation algebras, as well as pointers to earlier work, the reader will find [6] and [55] valuable sources.

## 4 The expressiveness of BRAs

The question arises what can be expressed by the logic of relation algebras. To answer this question needs some preparation. A first order language consists of predicate symbols, logical connectives $\wedge$, $\neg$, the existential quantifier $\exists$ and equality, and the usual abbreviations. When considering relational structures $\langle U, R\rangle$ as first order models, I tacitly assume that an appropriate first order language $L$ is given. For notational convenience, I shall sometimes identify predicate symbols with the predicates which interpret them, when no confusion is likely to arise.

If $\varphi(x, y)$ is a formula with the free variables $x, y$, and $\langle U, R\rangle$ is a model of the language $L$, then the truth set of $\varphi(x, y)$ in the model $\langle U, R\rangle$ is the relation

$$
\operatorname{def} \varphi(x, y)=\left\{\langle a, b\rangle \in U^{2}:\langle U, R\rangle \models \varphi(x / a, y / b)\right.
$$

If $S \subseteq U^{2}$ and $S=\operatorname{def} \varphi(x, y)$ for some $\varphi$, then $S$ is called definable in the model $\langle U, R\rangle$. Similarly, we extend this definition to languages with more than one predicate symbol and formulas with other than two free variables. For example, (the result of) relative composition is definable by the formula

$$
\varphi(x, y):(\exists z)[x R z \wedge z S y]
$$

and the fact that $x$ is $R$ connected to every element is expressed by

$$
\varphi(x):(\forall y) x R y
$$

A relation $S$ is $R A$ definable from $R_{0}, \ldots, R_{k}$, if it is in the BRA generated by the $R_{i}$. In other words, $S$ is RA definable from the $R_{i}$, if it is equal to a relational term constructed from the $R_{i}$ and the relational operators and constants.

As an example which we will need later, I shall show how extreme elements of an ordered set can be relationally defined. As a consequence of this, when considering relation algebras which contain an order relation, it is enough to look at the base set with the extreme elements removed.

Let $\langle U, P\rangle$ be an ordered set with largest element 1 and smallest element 0 ; furthermore, set $P P=$ $P \cap 0^{\prime}$. Let $U_{0}=U \backslash\{1\}, U_{1}=\{1\}$, and $U_{i j}=U_{i} \times U_{j}$ for $i, j \leq 1$. Now, we first observe that $\langle 1,1\rangle \notin P P \circ P P^{\hookrightarrow}$, since there is no element of $U$ which is strictly greater than 1 . On the other hand, $\langle x, x\rangle \in P P \circ P P^{\hookrightarrow}$ for all $x \neq 1$. Thus,

$$
U_{11}=1^{\prime} \cap-\left[\left(P P \circ P P^{\natural}\right) \cap 1^{\prime}\right],
$$

defines $\{\langle 1,1\rangle\}$. Now, set

$$
1_{u}^{\prime}=1^{\prime} \cap-U_{11} .
$$

Then,

$$
\begin{aligned}
& U_{00}=1_{u}^{\prime} \circ U^{2} \circ 1_{u}^{\prime}, \\
& U_{10}=U_{11} \circ U^{2} \circ U_{00}, \\
& U_{01}=U_{00} \circ U^{2} \circ U_{11},
\end{aligned}
$$

which shows that all $U_{i j}$ and $1_{u}^{\prime}$ are RA definable from $P$. The equation which tells us that 1 is the largest element with respect to $P$ now is

$$
\begin{equation*}
U_{01} \subseteq P . \tag{4.1}
\end{equation*}
$$

Similarly, we can RA define $\{0\}$.
In order to fathom the expressive power of the relational calculus, I will translate the relational properties listed on page 3 into relational equations:

1. $R$ is reflexive iff $1^{\prime} \cap R=1^{\prime}$.
2. $R$ is irreflexive iff $1^{\prime} \cap R=\emptyset$.
3. $R$ is antisymmetric iff $R \cap R^{\hookrightarrow} \subseteq 1^{\prime}$.
4. $R$ is asymmetric iff $R \cap R^{\llcorner }=\emptyset$.
5. $R$ is transitive iff $R \circ R \subseteq R$.
6. $R$ is functional, iff $R^{\llcorner } \circ R \subseteq 1^{\prime}$.

The expressiveness of BRAs corresponds to a fragment of first order logic, and the following fundamental result is due to A. Tarski [see 86]:

Proposition 4.1. If $\mathcal{R} \subseteq \operatorname{Rel}(U)$, then $[\mathcal{R}]$ is the set of all binary relations on $U$ which are definable in the (language of the) relational structure $\langle U, \mathcal{R}\rangle$ by first order formulas using at most three variables, two of which are free.

The question arises: Is this as good as it gets? Let us call a BRA A first order closed, if it contains every relation which is first order definable in $A$, regarded as a relational structure. It is worthy to point out that first order closedness is a property of BRAs, i.e. relational representations of (abstract) RAs.

The first result in this direction is
Proposition 4.2. [3] Every BRA on a set with at most six elements is first order closed.
Hence, on small sets, RA logic, i.e. the three variable fragment of first order logic, is as powerful as full first order logic. In the sequel, we will meet many other first order closed BRAs.

Look again at the RA of Table 1 on page 6, and its two representations. The $K_{3}$ in the right representation is first order definable by

$$
\varphi(x, y): x S z \wedge(\forall u)(\forall z)(x R u \wedge y R u \wedge x R z \wedge y R z \Rightarrow x=u \vee x=z \vee y=u \vee y=z \vee u=z) .
$$

As a relation, the $K_{3}$ is not in the BRA $A$ generated by $S_{1}$, and thus, $A$ is not first order closed. On the other hand, the representation of $A$ shown in Figure 2 on page 6 is first order closed by Proposition 4.2

Let us consider the quinary operation $Q$ on $\operatorname{Rel}(U)$ of [49], which is defined by

$$
x Q\left(R_{0}, \ldots, R_{4}\right) y \Longleftrightarrow(\exists z)(\exists u)\left(x R_{0} z R_{1} y \wedge x R_{2} u R_{3} y \wedge z R_{4} u\right) .
$$

The situation that $\langle x, y\rangle \in Q\left(R_{0}, \ldots, R_{4}\right)$ is pictured below:


Looking again at the BRAs associated with Table 1 on page 6, we see that

$$
\begin{aligned}
& Q\left(S_{1}, S_{1}, S_{1}, S_{1}, S_{1}\right)=\text { The } K_{4}, \\
& Q\left(S_{2}, S_{2}, S_{2}, S_{2}, S_{2}\right)=\emptyset .
\end{aligned}
$$

Hence, the network defined by $Q\left(S_{1}, S_{1}, S_{1}, S_{1}, S_{1}\right)$ is satisfiable, while the network defined by $Q\left(S_{2}, S_{2}, S_{2}, S_{2}, S_{2}\right)$ is not.

More generally, if $\underline{R}=\left\{R_{i, j}: i, j \lesseqgtr n\right\}$, let

$$
\begin{equation*}
x Q_{n}(\underline{R}) y \Longleftrightarrow\left(\exists z_{0}, \ldots, z_{n-1}\right)\left[x=z_{0} \wedge y=z_{n-1} \wedge \bigwedge_{i, j \lesseqgtr n}{z_{i}}_{i} R_{i j} z_{j}\right] \tag{4.2}
\end{equation*}
$$

Note that the formula on the right hand side of (4.2) is existential, and thus, it asserts the existence of a certain network on a complete digraph on $n$ nodes. I say that a BRA $A$ is $Q_{n}$ closed, if $Q_{n}(\underline{R}) \in A$ for every choice of $\underline{R} \subseteq A$; note that each BRA is $Q_{3}$ closed. $A$ is $Q$ closed, if it is $Q_{n}$ closed for every $n \geq 4$.

We now have
Proposition 4.3. 1. [49] If $U$ is finite and $A \leq \operatorname{Rel}(U)$, then

$$
A \text { is first order closed } \Longleftrightarrow A \text { is } Q \text { closed. }
$$

2. [9] If $U$ is infinite, then there is a $Q$ closed $A \leq \operatorname{Rel}(U)$ which is not first order closed.

It may be interesting to note that the formula $\varphi(x, y)$ in the proof of 2 . above, which exhibits that $A$ is not first order closed, contains only four variables:

$$
\varphi(x, y):(\exists z)(\forall w)[z P x \wedge z P y \wedge((w P x \wedge w P y) \rightarrow w P z)] .
$$

Thus, $\langle a, b\rangle \in \operatorname{def} \varphi(x, y)$, if $a$ and $b$ have a minimum with respect to $P$, which is a partial order in this example.

A property which is stronger than first order closedness has been introduced in [49] and further investigated in $[3,17,48]$. Let us first define invariant relations: Let $\Sigma_{U}$ be the symmetric group of $U$, $R \in \operatorname{Rel}(U)$ and $f \in \Sigma_{U}$, i.e. $f$ is a permutation of $U$. I will write $f(x, y)$ instead of $\langle f(x), f(y)\rangle$, and set

$$
R^{f}=\{f(x, y): x R y\}
$$

$R$ is called invariant under $f$, if $R=R^{f}$. There are only four relations which are invariant under every permutation, namely, $\emptyset, V, 1^{\prime}$ and $0^{\prime} .{ }^{2}$

If $A \leq \operatorname{Rel}(U)$, and $f \in \Sigma_{U}$, then, $f$ is called a base automorphism of $A$ if $R^{f}=R$ for every $R \in A$. It is not hard to see that the collection of all base automorphisms of $A$ is a subgroup of $\Sigma_{U}$, denoted by $A^{\rho}$. Conversely, if $G$ is a subgroup of $\Sigma_{U}$, then the sets of the form

$$
G_{x, y}=\{f(x, y): f \in G\}
$$

[^1]with $x, y \in U$ are the atoms of a subalgebra of $\operatorname{Rel}(U)$, denoted by $G^{\sigma}$. The pair of assignments $\left\langle{ }^{\rho},{ }^{\sigma}\right\rangle$ forms a Galois connection between the subgroups of $\Sigma_{U}$ and the subalgebras of $\operatorname{Rel}(U)$, and $A \leq \operatorname{Rel}(U)$ is called Galois closed, if $A=A^{\rho \sigma}$. The following is known about the connection between first order closure and Galois closure:

Proposition 4.4. 1. [49] If $A \leq \operatorname{Rel}(U)$ is Galois closed, then it is first order closed.
2. [9] If $U$ is finite, then the two notions coincide.
3. [9] There is a BRA on an infinite set $U$ which is first order closed, but not Galois closed.

If one is able to find a suitable subgroup of the group of base automorphisms of some $A$, it is sometimes easier to show first order closure by showing that $A$ is Galois closed. If $G \leq A^{\rho}$, then $A^{\rho \sigma} \leq G^{\sigma}$, and every element of $A$ is a union of atoms of $G^{\sigma}$. If $A$ is atomic as well, and one can exhibit for each atom $R$ of $A$ a pair $\langle x, y\rangle \in U^{2}$ such that $G_{x, y}=R$, then $A$ is Galois closed, and hence, first order closed.

The situation of the definability of sets instead of relations (or, if you like, subsets of the identity relation) is understood. Suppose that $A \leq \operatorname{Rel}(U)$; we regard $\langle U, A\rangle$ as a first order structure of a suitable language $\mathcal{L}$. We denote by $\mathcal{L}_{3}$ the three variable fragment of $\mathcal{L}$, i.e. the collection of those $\mathcal{L}$ formulas which contain at most three distinct variables. $A$ is called permutational [64] if $A^{\rho}$ is transitive, i.e.

$$
\text { For all } x, y \in U \text { there is some } f \in A^{\rho} \text { such that } f(x)=y \text {. }
$$

## Proposition 4.5. [4]

1. $A$ is integral if and only if for any $\varphi(x) \in \mathcal{L}_{3}$,

$$
\langle U, A\rangle \models(\exists x) \varphi(x) \rightarrow(\forall x) \varphi(x) .
$$

In other words, no proper nonempty subset of $U$ is definable by a formula with at most three variables.
2. $A$ is permutational if and only if for any $\varphi(x) \in \mathcal{L}$,

$$
\langle U, A\rangle \models(\exists x) \varphi(x) \rightarrow(\forall x) \varphi(x) .
$$

In this case, no proper nonempty subset of $U$ is definable by any $\mathcal{L}$ formula.
3. If $A$ is integral and Galois closed, then $A$ is permutational.

McKenzie [63] posed the problem, whether every integral RRA had a permutational representation. This was solved by Andréka et al. [2] who exhibited an RA on a set of 32 elements which is integral and has no permutational representation.

Table 6: Interval relations

| before: $\left\{\left\langle[q, r],\left[q^{\prime}, r^{\prime}\right]\right\rangle: q<r<q^{\prime}<r^{\prime}, q, r, q^{\prime}, r^{\prime} \in \mathbb{R}\right\}$ |
| ---: |
| meets: $\left\{\left\langle[q, r],\left[q^{\prime}, r^{\prime}\right]\right\rangle: q<r=q^{\prime}<r^{\prime}, q, r, q^{\prime}, r^{\prime} \in \mathbb{R}\right\}$ |
| overlaps: $\left\{\left\langle[q, r],\left[q^{\prime}, r^{\prime}\right]\right\rangle: q<q^{\prime}<r<r^{\prime}, q, r, q^{\prime}, r^{\prime} \in \mathbb{R}\right\}$ |
| starts: $\left\{\left\langle[q, r],\left[q^{\prime}, r^{\prime}\right]\right\rangle: q=q^{\prime}<r<r^{\prime}, q, r, q^{\prime}, r^{\prime} \in \mathbb{R}\right\}$ |
| ends: $\left\{\left\langle[q, r],\left[q^{\prime}, r^{\prime}\right]\right\rangle: q^{\prime}<q<r=r^{\prime}, q, r, q^{\prime}, r^{\prime} \in \mathbb{R}\right\}$ |
| contains: $\left\{\left\langle[q, r],\left[q^{\prime}, r^{\prime}\right]\right\rangle: q<q^{\prime}<r^{\prime}<r, q, r, q^{\prime}, r^{\prime} \in \mathbb{R}\right\}$ |

Figure 5: Interval relations


Figure 6: Disk relations



ес

PO

NTPP


TPP

## 5 Relations of time and plane

Allen [1] has presented a set of 13 relations which characterise the possible relations between intervals of time. These are the six relations of Table 6, their converses, and the identity. They are the atoms of an integral BRA $\mathcal{I}$ on the set of all closed intervals on the real line; its composition table can be found in e.g. [56]. The countable representation of $\mathcal{I}$ given in Table 6 is Galois closed [45], and thus RA logic is sufficient to describe the interval relations.

If we want to extend the time interval relations to two dimensional Euclidean space, a natural domain to choose is the set $D$ of closed disks. In the plane, we do not have the unique directions "left - right" of the real line any more, and thus, for example, we cannot distinguish between the "starts" and the "ends" relations, and between the "before" relation and its converse. In this spirit, we obtain the plane relations which are defined in Table 7, and pictured in Figure 6.

Table 7: Disk relations

| Disconnected (DC) | $:\{\langle a, b\rangle: a \cap b=\emptyset\}$ |
| :--- | :--- |
| Externally connected (EC) | $:\{\langle a, b\rangle: a \cap b \neq \emptyset, \operatorname{int}(a \cap b)=\emptyset\}$ |
| Partial overlap (PO) | $:\{\langle a, b\rangle: a \nsubseteq b, b \nsubseteq a, \operatorname{int}(a \cap b) \neq \emptyset\}$ |
| Tangential proper part (TPP) | $:\{\langle a, b\rangle: a \subsetneq b, \operatorname{Fr}(a) \cap \operatorname{Fr}(b) \neq \emptyset\}$ |
| Nontangential proper part (NTPP): $:\{\langle a, b\rangle: a \subseteq \operatorname{int}(b)\}$. |  |

In Table 7, $\operatorname{int}(a)$ is the topological interior of $a$, and $\operatorname{Fr}(a)$ its boundary, i.e. $\operatorname{Fr}(a)=a \cap-\operatorname{int}(a)$. Note that $D C, E C$, and $P O$ are symmetric, while $T P P$ and $N T P P$ are not; this gives us the additional disk relations $T P P^{\llcorner }$and $N T P P^{\llcorner }$. Along with $1^{\prime}$, they are the atoms of a BRA $\mathcal{D}_{c}$ on $D$ whose composition is given in Table 9 [34]. These relations are exactly the topological spatial relations on convex regular open sets of [36] obtained by classifying pairs of such regions by the intersection of their respective interiors and boundaries, shown in Table 8. The composition table of $\mathcal{D}_{c}$ has previously appeared in [39, 80]. I do not know, whether $\mathcal{D}_{c}$ is first order closed.

Table 8: Topological properties of pairs of convex regions [36]

| $R$ | $f r \cap f r$ | int $\cap$ int | $f r \cap i n t$ | int $\cap f r$ |
| :---: | :---: | :---: | :---: | :---: |
| $1^{\prime}$ | $\neq \emptyset$ | $\neq \emptyset$ | $\emptyset$ | $\emptyset$ |
| $D C$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $E C$ | $\neq \emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $P O$ | $\neq \emptyset$ | $\neq \emptyset$ | $\neq \emptyset$ | $\neq \emptyset$ |
| $T P P$ | $\neq \emptyset$ | $\neq \emptyset$ | $\neq \emptyset$ | $\emptyset$ |
| $N T P P$ | $\emptyset$ | $\neq \emptyset$ | $\neq \emptyset$ | $\emptyset$ |

Table 9: Closed disk algebra $\mathcal{D}_{c}$

| 。 | $T P P$ | $T P P^{*}$ | NTPP | NTP ${ }^{\sim}$ | $P O$ | EC | $D C$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T P P$ | $P P$ | $-\left(N T P P \cup N T P P^{\prime}\right)$ | $N T P P$ | $-P$ | $-P^{-}$ | EC, DC | $D C$ |
| $T P P^{\prime}$ | $1^{\prime}, T P P, T P P^{\mu}, P O$ | $P P^{\sim}$ | $P P^{\mu}, P O$ | $N T P P^{*}$ | $P P^{\sim}, P O$ | $P P^{\sim}, P O, E C$ | $-P$ |
| $N T P P$ | NTPP | $-P^{\sim}$ | $N T P P$ | 1 | $-P^{\sim}$ | $D C$ | $D C$ |
| $N T P P^{*}$ | $P P^{\lrcorner}, P O$ | $N T P P^{*}$ | $-(E C \cup D C)$ | $N T P P^{*}$ | $P P^{*}, P O$ | $P P^{\lrcorner}, P O$ | $-P$ |
| $P O$ | $P P, P O$ | - $P$ | $P P, P O$ | -P | 1 | $-P$ | $-P$ |
| $E C$ | $P P, P O, E C$ | $E C \cup D C$ | $P P, P O$ | $D C$ | $-P^{\prime}$ | $-\left(N T P P \cup N T P P^{-}\right)$ | $-P$ |
| $D C$ | $-P^{\sim}$ | $D C$ | $-P$ | $D C$ | $-P^{\sim}$ | $-P^{\sim}$ | 1 |

$\mathcal{D}_{c}$ is isomorphic to the subalgebra of $\mathcal{I}$ generated by the union of the "before" relation and its converse, but its circle representation cannot be embedded into any representation of $\mathcal{I}$ : Consider the square and its diagonals in Figure 7 on the next page, and label the sides of the square with $P O$ and its diagonals with $D C$. This network cannot be satisfied in any representation of $\mathcal{I}$ as shown in [56],
but it can be satisfied in the closed circle algebra by the indicated configuration.
Figure 7: Satisfiable circle network


## 6 Contact relation algebras

In this Section we shall look at relation algebras generated by a contact relation; results are from [34], unless otherwise indicated. To avoid trivialities, we always assume that the structures under consideration have at least two elements. Suppose that $U$ is a nonempty set of regions, and that $C$ is a binary relation on $U$ which satisfies

$$
\begin{equation*}
C \text { is reflexive and symmetric, } \tag{6.1}
\end{equation*}
$$

$$
\begin{equation*}
C x=C y \text { implies } x=y . \tag{6.2}
\end{equation*}
$$

These are the axioms A0.1 and A0.2 given by Clarke [20] for the mereological part of his calculus of individuals, but these properties are already mentioned in [24]. (6.2) is an extensionality axiom, which says that each region is completely determined by those regions to which it is $\mathrm{C}-$ related. We call a binary relation $C$ which satisfies (6.1) and (6.2) a contact relation; an RA generated by a contact relation will be called a contact $R A$ (CRA). It is easily seen that the identity is a contact relation; in the sequel we will assume that a contact relation is different from the identity. We note that (6.2) is a statement about binary relations, and the question arises if there is an equivalent RA expression. An answer is given by

Proposition 6.1. [34] $C$ is a contact relation iff $C$ is reflexive and symmetric, and
$C \backslash C$ is antisymmetric.

Condition (6.2) thus is equivalent to the RA inclusion

$$
\begin{equation*}
-(c \circ-c) \cdot-(c \cdot-c)^{-} \leq 1^{\prime} . \tag{6.4}
\end{equation*}
$$

I have used RA symbols to emphasise that (6.4) is independent of binary relations. Together with Lemma 3.3(1), we obtain the equivalence of (6.3) and

$$
\begin{equation*}
c \backslash c \text { is a partial order. } \tag{6.5}
\end{equation*}
$$

It is not hard to see that the closed and open disk algebra are CRAs with contact defined by

$$
x C y \Longleftrightarrow x \cap y \neq \emptyset,
$$

whereas the interval algebra $\mathcal{L}$ is not.
In the rest of the paper, I assume that $C$ is a contact relation on a set $U$ with at least two elements, and that $C$ is not the identity.

All disk relations of Table 7 are RA definable from $C$ and the relational constants, and I shall use these definitions for the rest of the paper:

$$
\begin{align*}
D C & =-C & & \text { disconnected }  \tag{6.6}\\
P & =-(C \circ D C) & & \text { Part of }  \tag{6.7}\\
P P & =P \cdot 0^{\prime} & & \text { Proper part of }  \tag{6.8}\\
O & =P^{\checkmark} \circ P & & \text { Overlap }  \tag{6.9}\\
P O & =O \cdot-\left(P+P^{\hookrightarrow}\right) & & \text { Partial overlap }  \tag{6.10}\\
E C & =C \cdot-O & & \text { External contact }  \tag{6.11}\\
T P P & =P P \cdot(E C \circ E C) & & \text { Tangential proper part }  \tag{6.12}\\
N T P P & =P P \cdot-T P P & & \text { Non-tangential proper part } \tag{6.13}
\end{align*}
$$

Intuitive interpretations of these relation can be taken from the closed disk algebra. However, as we shall see presently, there are highly non-standard models of CRAs.

Our first non-standard example is a CRA of minimal cardinality known as $\mathcal{N}_{1}$ with composition as in Table 10 [23]. It is integral, has four atoms, and is generated by a strict partial order $P P$. Two elements are in contact, if and only if they are comparable. A concrete representation of this algebra was first given in [27], and a sketch of $P P$ from the slightly different representation in [5] is shown in Figure 8: Think of $P P$ as a fractal-like structure with a copy $\mathbb{Q}$ of the rational numbers as its "backbone", and ever branching at each point into two copies of $\mathbb{Q} ; P P$ is like a time-structure, where the past is fixed, and there are three possibilities for the future at each moment in time.


$$
C \backslash C=-(C \circ D C)=-\left(\left(P+P P^{\lrcorner}\right) \circ D C\right)=-\left(D C+P^{\lrcorner}\right)=P .
$$

It may be worthy of mention that this representation of $\mathcal{N}_{1}$ is first order closed [45].
As a next step, we look for a CRA where $O \neq C$, and hence, $E C \neq 0$; thus, our algebra should have the five atoms $1^{\prime}, P P, P P^{\llcorner }, E C$, and $D C$. There are 14 isomorphism types of such algebras. As an example, I present $\mathcal{S}_{0}$ in Table 11, with a representation as follows: Let

$$
\begin{aligned}
& S=\left\{\frac{a}{3^{k}}: a \lesseqgtr 3^{k}, a \text { odd, } k=1,2,3, \ldots\right\}, \\
& T=\left\{\frac{a}{3^{k}}: 0 \lesseqgtr a \lesseqgtr 3^{k}, a \text { even, } k=1,2,3, \ldots\right\} .
\end{aligned}
$$

Table 10: The algebra $\mathcal{N}_{1}$

| $\circ$ | $P P$ | $P P^{\vee}$ | $D C$ |
| :---: | :---: | :---: | :---: |
| $P P$ | $P P$ | 1 | $D C$ |
| $P P^{\vee}$ | $-D C$ | $P P^{\vee}$ | $P P^{\vee}, D C$ |
| $D C$ | $P P, D C$ | $D C$ | 1 |


| $\circ$ | $P P$ | $P P^{\checkmark}$ | $E C$ | $D C$ |
| :---: | :---: | :---: | :---: | :---: |
| $P P$ | $P P$ | $P P, P P^{\checkmark}, 1^{\prime}$ | $E C, D C$ | $D C$ |
| $P P^{\checkmark}$ | $P P, P P^{\lrcorner}, 1^{\prime}$ | $P P^{\checkmark}$ | $E C$ | $E C, D C$ |
| $E C$ | $E C$ | $E C, D C$ | $P P, P P^{\checkmark}, 1^{\prime}$ | $P P^{\sim}$ |
| $D C$ | $E C, D C$ | $D C$ | $P P$ | $P P, P P^{\lrcorner}, 1^{\prime}$ |

Figure 8: An ordering for $\mathcal{N}_{1}$


Figure 9: An ordering for $\mathcal{S}_{0}$


It is not hard to see that

$$
\begin{align*}
& S \cap T=\emptyset, S, T \cong \mathbb{Q}  \tag{6.14}\\
& S \text { and } T \text { are dense in each other, }  \tag{6.15}\\
& x \in S \Rightarrow x=\inf \{y \in T: x \lesseqgtr y\}=\sup \{y \in T: y \lesseqgtr x\}  \tag{6.16}\\
& x \in T \Rightarrow x=\inf \{y \in S: x \lesseqgtr y\}=\sup \{y \in S: y \leq x\},  \tag{6.17}\\
& x \in S \Longleftrightarrow 1-x \in T \tag{6.18}
\end{align*}
$$

Now, we let $\left\langle S_{0}, \leq\right\rangle,\left\langle S_{1}, \leq\right\rangle$ be two disjoint copies of $\langle S, \leq\rangle, U=S_{0} \cup S_{1}$, and let $P$ be extension of the orders on the $S_{i}$ to $U$. Furthermore,

$$
\begin{aligned}
& x E C y \Longleftrightarrow x \in S_{i}, y \in S_{i+1} \text { and } 1-x \lesseqgtr y \\
& x D C y \Longleftrightarrow x \in S_{i}, y \in S_{i+1} \text { and } 1-x \ngtr y
\end{aligned}
$$

Here, $i \in\{0,1\}$, and addition in the indices is mod 2. The BRA generated by $C=P \cup P^{\cup} \cup E C$ is just $\mathcal{S}_{0}$.

The non-identity atoms of this representation for $P$ are shown in Figure 9. The lines represent the two copies of $S$, and, for an $x$, the labels on the various section of the lines indicate the relation which a
point in this section has to $x$. Note that the white circle labelled $1-x$ is the "border point" between $E C$ and $D C$, but it is not an element of $S$.

The structures occurring in spatial reasoning do not only have a relational part, but also an underlying algebra. It has been shown by Tarski [82] that the algebraic structures arising from models of classical mereology are complete atomless Boolean algebras without 0, and Biacino \& Gerla [14] exhibit the models of Clarke's system as complete orthocomplemented lattices with the smallest element removed. In both cases,

$$
\begin{equation*}
x C y \Longleftrightarrow x \not \leq-y \tag{6.19}
\end{equation*}
$$

defines the contact relation. Recall that an operation - on a bounded lattice is called orthocomplementation, if

$$
\begin{equation*}
x \cdot-x=0,--x=x, x \leq y \Rightarrow-y \leq-x \tag{6.20}
\end{equation*}
$$

The ordering $P$ of (6.19) is compatible with the lattice ordering in the sense that $P=\leq$. More generally, let us call a contact relation $C$ on an ordered structure $\langle U, \leq\rangle$ compatible with $\leq$, if $P=\leq$. For the rest of this Section, we will consider compatible contact relations on orthocomplemented lattices. Since extreme elements are RA definable, we can suppose that the contact relations which are compatible with the ordering on a bounded lattice are defined on the base set of the model with the extreme elements 0,1 removed. This does not mean, however, that we may disregard these elements altogether: If

$$
\begin{equation*}
T=-\left(P \circ P^{\smile}\right) \tag{6.21}
\end{equation*}
$$

then $T$ has the property that

$$
\begin{equation*}
x T y \Longleftrightarrow x+y=1 \tag{6.22}
\end{equation*}
$$

Indeed, $x T y$ if and only if there is no element in $L \backslash\{1\}$ above both $x$ and $y$. Since $x+y$ exists, we must have $x+y=1$. Conversely, if $x+y=1$, then the smallest element above both $x$ and $y$ is 1 , and it follows that there is no element in $L \backslash\{1\}$ above both $x$ and $y$, i.e. $x T y$. If

$$
\begin{equation*}
D D=-O \cap T \tag{6.23}
\end{equation*}
$$

then

$$
x D D y \Longleftrightarrow x \cdot y=0 \text { and } x+y=1
$$

In the sequel, I will write $D N$ for $D C \cap-D D$, i.e.

$$
\begin{equation*}
D N=-O \cap-T=\{\langle x, z\rangle: x \cdot z=0, x+z \lesseqgtr 1\} \tag{6.24}
\end{equation*}
$$

Table 12: The scale algebra $\mathcal{S}_{1}$

| $\circ$ | $P P$ | $P P^{\sim}$ | $E C$ | $D N$ | $D D$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $P P$ | $P P$ | $C P$ | $-C P$ | $D N$ | $D N$ |
| $P P^{\checkmark}$ | $C P$ | $P P^{\checkmark}$ | $E C$ | $E C, D C$ | $E C$ |
| $E C$ | $E C$ | $-C P$ | $C P$ | $P P^{\checkmark}$ | $P P^{\checkmark}$ |
| $D N$ | $E C, D C$ | $D N$ | $P P$ | $C P$ | $P P$ |
| $D D$ | $E C$ | $D N$ | $P P$ | $P P^{\checkmark}$ | $1^{\prime}$ |

Figure 10: An ordering for $\mathcal{S}_{1}$


The relations $D D$ and $D D$ are present in any CRA whose associated partial order $P$ is the ordering of an orthocomplemented lattice. If $C=O$, then the underlying structure is a quasi - Boolean algebra, i.e. a Boolean algebra with the smallest element removed; we will consider this case in Section 7. Otherwise $E C=C \cap-O \neq \emptyset$, and a model, similar to the previous one, is as follows: Let $E_{0}, E_{1}$ be two copies of the real interval $(0,1)$ ordered as usual by $\leq$, and set $E=E_{0} \cup E_{1}, E^{+}=E \cup\{1\}$. Order $E^{+}$by

$$
x P y \Longleftrightarrow x, y \in E_{i} \text { and } x \leq y, \text { or } \mathrm{y}=1
$$

In the following, addition is modulo 2. Let $m: E \rightarrow E$ be defined in such a way that, if $x \in E_{i}$, then $m(x)$ is the value of $x$ in $E_{i+1}$. Now, the relation $C$ defined on $E$ by

$$
\begin{equation*}
\langle x, y\rangle \in C \Longleftrightarrow y \npreceq m(1-x) \tag{6.25}
\end{equation*}
$$

defines a contact relation, and

$$
\begin{aligned}
P P & =\lesseqgtr \\
O & =P^{\llcorner } \circ P=P+P^{\llcorner }+1^{\prime} \\
E C & =C \backslash O=\{\langle x, y\rangle: y \ngtr m(1-x)\} \\
D D & =-\left[\left(-P^{\llcorner } \circ D C\right) \cup\left(P^{\llcorner } \circ C\right)\right]=\{\langle x, y\rangle: y=m(1-x)\} \\
D N & =D C \cap-D D=\{\langle x, y\rangle: y \lesseqgtr m(1-x)\}
\end{aligned}
$$

The composition of the RA $\mathcal{S}_{1}$ generated by $C$ is given in Table 12 . I call $\mathcal{S}_{1}$ a scale algebra, since $x$ is related to its complement like a scale, as indicated in Figure 10.

An algebra where $E C$ splits into two atoms $E N$ and $E D$, and $D C$ splits into $D N$ and $D D$ is given in Table 13. Since $D D$ is a one-one function of order two disjoint from $P \cup P^{\checkmark}$, there must be an even number of components of $P$. Furthermore, the Table tells us that, if $x E N y$ or $x D C y$, then $y$ is in the same component as $D D(x)$, and, if $x E D y$, then $y$ is in a component different from those of $x$ or $D D(x)$. Let $S_{i}, i<4$ be disjoint copies of the rational interval $(0,1)$. The mapping $m$ is defined

Table 13: Algebra $\mathcal{S}_{2}$ with complement and split EC

| $\circ$ | $P P$ | $P P^{\checkmark}$ | $E N$ | $E D$ | $D N$ | $D D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $P P$ | $P P$ | $P P, P P^{\checkmark}, 1^{\prime}$ | $E N, D C$ | $E D$ | $D N$ | $D N$ |
| $P P^{\checkmark}$ | $P P, P P^{\checkmark}, 1^{\prime}$ | $P P^{\checkmark}$ | $E N$ | $E D$ | $E N, D C$ | $E N$ |
| $E N$ | $E N$ | $E N, D C$ | $P P, P P^{\checkmark}, 1^{\prime}$ | $E D$ | $P^{\checkmark}$ | $P^{\checkmark}$ |
| $E D$ | $E D$ | $E D$ | $E D$ | $-E D$ | $E D$ | $E D$ |
| $D N$ | $E N, D C$ | $D N$ | $P P$ | $E D$ | $P P, P P^{\checkmark}, 1^{\prime}$ | $P P$ |
| $D D$ | $E N$ | $D N$ | $P P$ | $E D$ | $P P^{\checkmark}$ | $1^{\prime}$ |

Figure 11: An ordering for $\mathcal{S}_{2}$

from

$$
m: \begin{cases}S_{0} & \rightarrow S_{1} \\ S_{1} & \rightarrow S_{0} \\ S_{2} & \rightarrow S_{3} \\ S_{3} & \rightarrow S_{2}\end{cases}
$$

and $m$ puts $x \in(0,1)$ onto its twin in the other component. Let us now define

$$
\begin{aligned}
& x P P y \Longleftrightarrow x, y \in S_{i} \text { and } x \npreceq y \\
& x D D y \Longleftrightarrow y=m(1-x) \\
& x E N y \Longleftrightarrow m(1-x) \nLeftarrow y \\
& x D N y \Longleftrightarrow m(1-x) \nLeftarrow y \\
& x E D y \Longleftrightarrow y \text { is in a component different from that of } x \text { or } D D(x)
\end{aligned}
$$

If $C=-(D N \cup D D)$, then $\mathcal{S}_{2}$ is isomorphic to the algebra generated by $C$. An indication of the atoms of $\mathcal{S}_{2}$ is given in Fig. 11.

We can also have $E D \circ E D=1$; in this case, we need (at least) six components, and, otherwise, use the same definitions as for $S_{2}$.

## 7 Boolean contact algebras

The standard model of a contact structure is the collection $R O(X)$ of regular open sets of a connected regular $T_{0}$ space $X$, with contact given by

$$
\begin{equation*}
x C y \Longleftrightarrow \operatorname{cl}(x) \cap \operatorname{cl}(y) \neq \emptyset . \tag{7.1}
\end{equation*}
$$

$R O(X)$ is a complete Boolean algebra with the operations

$$
\begin{aligned}
x+y & =\operatorname{int}(c l(x \cup y)), \\
x \cdot y & =x \cap y, \\
x^{*} & =\operatorname{int}(X \backslash x) .
\end{aligned}
$$

I use * for complementation in $R O(X)$ to distinguish it from the set complement.
It is not hard to see, that

$$
P=\subseteq .
$$

A Boolean contact algebra $(\mathrm{BCA})$ is a pair $\langle B, C\rangle$, where $B$ is an atomless Boolean algebra, and $C$ is a contact relation on $U=B \backslash\{0,1\}$. Since 0 and 1 are RA definable from $C$, I will not include them in the field $U$ of $C$. If $\langle B, C\rangle$ and $\left\langle B^{\prime}, C^{\prime}\right\rangle$ are BCAs, I call $\left\langle B^{\prime}, C^{\prime}\right\rangle$ a substructure of $\langle B, C\rangle$, if $B^{\prime}$ is a Boolean subalgebra of $B$, and $C^{\prime}=C \cap(B \times B)$. As a note of caution I want to point out that the BRA generated by a substructure $\left\langle B^{\prime}, C^{\prime}\right\rangle$ of $\langle B, C\rangle$ is not necessarily a subalgebra of the BRA of $\langle B, C\rangle$.

The relation algebra generated by the contact relation of any $B C A$ must include the relation algebra generated by the Boolean order $\leq$ on $B_{0}$, since $C$ is compatible, and thus, $\leq=P \in[C]$. To find this algebra, I first define the following relations in addition to the relations defined in Section 6:

$$
\begin{aligned}
\# & =-\left(P \cup P^{\smile}\right) & & =\{\langle x, z\rangle: x \text { and } z \text { are incomparable w.r.t. } \leq\} \\
T & =-\left(P \circ P^{\cup}\right) & & =\{\langle x, z\rangle: x+z=1\} \\
P O N & =O \cap \# \cap-T & & =\{\langle x, z\rangle: x \# z, x \cdot z \neq 0, x+z \neq 1\} \\
P O D & =O \cap \# \cap T & & =\{\langle x, z\rangle: x \# z, x \cdot z \neq 0, x+z=1\} \\
D D & =-O \cap T & & =\{\langle x, z\rangle: x \cdot z=0, x+z=1\}, \\
D N & =-O \cap-T & & =\{\langle x, z\rangle: x \cdot z=0, x+z \lesseqgtr 1\},
\end{aligned}
$$

where $x, z \in U$. We also note that compatibility of $C$ implies

$$
x O y \Longleftrightarrow x \cdot y \geqslant 0 .
$$

Proposition 7.1. [34] Let B be an atomless Boolean algebra. Then, the relations

$$
1^{\prime}, P P, P P^{\triangleleft}, P O N, P O D, D N, D D
$$

as defined above are the atoms of the algebra $\mathcal{G}$ on $B \backslash\{0,1\}$ generated by the Boolean order $P$ whose composition is given in Table 14.

Table 14: The algebra $\mathcal{G}$

| $\circ$ | $O$ |  |  |  | $D$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $P P$ | $P P^{\checkmark}$ | $P O N$ | $P O D$ | $D N$ | $D D$ |
| $P P$ | $P P$ | $-(P O D \cup D D)$ | $P P, P O N, D N$ | $P P, P O, D$ | $D N$ | $D N$ |
| $P P^{\checkmark}$ | $1^{\prime}, O$ | $P P^{\checkmark}$ | $P P^{\checkmark}, P O$ | $P O D$ | $P P^{\checkmark}, P O, D$ | $P O D$ |
| $P O N$ | $P P, P O$ | $P P^{\checkmark}, P O N, D N$ | 1 | $P P, P O$ | $P P^{\checkmark}, P O N, D N$ | $P O N$ |
| $P O D$ | $P O D$ | $P P^{\checkmark}, P O, D$ | $P P^{\checkmark}, P O$ | $1^{\prime}, O$ | $P P^{\checkmark}$ | $P P^{\sim}$ |
| $D N$ | $P P, P O, D$ | $D N$ | $P P, P O N, D N$ | $P P$ | $-(P O D \cup D D)$ | $P P$ |
| $D D$ | $P O D$ | $D N$ | $P O N$ | $P P$ | $P P^{\checkmark}$ | $1^{\prime}$ |

In the algebra $\mathcal{G}$, there are two possibilities to define a contact relation: We can take either $C=O$ or $C=O \cup D D$. In both cases, $P=C \backslash C$.

In order to answer the question when a representation of $\mathcal{G}$ is Galois closed, we need some preparation. If $B$ is a Boolean algebra and $x \in B$, then $B \mid x$ is the Boolean algebra with base set $\{y \in B: y \leq x\}$, meet and join inherited from $B$, and complementation relative to $x . B$ is called homogeneous, if $B \mid x \cong B$ for every $x \in B, x>0$. We now have

Proposition 7.2. [33] An atomless representation $\langle B, C\rangle$ of $\mathcal{G}$ is Galois closed if and only if $B$ is a homogeneous Boolean algebra. In particular, $\mathcal{G}$ is Galois closed, if $B$ is the set of all regular open sets of the two dimensional Euclidean space.

The atoms of $\mathcal{G}$ constitute a refined version of what is known as RCC5 [57].

## 8 The standard model and the RCC

A special instance of a BCA is the region connection calculus (RCC) [77]. Since it can be shown from the original axioms that any RCC model is a quasi - Boolean algebra [33, 81], I use an axiom system, which incorporates this and which is equivalent to the original one; I also restrict the contact relation to the non-extremal elements: A model of the RCC is a structure $\langle B, C\rangle$ such that for all $x, y, z \in U=B \backslash\{0,1\}$,

RCC 1. $B$ is a Boolean algebra, and $C$ is a compatible contact relation on $U$.
RCC 2. $x C-x$.
RCC 3. If $y+z=1$ or $x C(y+z)$, then $x C y$ or $x C z$.
RCC 4. $x C-y \Longleftrightarrow x(-N T P P) y$.
RCC 5. $x O-y \Longleftrightarrow x(-P) y$.

The original RCC axioms asserted that there are no $N T P P$-minimal elements: Using the relational formalism, one can give a simple proof of this from the remaining axioms:

Proposition 8.1. [33]

$$
\begin{equation*}
(\forall x \in U)(\exists y \in U) y N T P P x \tag{8.1}
\end{equation*}
$$

Proof. Assume that there is some $x \in U$ such that for all $y \in U, y(-N T P P) x$; by RCC 4 , this implies that $y C-x$ for all $y \in R$. Since $P=C \backslash C$ (i.e. $P$ is the largest relation $S$ on U with $C \circ S \leq C$ ), and $\langle x,-x\rangle \notin P$, we obtain that $C \circ\{\langle x,-x\rangle\} \not \subset C$. Hence, there is some $t \in U$ such $\langle t,-x\rangle \notin C$, a contradiction.

Corollary 8.2. Each model of the RCC is a BCA.

It is known that $R O(X)$ with $X$ connected and regular $T_{0}$ is a model for the RCC [41]. In fact, more is true:

Proposition 8.3. [32] If $X$ is a connected regular $T_{0}$ space, then each substructure $B$ of $\langle R O(X), C\rangle$ is a model of the RCC axioms.

This shows that the polygonal algebras of [74] are RCC models. I do not know, whether the converse of Proposition 8.3 holds, i.e. whether every RCC model is isomorphic to a substructure of some $R O(X)$ with contact as in (7.1). At any rate, each atomless Boolean algebra can be made into an RCC model:

Proposition 8.4. [53] Every BA can be embedded into the algebra of regular open sets of a connected regular $T_{0}$ space.

Corollary 8.5. On each atomless Boolean algebra $B$ there is some contact relation $C$ such that $\langle B, C\rangle$ is an RCC model.

The disk relations shown in Figure 6, together with the identity and the converses of TPP and NTPP are usually taken as the base relations of the RCC, called RCC8. These relations were defined in [20] without a pictorial representation. Indeed, from a relational point of view, to take Figure 6 as an example for these relations, is somewhat misleading. The circles used to exemplify the connections, are much too special to get an intuitive feeling of the relations in the standard model. As we shall see below, the situations where the relations hold, and the landscape of relations which must exist in an RCC model is much richer than the picture indicates. As a simple example, let $x$ be the disjoint union of two disks $y$ and $z$. Then, $y T P P x, z T P P x$, which, topologically, is a totally different situation from the "touching circles" of Figure 6. Indeed, the weak composition table for the RCC8 relations of [77] is exactly the ("real") composition table of the closed circle algebra.

In the rest of this Section I shall exhibit nonempty relations which must exist in any RCC model $\langle B, C\rangle$, and thus, in particular, in a standard model of any dimension; unless stated otherwise, all of the material is drawn from [32]. Only relational operations and constants are used in deriving these relations and showing that they are not zero.

A different approach to obtain (more special) topological relations based on the concepts of interior, boundary, exterior is the generalisation of the 4 - intersection model of [36] by Egenhofer \& Herring [37]. This leads to the $3 \times 3$ matrix given in Table 15. Many examples of instances of these configurations are given in [35]. At the time of writing, I do not know the relationship between the expressiveness of the relational calculus and that of the 9 - intersection model, and more research is needed to clarify the situation.

Table 15: 9 relation configuration

$$
\left(\begin{array}{ccc}
\operatorname{int}(x) \cap \operatorname{int}(y) & \operatorname{int}(x) \cap \operatorname{Fr}(y) & \operatorname{int}(x) \cap-y \\
\operatorname{Fr}(x) \cap \operatorname{int}(y) & \operatorname{Fr}(x) \cap \operatorname{Fr}(y) & \operatorname{Fr}(x) \cap-y \\
-x \cap \operatorname{int}(y) & -x \cap \operatorname{Fr}(x) & -x \cap-y
\end{array}\right)
$$

Since I assume $C$ to be defined on $U=B \backslash\{0,1\}$, we immediately obtain the complement relation from (6.23). Since each element is connected to its complement, I will change the notation to

$$
\begin{array}{ll}
E C D=-\left(P \circ P^{\cup}\right) \cap-\left(P^{\cup} \circ P\right) & x E C D y \Longleftrightarrow y=-x, \\
E C N=E C \cap-E C D & x E C N y \Longleftrightarrow x \cdot y=0, x+y \lesseqgtr 1, x C y . \tag{8.3}
\end{array}
$$

Partial overlap splits as well into

$$
\begin{array}{ll}
P O D=P O \cap-\left(P \circ P^{\cup}\right) & x P O D y \Longleftrightarrow x P O y, x+y=1, \\
P O N=P O \cap-P O D & x P O N y \Longleftrightarrow x P O Y, x+y \lesseqgtr 1 . \tag{8.5}
\end{array}
$$

We will use the 10 disjoint relations

$$
1^{\prime}, N T P P, N T P P^{\llcorner }, T P P, T P P^{\lrcorner}, P O N, P O D, E C N, E C D, D C
$$

as base relations in which we can express other relations.
In Table 16 on the following page I list some properties of the relations and their interplay with the algebraic operators.

The weak composition table for these 10 relations is not the composition table of a relation algebra. However, they are the atoms of a semi-associative relation algebra in the sense Maddux [61].

One can easily show that $P O D$ splits into the relations

$$
\begin{align*}
& P O D Z=E C D \circ N T P P  \tag{8.6}\\
& P O D Y=P O D \backslash(E C D \circ N T P P) . \tag{8.7}
\end{align*}
$$

Table 16: Some properties of the 10 base relations

| $\begin{aligned} N T P P & =E C D \circ N T P P^{\cup} \circ E C D \\ P \circ N T P P & \leq N T P P \\ N T P P \circ T P P & =N T P P \\ N T P P \circ P & =N T P P \\ T P P \circ N T P P & =N T P P \\ E C N & =T P P \circ E C D \\ x D C z & \Rightarrow x T P P(x+z) \\ x N T P P z \text { and } y N T P P z & \Longleftrightarrow(x+y) N T P P z \\ x N T P P z & \Rightarrow-x \cdot z T P P z \\ x N T P P y \text { and } x N T P P z & \Longleftrightarrow x N T P P y \cdot z \\ E C D \circ D C & \leq N T P P^{\checkmark} \\ x E C N \circ T P P z & \Longleftrightarrow x E C N x^{*} \cdot z T P P z \\ x T P P^{\llcorner } \circ T P P z & \Longleftrightarrow x T P P^{\lrcorner} x \cdot z T P P z \\ x T P P \circ T P P \backsim & \Longleftrightarrow x T P P(x+z) T P P^{\llcorner } z \\ y N T P P(x+z) \text { and } y D C z & \Rightarrow y N T P P x \\ E C D \circ N T P P & \Longleftrightarrow P O D \end{aligned}$ |
| :---: |

Table 17: The RCC 11 weak composition

| $\circ_{w}$ | TPP | TPP ${ }$ | NTPP | NTPP ${ }^{\text {® }}$ | PON | PODY | PODZ | ECN | ECD | DC |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| TPP | TPP, NTPP | $\begin{array}{lr} \hline \hline 1^{\prime}, \text { TPP, TPP }{ }^{\circ}, \\ \text { PON, } & \text { ECN, } \\ \text { DC }, \neq & \\ \hline \end{array}$ | NTPP,= | $\begin{array}{lr} \hline \hline \text { TPP } \smile, ~ N T P P ~ \\ \text { PON, } & \text { ECN, } \\ \text { DC }, \neq & \\ \hline \end{array}$ | TPP, NTPP, PON, ECN, DC | TPP, NTPP, PON, PODY, ECN, ECD | TPP, NTPP, <br> PON, PODY, <br> PODZ  | ECN, DC | ECN,= | DC, $=$ |
| TPP ${ }^{\text { }}$ | $\begin{array}{ll} \hline 1^{\prime}, \mathrm{TPP}, \mathrm{TPP}{ }^{\wedge}, \\ \text { PON, } & \text { PODY, } \\ \text { PODZ } & \\ \hline \end{array}$ | TPP ${ }^{\text {, }}$ NTPP ${ }^{\text { }}$ |   <br> TPP, NTPP, <br> PON, PODY, <br> PODZ  | $\mathrm{NTPP}^{\sim}$,= | $\begin{array}{lr} \hline \text { TPP }{ }^{\wedge}, & \text { NTPP } \\ \text { PON, } & \text { PODY, } \\ \text { PODZ } & \\ \hline \end{array}$ | PODY, PODZ | PODZ |  | PODY | $\begin{aligned} & \mathrm{TPP}^{\checkmark}, \mathrm{NTPP}^{{f23464b20-57d3-40f6-9adb-fdf0c0f3aae0}}\), \\ NTPP, NTPP \({ }^{{f621d507b-fd5b-430a-a299-05ded9981b1d}}, \\ & \text { PON, PODY, } \\ & \text { PODZ, } \\ & \text { ECD, }, \end{aligned}$ |
| PON |   <br> TPP, NTPP, <br> PON, PODY, <br> PODZ  | TPP ${ }^{\checkmark}$, NTPP ${ }^{\smile}$, PON, ECN, DC | TPP, NTPP, <br> PON, PODY, <br> PODZ  | $\begin{array}{lr} \hline \mathrm{TPP}^{\wedge}, & \mathrm{NTPP}^{\wedge}, \\ \mathrm{PON}, & \mathrm{ECN}, \\ \mathrm{DC},= & \end{array}$ |  |   <br> TPP, NTPP, <br> PON, PODY, <br> PODZ  |   <br> TPP, NTPP, <br> PON, PODY, <br> PODZ  | $\begin{array}{lr} \hline \mathrm{TPP}^{\cup}, & \mathrm{NTPP}^{\cup}, \\ \mathrm{PON}, & \mathrm{ECN}, \\ \mathrm{DC}, \neq & \end{array}$ | PON,= | $\begin{array}{lr} \mathrm{TPP}^{\wedge}, & \mathrm{NTPP}^{\wedge}, \\ \mathrm{PON}, & \mathrm{ECN}, \\ \mathrm{DC},= \end{array}$ |
| PODY | PODY, PODZ | $\begin{aligned} & \hline \mathrm{TPP}^{\smile}, \mathrm{NTPP}^{{fe4fad010-42e3-4bb4-8e57-c58a050969c3}} \\ & \text { PON, ECN, DC } \end{aligned}$ | TPP,= | $\begin{aligned} & \mathrm{TPP}^{\checkmark}, \mathrm{NTPP}^{`} \\ & \text { PON, ECN, DC } \end{aligned}$ |  |  |  |  |  |  |
| ECD | PODY,= | ECN,= | PODZ,= | DC | PON,= | TPP,= | NTPP,= | TPP ${ }^{\text {, }}=$ | 1',= | NTPP ${ }^{\text {, }}$ = |
| DC | $\begin{aligned} & \text { TPP, NTPP, } \\ & \text { PON, ECN, DC } \end{aligned}$ | DC,= | TPP, NTPP, <br> PON, PODY, <br> PODZ, ECN, <br> ECD, DC, $=$  | DC | TPP, NTPP, <br> PON, ECN, <br> DC, $=$  | NTPP,= | NTPP,= | TPP, NTPP, <br> PON, ECN, <br> DC, $=$  | NTPP,= | $\begin{aligned} & \text { 1', TPP, TPP } \\ & \text { NTPP, NTPP } \\ & \text { PON, ECN, DC } \end{aligned}$ |

A weak composition table for the 11 relations can be found in Table 17. For cells containing $=$, the RCC axioms together with general RA properties imply that equality holds; for cells containing $\neq$, there is a model in which the composition is strictly smaller than the cell entry. In this way, one can indicate in which cells the composition may be weak, and when it is not.

It turns out that there is a relation algebra $A$ whose composition is represented by the $\mathrm{RCC11}$ table. $A$, however, cannot come from an RCC model as Proposition 8.6 shows, and no representation of $A$ is known.

Proposition 8.6. [32] The relations given in Table 18 are present and not zero in any RCC model, and they are the atoms of an integral relation algebra.

Table 18: RCC necessary relations

$$
\begin{aligned}
& 1^{\prime} \\
& T P P A=T P P \cap(E C N \circ T P P) \\
& T P P A^{\llcorner }=T P P^{\llcorner } \cap(E C N \circ T P P)^{\llcorner } \\
& T P P B=T P P \cap-(E C N \circ T P P) \\
& T P P B^{\checkmark}=T P P^{\llcorner } \cap-(E C N \circ T P P)^{\checkmark} \\
& \text { NTPP } \\
& N T P P^{\checkmark} \\
& P O N Y A 1=P O N \cap(E C N \circ T P P) \cap-(E C N \circ T P P)^{\wedge} \cap\left(T P P \circ T P P^{\triangleleft}\right) \cap\left(T P P^{\wedge} \circ T P P\right) \\
& P O N Y A 1^{\smile}=P O N \cap(E C N \circ T P P)^{\wedge} \cap-(E C N \circ T P P) \cap\left(T P P \circ T P P^{\wedge}\right) \cap\left(T P P^{\wedge} \circ T P P\right) \\
& P O N Y A 2=P O N \cap(E C N \circ T P P) \cap-(E C N \circ T P P)^{\wedge} \cap\left(T P P \circ T P P^{\smile}\right) \cap-\left(T P P^{\backsim} \circ T P P\right) \\
& P O N Y A 2^{\llcorner }=P O N \cap(E C N \circ T P P)^{\llcorner } \cap-(E C N \circ T P P) \cap\left(T P P \circ T P P^{\vee}\right) \cap-\left(T P P^{\llcorner } \circ T P P\right) \\
& P O N Y B=P O N \cap(E C N \circ T P P) \cap-(E C N \circ T P P)^{\cup} \cap-\left(T P P \circ T P P^{\cup}\right) \\
& P O N Y B^{\hookrightarrow}=P O N \cap(E C N \circ T P P)^{\wedge} \cap-(E C N \circ T P P) \cap-\left(T P P \circ T P P^{\cup}\right) \\
& P O N X A 1=P O N \cap(E C N \circ T P P) \cap(E C N \circ T P P)^{\cup} \cap\left(T P P \circ T P P^{\vee}\right) \cap\left(T P P^{\backsim} \circ T P P\right) \\
& P O N X A 2=P O N \cap(E C N \circ T P P) \cap(E C N \circ T P P)^{\wedge} \cap\left(T P P \circ T P P^{\wedge}\right) \cap-\left(T P P^{\wedge} \circ T P P\right) \\
& P O N X B 1=P O N \cap(E C N \circ T P P) \cap(E C N \circ T P P)^{\wedge} \cap-\left(T P P \circ T P P^{\wedge}\right) \cap\left(T P P^{\wedge} \circ T P P\right) \\
& P O N X B 2=P O N \cap(E C N \circ T P P) \cap(E C N \circ T P P)^{\wedge} \cap-\left(T P P \circ T P P^{\lrcorner}\right) \cap-\left(T P P^{\llcorner } \circ T P P\right) \\
& P O N Z=P O N \cap-(E C N \circ T P P) \cap-(E C N \circ T P P)^{\llcorner } \\
& P O D Y A=E C D \circ(T P P \cap(E C N \circ T P P)) \\
& P O D Y B=E C D \circ(T P P \cap-(E C N \circ T P P)) \\
& P O D Z=E C D \circ N T P P \\
& E C N A=E C N \cap\left(T P P \circ T P P^{\cup}\right) \\
& E C N B=E C N \cap-\left(T P P \circ T P P^{\cup}\right) \\
& E C D \\
& D C
\end{aligned}
$$

We have not found a representation of this algebra. In particular, we have as yet not been able to determine whether this algebra is the BRA generated by the contact relation on a standard model. If

Table 19: Splitting of $P O N$

|  | ECNoTPP | $(E C N \circ T P P)^{\checkmark}$ | $T P P \circ T P P^{\checkmark}$ | $T P P^{\sim} \circ T P P$ |
| :---: | :---: | :---: | :---: | :---: |
| PONY A1 | + | - | + | + |
| PONY A1 | - | + | + | + |
| PONY A2 | $+$ | - | + | - |
| PONY A2 | - | $+$ | $+$ | - |
| PONYB | + | - | - |  |
| $P O N Y B^{\checkmark}$ | - | $+$ | - |  |
| PONX A1 | + | + | + | + |
| PONX A2 | + | + | + | - |
| PONX B1 | + | $+$ | - | + |
| PONX B2 | $+$ | + | - | - |
| PONZ | - | - |  |  |

the CRA of some $R O(X)$ is integral, then it is not permutational (and thus, not first order closed), since connectivity is relationally definable by the formula

$$
\varphi(x):(\forall y)(\forall z)(y P x \wedge z P x \wedge(\forall w)((y P w \wedge z P w) \rightarrow x P w)) \rightarrow y C z)
$$

and not all regular open sets are connected.
To give an impression of some of these relations in a standard model, consider Figures $12-14$ on the following page.

$$
s+t+w \preceq 1
$$

aNTPPs, bNTPPt.


Figure 13: $a N T P P b N T P P s \lesseqgtr 1$

$x T P P A z:$ In Figure 13 set $x=a+t, z=s+t$.
$x T P P B z:$ In Figure 12, set $x=s, z=s+t$ or $x=a^{*} \cdot s, z=s$.
$x P O N Y A 1 z$ : In Figure 12, set $x=a+t+w, z=s+w$.
$x P O N Y A 2 z:$ In Figure 12, set $x=t+a, z=s$.

Figure 14: $a N T P P b, b N T P P s, c N T P P s, b D C c$

$x$ PONY Bz: In Figure 13, set $x=b, z=s \cdot a^{*}$.
$x P O N X A 1 z:$ In Figure 12, set $x=t+a, z=s+b$.
$x P O N X A 2 z$ : In Figure 12, set $x=s, z=a+c$.
$x P O N X B 1 z$ : In Figure 14, set $x=s \cdot(a+c)^{*}, z=s^{*}+b$.
$x P O N X B 2 z$ : In Figure 13, set $x=b, z=a+s \cdot b^{*}$.
$x P O N Z z:$ In Figure 12, set $x=s+t \cdot b^{*}, z=t+s \cdot a^{*}$.
The topological properties of some of these relations are shown in Table 20 on the next page. From these, the topological characterisations of most of the remaining ones can be determined, since they are intersections, respectively, complements of the given ones. For example,

$$
\begin{aligned}
x T P P A z & \Longleftrightarrow x(T P P \cap(E C N \circ T P P)) z \\
& \Longleftrightarrow x \subsetneq z, \operatorname{Fr}(x) \cap \operatorname{Fr}(-x \cap z) \neq \emptyset, \operatorname{Fr}(z) \cap \operatorname{Fr}(-x \cap z) \neq \emptyset, \operatorname{cl}(x) \cup \operatorname{cl}(z) \neq X .
\end{aligned}
$$

Our final example in this Section from [32] shows that not every RCC model is integral: Consider $\mathbb{R}^{2}$ and define $K$ as the collection of sets of the form

$$
K(a, b)= \begin{cases}\left\{p \in \mathbb{R}^{2}: a \lesseqgtr|p| \lesseqgtr b\right\}, & \text { if } 0 \neq a \\ \left\{p \in \mathbb{R}^{:}|p| \lesseqgtr b\right\}, & \text { if } a=0\end{cases}
$$

where $a \in \mathbb{R}, b \in \mathbb{R} \cup\{\infty\}$, and $|p|$ is the Euclidian distance of $p \in \mathbb{R}^{2}$ to $\left.(0,0)\right)$. Let $R$ be the set of all finite unions of elements of $K$ including $\emptyset$. Then $R$ is a subalgebra of $R O\left(\mathbb{R}^{2}\right)$, and, by Proposition $8.3,\langle R, C\rangle$ is a model of the RCC.

Now, consider $x=K(0,1)$. I want to show that there is no $y \in R$ with $x T P P A y$.
Every element $y$ of $R$ with $x T P P y$ is of the form $x \cup\{K(a, b): 1 \leq a\}$. We conclude $x T P P B y$ because $-x \cdot y=\{K(a, b): 1<a\}$ and $\{K(a, b): 1 \lesseqgtr a\}$ is disconnected to $x$.

Table 20: Topological interpretation of RCC25 relations

| Atom | Name | $x, z \in R O(X) \backslash\{\emptyset, X\}$ |
| :---: | :---: | :---: |
| Base relations |  |  |
| * | $T P P$ <br> NTPP <br> PON <br> POD <br> $E C N$ <br> ECD <br> DC | $\begin{aligned} & x \subsetneq z, \operatorname{Fr}(x) \cap \operatorname{Fr}(z) \neq \emptyset \\ & \operatorname{cl}(x) \subsetneq z \\ & x \not \subset z, z \not \subset x, x \cap z \neq \emptyset, \operatorname{cl}(x) \cup \operatorname{cl}(z) \neq X \\ & x \nsubseteq z, z \not \subset x, x \cap z \neq \emptyset, \operatorname{cl}(x) \cup \operatorname{cl}(z)=X \\ & x \cap z=\emptyset, \operatorname{Fr}(x) \cap \operatorname{Fr}(z) \neq \emptyset, \operatorname{cl}(x) \cup \operatorname{cl}(z) \neq X \\ & x \cap z=\emptyset, \operatorname{Fr}(x) \cap \operatorname{Fr}(z) \neq \emptyset, \operatorname{cl}(x) \cup \operatorname{cl}(z)=X \\ & \operatorname{cl}(x) \cap \operatorname{cl}(z)=\emptyset \end{aligned}$ |
| Other relations |  |  |
|  | $\begin{aligned} & E C N \circ T P P \\ & T P P \circ T P P^{\backsim} \\ & T P P^{\backsim} \circ T P P \\ & E C D \circ N T P P \end{aligned}$ | $\begin{aligned} & \operatorname{Fr}(x) \cap \operatorname{Fr}(-x \cap z) \neq \emptyset, \operatorname{Fr}(z) \cap \operatorname{Fr}(-x \cap z) \neq \\ & \emptyset, \operatorname{cl}(x) \cup \operatorname{cl}(z) \neq X \\ & \operatorname{Fr}(x) \cap \operatorname{Fr}(\operatorname{int}(\operatorname{cl}(x \cup z))) \neq \emptyset, \operatorname{Fr}(z) \cap \\ & \operatorname{Fr}(\operatorname{int}(\operatorname{cl}(x \cup z))) \neq \emptyset \\ & \operatorname{Fr}(x) \cap \operatorname{Fr}(x \cap z) \neq \emptyset, \operatorname{Fr}(z) \cap \operatorname{Fr}(x \cap z) \neq \emptyset \\ & x \cup z=X \end{aligned}$ |
| $*$ $*$ $*$ | $\begin{aligned} & \hline P O D Z \\ & E C N A \\ & E C N B \end{aligned}$ | $\begin{aligned} & x \cup z=X \\ & x E C N z, \operatorname{Fr}(x) \cap \operatorname{Fr}(x+z) \neq \emptyset, \operatorname{Fr}(z) \cap \\ & \operatorname{Fr}(x+z) \neq \emptyset \\ & x E C N z, \operatorname{cl}(x) \subseteq x+z \text { or } \operatorname{cl}(z) \subseteq x+z \end{aligned}$ |

## 9 A relational logic for CRAs

Semantics for modal logic are nowadays mostly given by frames $\left\langle W, R_{i}\right\rangle$, i.e. sets with accessibility relations [54]. The meaning function assigns subsets of $W$ to formulas, where the classical logical operators are interpreted by set operations. The meaning of the modal operators is interpreted by properties of the accessibility relations. Equivalent to the frame semantics are the algebraic semantics which translates the modal operators to normal and additive operators on suitable Boolean algebras [51]. Orłowska [69] has shown that any classical modal logic can be interpreted in a purely relation algebraic setting, and has exhibited a sound and complete proof system for the logic.

Such systems are in the style of Rasiowa \& Sikorski [78], and consist of decomposition rules, specific rules and (sequences of) axiomatic expressions. A decomposition rule when applied to an expression of the theory returns a set of expressions which are syntactically simpler than the original one. These rules provide definitions of relational operators. The specific rules are the counterparts of relational constraints. It is worth mentioning that in the Hilbert-style proof systems for applied modal logics it is often the case that not all the relational constraints can be explicitly expressed and axiomatised. The
experience with relational proof systems designed until now shows that many constraints which are not modally expressible receive an explicit representation in the form of a relational rule or a relational axiomatic sequence. As a case in point, it was shown in [30] that the extensionality axiom (6.2) of contact relations is not expressible in a classical modal logic, nor, as shown in [29], in its sufficiency counterpart of [40]. Another example is that the fact that a relation is an intersection of other relations is not expressible in the standard modal language, but it is expressible in the form of a relational rule. More details on relational proof systems can be found in [70].

I shall present a relational proof system for CRAs, which was put forward in [30], from which all material in this Section is taken.

The alphabet of the language $\mathcal{L}$ consists of the disjoint union of the following sets:

1. A set $\left\{C, 1^{\prime}\right\}$ of constants, representing, respectively, the contact relation and the identity.
2. A countably infinite set $V I$ of individuum variables.
3. A set $\{\cup, \cap,-, ;$,$\} of names for the relational operators.$
4. A set $\{()$,$\} of delimiters.$

With some abuse of language, I use the same symbols as for the actual operations; it will be clear from the context which meaning is intended.

The set $C E$ of terms ("contact expressions") is defined as follows:

1. $C$ and $1^{\prime}$ are terms.
2. If $R$ and $S$ are terms, so are

$$
(R \cup S),(R \cap S),(-R),(R ; S),\left(R^{\cup}\right) .
$$

3. No other string is a term.

I will use the usual conventions of reducing brackets. Note that $C E$ can be regarded as the absolutely free algebra of type $\langle 2,2,1,2,1\rangle$ over $\left\{C, 1^{\prime}\right\}$.

The set of $\mathcal{L}$-formulas is

$$
\{x R y: R \in C E, x, y \in V I\} .
$$

A model of $\mathcal{L}$ is a pair $M=\langle W, m\rangle$, where $W$ is a nonempty set, and $m: C E \rightarrow W \times W$ is a mapping such that

$$
\begin{align*}
& m(C) \text { is a contact relation. }  \tag{9.1}\\
& m\left(1^{\prime}\right) \text { is the identity relation on } W .  \tag{9.2}\\
& m \text { is a homomorphism from } C E \text { to }\langle\operatorname{Rel}(W), \cup, \cap,-,,,\rangle . \tag{9.3}
\end{align*}
$$

Table 21: Decomposition rules


A valuation $v$ is a mapping from $V I$ to $W$. If $x R y$ is a formula, then I say that $M$ satisfies $x R y$ under $v$, written as $M, v \models x R y$, if $\langle v(x), v(y)\rangle \in m(R) . x R y$ is called true in the model $M$, if $M, v \models x R y$ for all valuations $v$, i.e. if $m(R)=W^{2} . x R y$ is called valid, if it is true in all models.

Proofs have the form of trees: Given a formula $x R y$, we successively apply decomposition or specific rules; in this way we obtain a tree whose root is $x R y$, and whose nodes consist of sequences of formulas. A branch of a tree is closed if it contains a node which contains an axiomatic sequence as a subsequence. A tree is called closed if all its branches are closed.

Rasiowa-Sikorski (RS) proof systems are, in a way, dual to tableaux systems: Whereas in the latter one tries to refute the negation of a formula, the RS systems attempt to verify a formula by closing the branches of a decomposition tree with axiomatic sequences. Rules in RS systems go in both directions: I call a rule admissible, if

The upper sequence is valid iff the lower sequence(s) is (are) valid.
Here, a sequence of formulas is valid if its meta-level disjunction is valid.
The rules of our system are given in Tables 21 and 22, and the axiomatic sequences are

$$
\begin{align*}
& x R y, x(-R) y  \tag{9.4}\\
& x 1^{\prime} x \tag{9.5}
\end{align*}
$$

where $R \in C E$.
A variable $z$ is called restricted in a rule if it does not occur in the upper part of that rule.
As an example of a derivation, I want to show that the relation $P$ as defined by (6.7) is antisymmetric, i.e. that

$$
P \cap P^{\llcorner } \subseteq 1^{\prime}
$$

Table 22: Specific rules

| (sym 1') | $\frac{K, x 1^{\prime} y, H}{K, y 1^{\prime} x, H}$ |  |
| :---: | :---: | :---: |
| $\left(\operatorname{tran} 1^{\prime}\right)$ | $\frac{K, x 1^{\prime} y, H}{K, x 1^{\prime} z, H, x 1^{\prime} y \mid K, z 1^{\prime} y, H, x 1^{\prime} y},$ | $z$ a variable |
| $\left(1_{1}^{\prime}\right)$ | $\frac{K, x R y, H}{K, x 1^{\prime} z, H, x R y \mid K, z R y, H, x R y},$ | $z$ a variable |
| $\left(1_{2}^{\prime}\right)$ | $\frac{K, x R y, H}{K, x R z, H, x R y \mid K, z 1^{\prime} y, H x R y},$ | $z$ a variable |
| (refl $C$ ) | $\frac{K, x C y, H}{K, x 1^{\prime} y, x C y, H}$ |  |
| $(\operatorname{sym} C)$ | $\frac{K, x C y, H}{K, y C x, H}$ |  |
| (ext $C$ ) | $\frac{K}{K, x(-C) z, y C z\|K, y(-C) t, x C t\| K, x\left(-1^{\prime}\right) y}$ | $z$ and $t$ restricted variables |
| (cut $C$ ) | $\frac{K}{K, x C y \mid K, x(-C) y}$ |  |

that is, by definition of $P$,

$$
(C ;-C) \cup(-C ; C) \cup 1^{\prime}=V .
$$

W use the same symbols as in $\mathcal{L}$ with some abuse of notation. To prove the claim, one must find a closed proof tree for the formula

$$
\begin{equation*}
x\left((C ;-C) \cup(-C ; C) \cup 1^{\prime}\right) y \tag{9.6}
\end{equation*}
$$

Applying rule ( $\cup$ ) to (9.6) and again to the resulting formula, we obtain

$$
\begin{equation*}
x(C ;-C) y, x(-C ; C) y, x 1^{\prime} y . \tag{9.7}
\end{equation*}
$$

Rule (ext $C$ ) with $K$ given by (9.7) leads to three branches:

$$
\begin{align*}
& x(C ;-C) y, x(-C ; C) y, x 1^{\prime} y, x(-C) z, y C z  \tag{9.8}\\
& x(C ;-C) y, x(-C ; C) y, x 1^{\prime} y, y(-C) t, x C t  \tag{9.9}\\
& x(C ;-C) y, x(-C ; C) y, x 1^{\prime} y, x\left(-1^{\prime}\right) y \tag{9.10}
\end{align*}
$$

Node (9.10) is closed, and we look at node (9.8). Decomposing $x(C ;-C) y$ with rule (;) gives two more branches:

$$
\begin{align*}
& x C z, x(-C ; C) y, x 1^{\prime} y, x(-C) z, y C z, x(C ;-C) y,  \tag{9.11}\\
& z(-C) y, x(-C ; C) y, x 1^{\prime} y, x(-C) z, y C z, x(C ;-C) y . \tag{9.12}
\end{align*}
$$

Node (9.11) is closed. If we apply rule (sym $C$ ) to $y C z$ in (9.12), we obtain

$$
\begin{equation*}
z(-C) y, x(-C ; C) y, x 1^{\prime} y, x(-C) z, z C y, x(C ;-C) y \tag{9.13}
\end{equation*}
$$

which is closed. Similarly, one shows that (9.9) leads to closed branches as well.
We now have

## Proposition 9.1. 1. All decomposition rules are admissible.

2. All specific rules are admissible.
3. The axiomatic sequences are valid.
4. If a formula is valid then it has a closed proof tree.

## 10 Approximating regions

The final part of these tutorial notes is taken from the forthcoming [31].
It is rarely the case that spatial regions can be determined up to their true boundaries, if, indeed, they have such boundaries; in most cases, we can only observe regions up to a certain granularity. Often, this is a desirable feature, since too much detail can disturb the view, and we will not be able to see the wood for the trees, if our desire is to see the wood.

Having as our basic assumption that regions can (or need to) be observed only approximately, we want to find an operationalisation of the domain of regions, which is broad enough to express the properties which we want to study, and, at the same time, has enough mathematical structure to serve as a reasoning mechanisms without being overly restrictive to our intuition.

I make three model assumptions:

1. The first assumption is that there is a collection of regions each of which can be observed only up to the granularity given by the elements of a set $B$ of crisp or definable regions; this power of observation is expressed by pairs of the form $\langle a, b\rangle, a \leq b$, where $a, b$ are definable regions. In other words, to each (unknown) region $x$ there is is a lower bound $i(x)=a$ and an upper bound $h(x)=b$, both of which are crisp, up to which $x$ is discernible. If $i(x)=h(x)$, then $x$ itself is definable. The pair $\langle i(x), h(x)\rangle$ is called an approximating region.

Figure 15: An approximating region

2. The next assumption is that the domain $B$ of definable regions forms a Boolean algebra $B$. In practice, $B$ will be finite, in particular, complete and atomic, but we will not use this in our theoretical approach.
3. The final assumption is that the bounds $\langle a, b\rangle$ are best possible; in other words, if $x$ is a region approximated by $\langle a, b\rangle$, then

No definable region $c$ with $a \lesseqgtr c$ is a part of $x$,

$$
\begin{equation*}
\text { If } c \lesseqgtr b \text {, then } x \text { overlaps with }-c \text {. } \tag{10.1}
\end{equation*}
$$

This implies that for each approximating region $x=\langle a, b\rangle$ there is a collection $m(x)$ of regions each of which is approximated by $x$, and for which (10.1) and (10.2) hold. Furthermore, if $y$ is an approximating region different from $x$, then $m(x) \cap m(y)=\emptyset$.

These assumptions may seem too strong, but the example below shows that they are fulfilled in an important area of application, namely, screen resolution.

Consider the region $X$ in the Euclidean plane, depicted in Figure 15. We suppose in our example that granularity in the plane is determined by an equivalence relation on the points, the classes of which are the atoms of the Boolean algebra $B$ of definable regions; these are drawn as squares. We can, for example, think of the squares as pixels on a computer screen. The region $X$ can only be discerned up to the bounds given by its lower and upper approximation, each of which is a union of squares, i.e.

$$
\begin{align*}
i(X) & =\{x \in U: \theta x \subseteq X\},  \tag{10.3}\\
h(X) & =\{x \in U: \theta x \cap X \neq \emptyset\} \tag{10.4}
\end{align*}
$$

is the lower, resp. upper approximation of $X$. Here, $\theta x=\{y: x \theta y\}$ is the equivalence class containing $x$. It is obvious that our three model assumptions are fulfilled. This is the rough set approach to data analysis of [72]; similar paradigms have been put forward in the field of spatial reasoning by [22,57, 88, 89]. An up to date introduction to rough set data analysis with many pointers to further reading is [28] ${ }^{3}$.

Our first task is to find appropriate algebraic structures in which our model assumptions can be expressed.

Throughout, $\langle B,+, \cdot,-, 0,1\rangle$ will denote a Boolean algebra (BA); we may think of $B$ as an algebra of definable (or crisp) objects within some domain as mentioned in the introduction. Since we intend to identify approximate objects with pairs of definable objects from below and above, we start by setting ${ }^{4}$

$$
\begin{equation*}
B^{[2]}=\left\{\langle a, b\rangle \in B^{2}: a \leq b\right\} . \tag{10.5}
\end{equation*}
$$

We regard $B^{[2]}$ as a sublattice of $B \times B$, so that

$$
\begin{aligned}
\langle a, b\rangle+\langle c, d\rangle & =\langle a+c, b+d\rangle \\
\langle a, b\rangle \cdot\langle c, d\rangle & =\langle a \cdot c, b \cdot d\rangle
\end{aligned}
$$

Lower and upper approximation are defined by

$$
\begin{aligned}
i(a, b) & =\langle a, a\rangle, \\
h(a, b) & =\langle b, b\rangle .
\end{aligned}
$$

We observe that

$$
\begin{equation*}
h(i(a, b))=i(a, b), i(h(a, b))=h(a, b) . \tag{10.6}
\end{equation*}
$$

We can recover $B$ by identifying $B$ with $\{\langle a, a\rangle: a \in B\}$. Thus, an approximating region is definable, if it is equal to its lower and upper approximation. The operators $i$ and $c$ are a co-normal multiplicative interior, respectively, a normal additive closure operator, i.e. for $x, y \in B^{[2]}$,

$$
\begin{align*}
i(1) & =1  \tag{10.7}\\
x \leq y & \Rightarrow i(x) \leq i(y),  \tag{10.8}\\
i(x) & \leq x,  \tag{10.9}\\
i(i(x)) & =i(x)  \tag{10.10}\\
i(x \cdot y) & =i(x) \cdot i(y), \tag{10.11}
\end{align*}
$$

[^2]and
\[

$$
\begin{align*}
h(0) & =0,  \tag{10.12}\\
x \leq y & \Rightarrow h(x) \leq h(y),  \tag{10.13}\\
x & \leq h(x),  \tag{10.14}\\
h(h(x)) & =h(x),  \tag{10.15}\\
h(x+y) & =h(x)+h(y), \tag{10.16}
\end{align*}
$$
\]

Furthermore, we see that for $x, y \in B^{[2]}$,

$$
\begin{equation*}
i(x)=i(y) \text { and } h(x)=h(y) \text { imply } x=y . \tag{10.17}
\end{equation*}
$$

This expresses the intuition that approximating regions are uniquely determined by their lower and upper bound. The algebra $B^{[2]}$ may be too large for certain situations. It describes the situation that for each $x=\langle a, b\rangle$ with $a \lesseqgtr b$ there are "true" regions which are approximated by $\langle a, b\rangle$; however, this may not be always the case. Thus, a less restrictive notion is required, and we generalise $B^{[2]}$ as follows:

An approximating algebra (AA) $\langle L,+, \cdot, 0,1, i, h\rangle$ is a structure of type $\langle 2,2,0,0,1,1\rangle$ such that for all $x, y \in L$,

$$
\begin{align*}
& \langle L,+, \cdot, 0,1\rangle \text { is a bounded distributive lattice. }  \tag{10.18}\\
& i \text { is a co-normal multiplicative interior operator on } L \text {. }  \tag{10.19}\\
& h \text { is a normal additive closure operator on } L . \tag{10.20}
\end{align*}
$$

$i(h(x))=h(x), h(i(x))=i(x)$.
$i(x)=i(y)$ and $h(x)=h(y)$ imply $x=y$.
Each closed element has a complement.
It is not hard to see that $B^{[2]}$ is an AA, and one can show that each AA is a subalgebra of some $B^{[2]}$ [26]. We will denote by $B(L)$ - or just by $B$ if no confusion can arise - the set of closed elements of $L$. By (10.21), $B$ is also the set of interior elements of $L$.

It has been shown in [31] that the class AA is term equivalent to the class of regular double Stone algebras, which, in turn are equipollent to three valued Łukasiewicz algebras.

Thus far, we have operationalised the notion of approximating region, and the question arises, how we can deal with a contact relation. Our first observation is

Proposition 10.1. On an $A A\langle L, \leq\rangle$ which is not a Boolean algebra there is no compatible proper contact relation.

Since each AA is obtained from its Boolean algebra $B$ of definable elements via the approximation functions, we suppose that we have a contact relation $C$ on $B$, which we want to approximate in a
similar way. It seems natural for $x, y \in L$ to say that

$$
\begin{aligned}
x \text { and } y \text { are certainly connected } & \Longleftrightarrow i(x) C i(y), \\
x \text { and } y \text { are possibly connected } & \Longleftrightarrow h(x) C h(y) .
\end{aligned}
$$

Thus, given $\langle B, C\rangle$, we let

$$
\begin{align*}
& x C^{i} y \Longleftrightarrow i(x) C i(y),  \tag{10.24}\\
& x C^{h} y \Longleftrightarrow h(x) C h(y) . \tag{10.25}
\end{align*}
$$

For formal reasons, however, we do not want to start with $C$ on $B$, but with the approximation relations $C^{i}, C^{h}$ on $L$, whose restriction to $B$ is a contact relation $C$ such that (10.24) and (10.25) hold.

An approximating contact algebra (ACA) is a structure $\left\langle L, i, h, C^{i}, C^{h}\right\rangle$ such that $\langle L, i, h\rangle$ is an AA and

$$
\begin{align*}
C^{i} & =C^{i}, C^{h}=C^{h \nu}  \tag{10.26}\\
1^{\prime} & \subseteq C^{i} \cap C^{h},  \tag{10.27}\\
x C^{i} y & \Longleftrightarrow i(x) C^{i} i(y),  \tag{10.28}\\
x C^{h} y & \Longleftrightarrow h(x) C^{h} h(y),  \tag{10.29}\\
h(x) C^{i} h(y) & \Longleftrightarrow h(x) C^{h} h(y),  \tag{10.30}\\
C^{h}(h(x)) \subseteq C^{h}(h(y)) & \Longleftrightarrow h(x) \leq h(y) . \tag{10.31}
\end{align*}
$$

The following theorem shows that the notion captures our intention:
Proposition 10.2. 1. Let $\left\langle L, i, h, C^{i}, C^{h}\right\rangle$ be an ACA, and set $C=C^{h} \mid B^{2}$. Then, $C$ is a compatible contact relation on $B$, and (10.24) and (10.25) hold. Furthermore, $C^{h} \mid B^{2}=$ $C^{i} \mid B^{2}$.
2. If $\langle L, i, h\rangle$ is an $A A$ and $C$ a contact relation on $B$, then, $C^{i}$ and $C^{h}$, defined by (10.24) and (10.25) satisfy (10.26) - (10.31). Furthermore, $C=C^{h} \mid B^{2}$.

In the sequel, we let $\left\langle L, i, h, C^{i}, C^{h}\right\rangle$ be a generic ACA, and $C=C^{h} \upharpoonright B^{2}$ be the associated contact relation on $B$.

Let

$$
\begin{gather*}
x \theta^{i} y \Longleftrightarrow i(x)=i(y),  \tag{10.32}\\
x \theta^{h} y \Longleftrightarrow h(x)=h(y) . . \tag{10.33}
\end{gather*}
$$

Then, $\theta^{i}$ is an $\langle L, \cdot, 0,1, i\rangle$ congruence, $\theta^{h}$ is an $\langle L,+, 0,1, h\rangle$ congruence, and $\theta^{i} \cap \theta^{h}=1^{\prime}$ by (10.17). The following is easy to prove, observing that $C \in \operatorname{Rel}(B)$ :

Figure 16: Approximate part of


Proposition 10.3. $C^{i}=\theta^{i} ; C ; \theta^{i}$ and $C^{h}=\theta^{h} ; C ; \theta^{h}$.

The question arises, how the ordering $P$ belonging to $C$ can be sensibly approximated, in other words, what does it mean to say that the approximating region $x$ is (approximately) a part of the approximating region $y$ ?

We know that the part-of relation $P$ on $B$ generated by $C$ is the Boolean order. Besides the lattice ordering $\leq$ on $L$ which extends $P$, there are, on first glance, several possibilities to generalise $P$ to $L$ :

$$
\begin{align*}
& x P_{0} y \Longleftrightarrow h(x) \leq i(y),  \tag{10.34}\\
& x P_{1} y \Longleftrightarrow h(x) \leq h(y),  \tag{10.35}\\
& x P_{2} y \Longleftrightarrow i(x) \leq i(y),  \tag{10.36}\\
& x P_{3} y \Longleftrightarrow i(x) \leq h(y) . \tag{10.37}
\end{align*}
$$

A sketch of the $\subseteq$ relationships among these part-of relations is given in Figure 16. Note that $P_{1} \cap P_{2}=$ $\leq$.

Following our assumptions (10.1) and (10.2), we think of lower bound as certainty and upper bound as possibility, where both bounds are best possible. With this in mind, we see that an approximated part-of $P^{a}$ relation on an ACA must satisfy

The restriction of $P^{a}$ to $B$ is equal to $P$.

$$
\begin{equation*}
x P^{a} y \Rightarrow i(x) \leq i(y) \text { and } h(x) \leq h(y) \tag{10.38}
\end{equation*}
$$

While the first part of condition (10.39) is clear, I should explain the second part: If $x$ is approximately a part of $y$, and if $z$ is certainly part of $x$, i.e. part of all regions which $x$ approximates, then $z$ should be certainly part of anything which is approximated by $y$. If $h(x) \nsubseteq h(y)$, then no region approximated by $y$ has a common part with $h(x) \cdot-h(y)$, but every region approximated by $x$ has. Thus, there can be no region in $m(x)$ which is part of some region in $m(y)$.

With these observations, there can be only two suitable ordering relations, namely, the lattice ordering $\leq$, and the ordering $P_{0}$, denoted by $\preceq$, which is defined by

$$
\begin{equation*}
x \preceq y \Longleftrightarrow h(x) \leq i(y) \tag{10.40}
\end{equation*}
$$

The lattice ordering can be interpreted as as "possible part of", while $x \preceq y$ says that anything which $x$ approximates is certainly a part of anything which $y$ approximates.

Proposition 10.4. For all $x, y \in L$,

$$
\begin{equation*}
x \leq y \Longleftrightarrow C^{i}(x) \subseteq C^{i}(y) \text { and } C^{h}(x) \subseteq C^{h}(y) . \tag{10.41}
\end{equation*}
$$

Work on this topic is still in progress. We have implemented a relational proof system for AA, some of whose specific rules are shown in Table 23.

Table 23: Interior operator rules of AA

| (10.42) | $\frac{\Gamma, x 1^{\prime} y, \Delta}{\Gamma, z I x, \Delta, x 1^{\prime} y \mid \Gamma, z H y, x 1^{\prime} y}$ | $z \in V I$, |
| :---: | :---: | :---: |
| (10.43) | $\frac{\Gamma, x I y, \Delta}{\Gamma, x I z, \Delta, x I y}$ | $z \in V I$, |
| (10.44) | $\frac{\Gamma, x \leq y, \Delta}{\Gamma, y I x, \Delta, x \leq y}$ |  |
| (10.45) | $\frac{\Gamma, x \leq y, \Delta}{\Gamma, z I x, \Delta, x \leq y\|\Gamma, x \leq t, \Delta, x \leq y\| \Gamma, t I y, \Delta, x \leq y}$ | $z, t \in V I$ |
| (10.46) | $\frac{\Gamma, x 1^{\prime} y, \Delta}{\Gamma, x R_{I} z, \Delta, x 1^{\prime} y \mid \Gamma, z I y, \Delta, x 1^{\prime} y}$ | $z \in V I$ |
| (10.47) | $\frac{\Gamma,(I \circ M)(x, y, z, t), \Delta}{\Gamma, x I z, \Delta \mid \Gamma, y I t, \Delta}$ |  |

## 11 Summary

I have given an introduction to the theory of algebras of binary relations, and have shown, how the relational calculus can aid in obtaining results in spatial reasoning. I have defined contact relation
algebras, and have given a sound and complete relational proof system for these structures. An investigation of Boolean contact algebras showed that there is exactly one RA which is obtained from classical mereology, and that the region connection calculus has at least 25 disjoint base relations which must be present in any of its models. The standard model for contact structures and any of its atomless Boolean subalgebras are RCC models with contact inherited from the whole space. Finally, I have described a class of algebras which models regions which are only known up to a crisp lower and a crisp upper bound, and have exhibited how contact and part of relations can be sensibly defined under these model assumptions.

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[^0]:    ${ }^{1}$ Available via http://www.math-inst.hu/pub/algebraic-logic/Contents.html

[^1]:    ${ }^{2}$ I would like to draw the reader's attention to Tarski's discussion of invariance in geometry, topology, and logic [85].

[^2]:    ${ }^{3}$ Technical report version available via html://www.infj.ulst.ac.uk/~cccz23/papers/papers.html
    ${ }^{4}$ To the best of my knowledge, the notation $B^{[2]}$ has been introduced by [42]

