Relation algebras and their application in temporal and spatial reasoning

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Abstract

Qualitative temporal and spatial reasoning is in many cases based on binary relations such as *before*, *after*, *starts*, *contains*, *contact*, *part of*, and others derived from these by relational operators. The calculus of relation algebras is an equational formalism; it tells us which relations must exist, given several basic operations, such as Boolean operations on relations, relational composition and converse. Each equation in the calculus corresponds to a theorem, and, for a situation where there are only finitely many relations, one can construct a *composition table* which can serve as a look up table for the relations involved. Since the calculus handles relations, no knowledge about the concrete geometrical objects is necessary. In this sense, relational calculus is "pointless".
Relation algebras were introduced into temporal reasoning by Allen [1] and into spatial reasoning by Egenhofer and Sharma [32]. The calculus of relation algebras is also well suited to handle binary constraints as demonstrated e.g. by Ladkin and Maddux [55]. In the present paper I will give an introduction to relation algebras, and an overview of their role in qualitative temporal and spatial reasoning.

1 Introduction

1.1 Qualitative temporal and spatial reasoning

Qualitative temporal and spatial reasoning (QTSR) aims to express non–numerical relationships among temporal and spatial objects. The basis for qualitative temporal reasoning (QTR) are relations between points in time or relations between time intervals, such as

- Points in time – *"before", "after", "at the same time"*. Situations different from linear time have also been considered, for example, when the past is fixed, but the future is indeterministic, see e.g. Figure 4.
- Time intervals – *"meets", "starts", "ends"*, see Figure 5,

Qualitative spatial reasoning (QSR) expresses relationships between regions, for example, in physical space, such as

- Regions in the plane – *"part of", "in external contact", "overlapping"*, see Figure 6,

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- Cardinal directions – *“North”, “East”, ...*.

Except for points in time, the basic entities of QTSR are sets of individual points, such as time intervals or regions in space. Points are now second order definable as sets of regions, similar to the representation of Boolean algebras, where points can be recovered as sets of ultrafilters. As pointed out by Gerla [38], as early as 1835, Lobachevskij [64] already exhibited a “pointless” geometry by considering solids as the basic entities and the relation of “contact” among them.

The formalization of the “part – of” relationship, together with the notion of “fusion”, goes back to the mereological systems of Stanisław Leśniewski, developed from 1915 onwards [59, 60, 65, 92]. It may be worthy of mention that Tarski was Leśniewski's doctoral student, indeed, his only one.

Based on earlier work of de Laguna [16], Whitehead [97] uses a relation “x is extensionally connected with y” as the basic relation between regions. His system includes Leśniewski's mereology, and was not formalized. Later, Grzegorczyk [39] and Clarke [14] gave axiomatizations of Whitehead's contact. More recent spatial calculi include the Region Connection Calculus of Randell et al. [86] and the Occlusion Calculus of Randell et al. [85] and Köhler [51].

Nowadays, “mereology” has almost become synonymous in the QSR community with the study of spatial relations in appropriate domains. Since the frequently studied models of the structures are interpreted in topological spaces, one also speaks of “mereotopology”, in particular, when topological properties of regions such as “connected” or “convex” are considered [see e.g. 7, 82, 83].

1.2 Relation algebras

Why would relation algebras be interesting to researchers in the area of temporal or spatial reasoning? As we have seen in the previous Section, a large part of (no pun intended) contemporary spatial reasoning is based on the investigations of the properties of “part of” relations and their extensions to “contact relations” in various domains.

The relational calculus (which should not be confused with relational algebra used in the theory of relational databases) is an equational formalism; it tells us which relations must exist, given several basic operations, such as Boolean operations on relations, relational composition and converse. Each equation in the calculus corresponds to a theorem, and, for a situation where there are only finitely many relations, one can construct a *composition table* (defined below) which can serve as a look up table for the relations involved. Since the calculus handles relations, no knowledge about the object domain is necessary. In this sense, relational calculus is truly “pointless”.

Relations and their algebras have been studied since the latter half of the last century, e.g. by de Morgan [17], Peirce [81] and Schröder [90]. In the seminal paper “On the calculus of relations” [93], Tarski picked up where Schröder had left off forty five years earlier. He gave two axiomatizations of a theory of binary relations, one in the style of Hilbert and Ackermann, and one as an equational formalism. At the end of this paper, Tarski raises some questions in the solution of which he would be engaged for the rest of his life:

1. Is every model of the axiom system of the calculus of relations isomorphic to an algebra of binary relations?
2. What is the expressive power of the calculus of relations? To what extent can this calculus provide a framework for the development of first order logic or, indeed, Mathematics?
3. Is there a decision procedure for expressible first order sentences?

Tarski had proved in the late 1940s that set theory and number theory could be formulated in the calculus of relation algebras:

> "It has even been shown that every statement from a given set of axioms can be reduced to the problem of whether an equation is identically satisfied in every relation algebra. One could thus say that, in principle, the whole of mathematical research can be carried out by studying identities in the arithmetic of relation algebras". [13]

A full account of this appeared for the first time in 1987 after Tarski's death [95].

Relation algebras were introduced into temporal reasoning by Allen [1] and into spatial reasoning by Egenhofer and Sharma [32], and Egenhofer [28] writes

> "Spatial databases will benefit from the composition table of topological relations if it is applied during data acquisition to integrate independently collected topological information and to derive new topological knowledge ; to detect consistency violations among spatial data about some otherwise non–evident topological facts; or during query processing, when spatial queries are less expensive to be executed or involve less objects."

For a brief overview of the history of relation algebras and algebraic logic, I invite the reader to consult [6] and also [72]. An introduction to relation algebras is given in [71], and for a more complete treatment the reader is invited to consult the excellent book by Hirsch and Hodkinson [45].

This paper is organized as follows: After a brief introduction to binary relations and their algebras in Section 2, I shall present several algebras of relations which have been studied in the context of temporal–spatio reasoning. In Section 3 I will introduce abstract relation algebras (RAs) and present their structural properties as well as their connection to algebras of binary relations. This will be followed in Section 4 by a discussion of the expressiveness of algebras of relations. Section 5 will be devoted to the question which relations must be necessarily present in any RCC model. I will also present a countable model for the RCC calculus. Finally, Section 6 will be concerned with network satisfaction problems in the relation algebraic context.

2 Binary relations and their algebras

A *binary relation* R on a set U is a subset of $U \times U$, i.e. a set of ordered pairs $\langle x, y \rangle$ where $x, y \in U$.

I shall usually just speak of R as a *relation*, and instead of $\langle x, y \rangle \in R$, I shall usually write xRy. The collection of all binary relations on U is denoted by $Rel(U)$. The smallest relation on U is the empty relation, and the largest one the universal relation $U \times U$ which I will denote by V.

A pictorial representation of the fact that xRy can be given by drawing an arrow from x to y, which is labeled R:

$$x \xrightarrow{R} y$$

Each relation R induces a mapping $U \to 2^U$ via the assignment $x \mapsto \{y : xRy\}$. With some abuse of notation I will write $R(x)$ for this set.

We are going to introduce two operators on relations and a constant, namely, *composition, converse*, and *identity*. The *composition* or *relative multiplication* $R \circ S$ of two relations is defined as

$$R \circ S = \{\langle x, y \rangle : (\exists z)[xRz \text{ and } zSy]\} \tag{2.1}$$

$$\begin{tikzcd} & z \arrow[dr, "S"] & \\ x \arrow[ur, "R"] \arrow[rr, "R \circ S"] & & y \end{tikzcd}$$

Observe that $x(R \circ S)y$ implies the existence of some z with xRz and zSy. This is sometimes called *existential import* [9].

Another distinguished unary operator is *relational converse* or just *converse*:

$$R^{\smile} = \{\langle y, x \rangle : xRy\}. \tag{2.2}$$

The structural interplay between $\circ$ and $^{\smile}$ is given by

Proposition 2.1. *$^{\smile}$ is an involution on the semigroup $\langle Rel(U), \circ \rangle$, i.e.*

1. *$^{\smile}$ is bijective and of order two, i.e. $x^{\smile\smile} = x$.*
2. *$(R \circ S)^{\smile} = S^{\smile} \circ R^{\smile}$ for all $R, S \in Rel(U)$.*

The *identity relation* $\{\langle x, x \rangle : x \in U\}$ will be denoted by $1'$, and its complement by $0'$. Then,

Proposition 2.2. *$\langle Rel(U), \circ, 1' \rangle$ is a monoid, i.e.*

1. *$\circ$ is associative.*
2. *$1' \circ R = R \circ 1' = R$ for all $R \in Rel(U)$.*

Besides the semigroup structure $\langle Rel(U), \circ, 1' \rangle$, the collection of all relations on U is a Boolean algebra $\langle Rel(U), \cap, \cup, -, \emptyset, U^2, \rangle$ under the set operations. Combining these structures we obtain the *full algebra of binary relations on U* as $\langle Rel(U), \cap, \cup, -, \emptyset, U^2, \circ, {}^{\smile}, 1' \rangle$. If $A \subseteq Rel(U)$ is closed under the distinguished operations and contains the constants, then we call it an *algebra of binary relations* (BRA).

I shall usually identify algebras with their base set, and, with some abuse of notation, I will also denote classes of algebras by the abbreviation of their type, e.g. BRA is also the class of all algebras of binary relations. If A is an algebra and B is a subalgebra of A, I denote this by writing $B \leq A$.

For $\mathcal{R} \subseteq Rel(U)$, let $[\mathcal{R}]$ be the subalgebra of $Rel(U)$ generated by $\mathcal{R}$. In other words, $[\mathcal{R}]$ is the smallest subset of $Rel(U)$ which is closed under the Boolean and relational operators, and contains the constants.

Many properties of relations can be expressed by equations (or inclusions) among relations. Here are a few examples:

$$\begin{aligned}
R \text{ is reflexive} &\iff (\forall x) xRx, \\
&\iff 1' \subseteq R. \\
R \text{ is symmetric} &\iff (\forall x, y)[xRy \iff yRx], \\
&\iff R = R^{\smile}. \\
R \text{ is transitive} &\iff (\forall x, y, z)[xRy \wedge yRz \implies xRz], \\
&\iff R \circ R \subseteq R. \\
R \text{ is dense} &\iff (\forall x) x(-R)x \wedge (\forall x, y)[xRy \implies (\exists z) xRzRy], \\
&\iff R \cap 1' = \emptyset \wedge R \subseteq R \circ R, \\
&\iff R \cap (1' \cup -(R \circ R)) = \emptyset. \\
R \text{ is extensional} &\iff (\forall x, y)[R(x) = R(y) \implies x = y], \\
&\iff [-(R \circ -R^{\smile}) \cap -(R^{\smile} \circ -R)] \subseteq 1'.
\end{aligned}$$

As the last equivalence is not completely trivial and will be relevant for things to come, I will give a proof:

"$\Longrightarrow$": Suppose that R is extensional and $x[-(R \circ -R^{\smile}) \cap -(R^{\smile} \circ -R)]y$; we need to show that $x1'y$, i.e. that $x = y$. Assume w.l.o.g. that $R(x) \not\subseteq R(y)$; then there is some z such that xRz and $y(-R)z$, i.e. $xR \circ -R^{\smile}y$, contradicting our hypothesis. Therefore, $R(x) = R(y)$, and the extensionality of R implies that $x = y$.

"$\Longleftarrow$": Suppose that the right hand side holds and that $R(x) = R(y)$. Assume that $x \neq y$. Then, $xR \circ -R^{\smile}y$ or $xR^{\smile} \circ -Ry$. In the first case, there is some z such that xRz and $z - R^{\smile}y$, i.e. xRz and $y(-R)z$. This contradicts $R(x) = R(y)$. Similarly, the second case cannot hold, and it follows that $x = y$. □

If A is a finite BRA, then, as a Boolean algebra, it is atomic, i.e.

$$(\forall b \in A)[b \neq 0 \Longrightarrow (\exists a \in A)[(0 \lneq a \leq b) \wedge (\forall c)[c \leq a \Longrightarrow c = 0 \vee c = a]]].$$

In this case, the actions of the Boolean operators are uniquely determined by the atoms. To determine the structure of A it is therefore enough to specify the relative multiplication and the converse operation. This is usually done in a *composition table* (CT), which is a quadratic array. Rows and columns are labeled with the atoms of A, and the cells contain sets of atoms (with the brackets usually omitted).

An entry $T_0, \dots, T_k$ in cell $\langle R, S \rangle$ of such a table means that $R \circ S = T_0 \cup \dots \cup T_k$. Such an entry leads to two kinds of theorem:

$$(\forall x, y, z)[xRz \wedge zSy \Longrightarrow xT_0y \vee \dots \vee xT_ky], \tag{2.3}$$

$$(\forall x, y)[xT_iy \Longrightarrow (\exists z)xRz \wedge zSy]. \tag{2.4}$$

If $1'$ is an atom of A, we call the algebra *integral*[1]. In this case, $1'$ need not be listed in the composition table, since it is clear how composition with $1'$ works. Observe that the converse of an atom is again an atom, and that each atom either is contained in $1'$ or disjoint from it; thus, in a relational representation of a BRA, we can obtain the converse of R by looking for the unique element Q of the table for which $(R \circ Q) \cap 1' \neq \emptyset$.

I need to mention another form of relational composition which has appeared in the literature [86], and which works only in the "$\Longrightarrow$" direction given in (2.3). I will call this *weak composition* to distinguish it from the "true" relational composition: Suppose that $\mathcal{R}$ is a set of relations on U, and $R, S \in \mathcal{R}$. Then,

$$R \circ_w S = \bigcup \{T \in \mathcal{R} : (R \circ S) \cap T \neq \emptyset\}. \tag{2.5}$$

A weak composition table (WCT) contains thus in $\langle R, S \rangle$ all relations $T \in \mathcal{R}$ for which $(R \circ S) \cap T \neq \emptyset$, and

$$x(R \circ_w S)y \Longleftrightarrow (\exists T \in \mathcal{R})[xTy \text{ and } (R \circ S) \cap T \neq \emptyset]. \tag{2.6}$$

I will call a WCT T *extensional for* A, if there is a BRA A whose composition table is T. We will see that a WCT can be non-extensional for some structures and extensional for others. It is of prime importance to find out whether a WCT can be the "real" composition table of a BRA, and, if so, what the relations look like. Necessary and sufficient conditions for a WCT to be the composition table of a relation algebra are given in Proposition 3.1.

2.1 Examples

Before I consider examples from the area of temporal and spatial reasoning, I want to present a composition table with different interpretations. Let S be the disjoint union of a K_3 and a K_4 on a seven element set U.

Figure 1: The relation S, first version

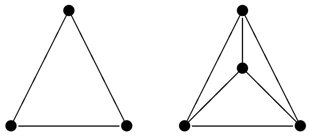
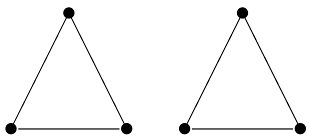


Table 1: The BRA $\mathcal{A}$

$\circ$	S	T
S	$-T$	T
T	T	$-T$

S generates a BRA $\mathcal{A}$ on U with atoms $S, T, 1'$ and the composition shown in Table 1. Note that T is the complement of $S \cup 1'$. If S is the relation shown in Figure 2, then the table of the BRA generated by S will also be given by Table 1. This shows that different BRAs can have the same algebraic structure, and that, in general, the algebraic structure of a BRA is too weak to determine the size of the base set or what the relations look like. Nevertheless, the expressive power of BRA can be surprisingly strong. We shall return to this theme in Section 4.

Figure 2: The relation S, second version



At the other end of the spectrum is the BRA generated by the relations shown in Figure 3.

The relation R is the pentagon, S is the pentagram, and together they generate the pentagonal algebra $\mathcal{P}$ whose composition is given in Table 2. It can be shown that every BRA with such a table must be defined on a base set U with five elements, say $U = \{0, 1, 2, 3, 4\}$, and consist of the relations given in Figure 3 [4]. Thus,

$$xRy \iff |x-y| \mod 5 \in \{1,4\},\ xSy \iff |x-y| \mod 5 \in \{2,3\}$$

It may be worthy to mention that, unlike the class of Boolean algebras, the class of BRAs is not locally finite, i.e. there are finitely generated BRAs which are infinite – take as an example the relation R between natural numbers where $nRm \iff |n-m| = 1$.

2.1.1 The point algebra

Our first temporal example is an algebra generated by the dense linear flow of time without beginning or end: Let U be the set $\mathbb{Q}$ of rational numbers, and let $\mathcal{P}t \leq Rel(U)$ be generated by the natural strict ordering PP on $\mathbb{Q}$. If $xPPy$, this can be interpreted as

[1] I will give another, but equivalent, definition in Section 3.

Figure 3: Pentagon and pentagram

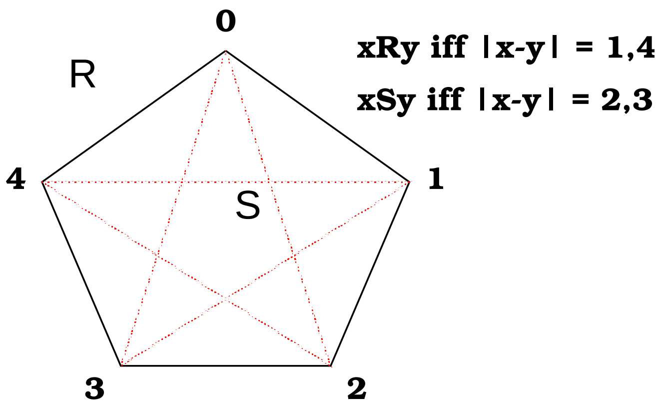


Table 2: The pentagonal algebra

$\circ$	R	S
R	$1', S$	$0'$
S	$0'$	$1', R$

Time point x occurs earlier than time point y.

The resulting relation algebra, called the *point algebra*, has the three atoms $PP, PP^{\smile}, 1'$ and composition as in Table 3.

Table 3: The point algebra $\mathcal{P}t$

$\circ$	PP	$PP^{\smile}$
PP	PP	V
$PP^{\smile}$	V	$PP^{\smile}$

Any representation of $\mathcal{P}t$ must be on an infinite set: Since $PP \cap PP^{\smile} = \emptyset$, we see that PP is asymmetric. The fact that $PP \circ PP = PP$ tells us two things:

1. PP is transitive, since $PP \circ PP \subseteq PP$.
2. PP is dense, since $PP \subseteq PP \circ PP$, i.e. between two different elements of U there is a third one:

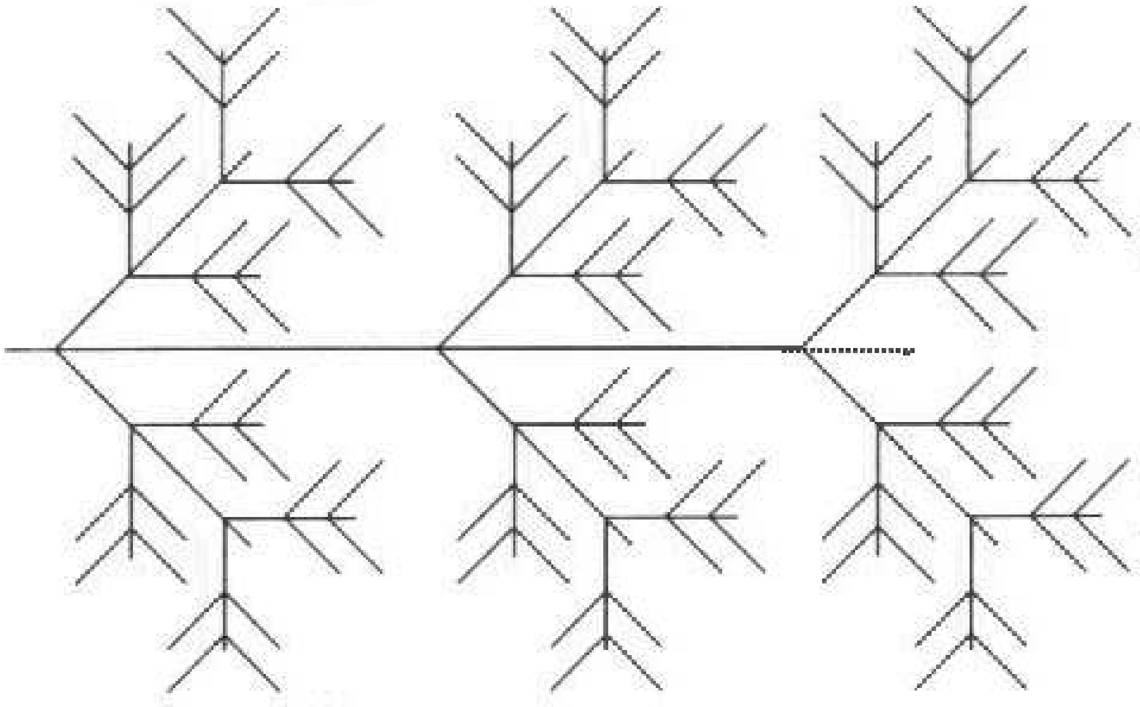
$$(\forall x, y)[xPPy \implies (\exists z)(xPPz \text{ and } zPPy)].$$

This implies that U is infinite.

2.1.2 The left linear point algebra

The representation of the point algebra given above was generated by a dense linear order which can be thought of as deterministic instance of time. This can be generalized to an indeterministic scenario: Consider the ordering $\prec$ indicated in Figure 4; it is a strict dense partial order without endpoints and

Figure 4: An ordering for $\mathcal{L}$



densely branching. Thus, at any given point there are infinitely many choices for the future. Furthermore, $\prec$ induces a sup – semilattice, and it is linearly ordered looking left into the past. These two properties can be interpreted in the sense that two events have a common ancestor, and that the past is uniquely determined. The BRA $\mathcal{L}$ generated by $\prec$ has four atoms, and its composition is given in Table 4; there we set $PP = \prec$. $\mathcal{L}$ was introduced by Comer [15], and the first concrete representation of $\mathcal{L}$ was given in [19].

Table 4: The algebra $\mathcal{L}$

$\circ$	PP	$PP^{\smile}$	DC
PP	PP	1	DC
$PP^{\smile}$	$-DC$	$PP^{\smile}$	$PP^{\smile}, DC$
DC	PP, DC	DC	1

2.1.3 The interval algebra

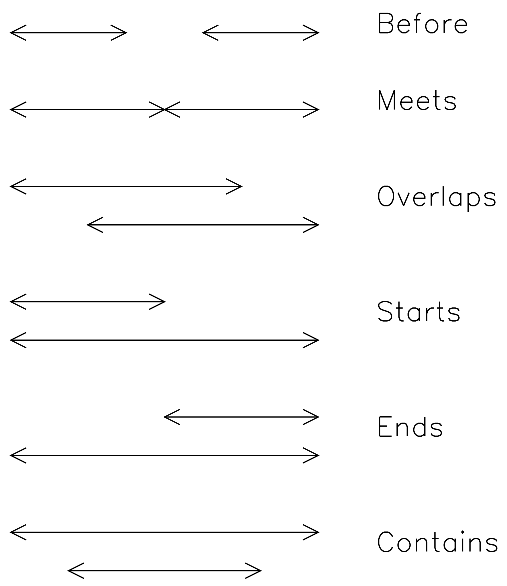
The most widely investigated relation algebra in the temporal context was given by Allen [1]. He has presented 13 distinguished relations, which characterize the possible relations between closed intervals of time. These are the six relations indicated in Figure 5, their converses, and the identity:

Together with their converses, they are the atoms of an integral BRA $\mathcal{I}$ on the set of all closed intervals on the real line; its composition table can be found e.g. in [55]. Observe that this model is "pointless" in the sense that the basic object in the ontology of time is the interval, as opposed to a point.

Table 5: Interval relations

before:	$\{\langle [q,r],[q',r']\rangle : q < r < q' < r', q,r,q',r' \in \mathbb{R}\}$
meets:	$\{\langle [q,r],[q',r']\rangle : q < r = q' < r', q,r,q',r' \in \mathbb{R}\}$
overlaps:	$\{\langle [q,r],[q',r']\rangle : q < q' < r < r', q,r,q',r' \in \mathbb{R}\}$
starts:	$\{\langle [q,r],[q',r']\rangle : q = q' < r < r', q,r,q',r' \in \mathbb{R}\}$
ends:	$\{\langle [q,r],[q',r']\rangle : q' < q < r = r', q,r,q',r' \in \mathbb{R}\}$
contains:	$\{\langle [q,r],[q',r']\rangle : q < q' < r' < r, q,r,q',r' \in \mathbb{R}\}$

Figure 5: Interval relations



2.1.4 Compass algebras

Compass algebras were introduced by Maddux [70, 73]. The domain of the relations are the points in an n–dimensional Euclidean plane $\mathbb{R}^n$. For each element $\vec{v}$ of $\mathcal{R}^n$ we define two relations

$$D_{\vec{v}} = \{\langle \vec{x}, \vec{y}\rangle \in \mathcal{R}^n : (\exists r \in \mathcal{R}^+)[\vec{x} + r \cdot \vec{v} = \vec{y}]\}, \tag{2.7}$$

$$E_{\vec{v}} = \{\langle \vec{x}, \vec{y}\rangle \in \mathcal{R}^n : (\exists r \in \mathcal{R})[\vec{x} + r \cdot \vec{v} = \vec{y}]\}. \tag{2.8}$$

Here, $\mathcal{R}^+$ is the set of all positive real numbers. Observe that $E_{\vec{v}} = D_{\vec{v}} \circ D_{\vec{v}}{}^{\smile}$. If $\vec{v_1}, \dots \vec{v_m} \in \mathcal{R}^n$, then $\mathfrak{C}_n[\vec{v_1}, \dots \vec{v_m}]$ is defined as the subalgebra of the full algebra of relations over $\mathcal{R}^n$ generated by $D_{\vec{v_1}}, \dots D_{\vec{v_m}}$. It is called the *n–dimensional compass algebra determined by* $\vec{v_1}, \dots \vec{v_m}$. The name for these algebras comes from the 2–dimensional compass algebra determined by $\langle 1, 0\rangle, \langle 0, 1\rangle$. One can interpret $E_{\langle 1,0\rangle}$ as the "east–west" direction, and $E_{\langle 0,1\rangle}$ as the "north–south" direction. Other directions are

$$
\begin{array}{rlrll}
1' = D_{\langle 0,0\rangle} & = \text{identity}, & & & \\
a = D_{\langle 1,0\rangle} & = \text{east}, & b & = a \circ c & = \text{northeasterly} \\
a^{\smile} = D_{\langle -1,0\rangle} & = \text{west}, & b^{\smile} & = a^{\smile} \circ c^{\smile} & = \text{southwesterly} \\
c = D_{\langle 0,1\rangle} & = \text{north} & d & = a^{\smile} \circ c & = \text{northwesterly} \\
c^{\smile} = D_{\langle 0,-1\rangle} & = \text{south}, & d^{\smile} & = c^{\smile} \circ a & = \text{southeasterly}.
\end{array}
$$

These relations are the atoms of an integral RA whose composition table is given in Table 6.

Table 6: Composition table for the compass algebra $\mathfrak{C}_2[\langle 1,0\rangle, \langle 0,1\rangle]$

$\circ$	a	$a^{\smile}$	b	$b^{\smile}$	c	$c^{\smile}$	d	$d^{\smile}$
a	a	$1', a, a^{\smile}$	b	$b^{\smile}, c^{\smile}, d^{\smile}$	b	$d^{\smile}$	b, c, d	$d^{\smile}$
b	b	b, c, d	b	1	b	$a, b, d^{\smile}$	b, c, d	$a, b, d^{\smile}$
c	b	d	b	$d, a^{\smile}, b^{\smile}$	c	$1', c, c^{\smile}$	d	$a, b, d^{\smile}$
d	b, c, d	d	b, c, d	$d, a^{\smile}, b^{\smile}$	d	$d, a^{\smile}, b^{\smile}$	d	1
$a^{\smile}$	$1', a, a^{\smile}$	$a^{\smile}$	b, c, d	$b^{\smile}$	d	$b^{\smile}$	d	$b^{\smile}, c^{\smile}, d^{\smile}$
$b^{\smile}$	$b^{\smile}, c^{\smile}, d^{\smile}$	$b^{\smile}$	1	$b^{\smile}$	$d, a^{\smile}, b^{\smile}$	$b^{\smile}$	$d, a^{\smile}, b^{\smile}$	$b^{\smile}, c^{\smile}, d^{\smile}$
$c^{\smile}$	$d^{\smile}$	$b^{\smile}$	$a, b, d^{\smile}$	$b^{\smile}$	$1', c, c^{\smile}$	$c^{\smile}$	$d, a^{\smile}, b^{\smile}$	$d^{\smile}$
$d^{\smile}$	$d^{\smile}$	$b^{\smile}, c^{\smile}, d^{\smile}$	$a, b, d^{\smile}$	$b^{\smile}, c^{\smile}, d^{\smile}$	$a, b, d^{\smile}$	$d^{\smile}$	1	$d^{\smile}$

An application based on a spatial planning problem is given in [70]. Cardinal directions were also investigated by Frank [37].

2.1.5 The containment algebra

An "extension" of the point algebra to the two–dimensional case is the *containment algebra* $\mathcal{C}$. Suppose that U is the collection of all open (or closed) disks in the plane,and that $xPPy \iff x \subsetneq y$. The BRA $\mathcal{C}$ generated by PP on $Rel(U)$ contains five atoms, and its composition is given in Table 7. According to Ladkin and Maddux [55], $\mathcal{C}$ was first investigated by Ladkin & Hayes in the early 1980s; it has later re–appeared as the "RCC5 table" in [10]. To see how the relations other than PP and $PP^{\smile}$ are generated

Table 7: The containment algebra $\mathcal{C}$

$\circ$	PP	$PP^{\smile}$	PO	DC
PP	PP	V	PP, PO, DC	DC
$PP^{\smile}$	$-DC$	$PP^{\smile}$	$PP^{\smile}, PO$	$PP^{\smile}, PO, DC$
PO	PP, PO	$PP^{\smile}, PO, DC$	V	$PP^{\smile}, PO, DC$
DC	PP, PO, DC	DC	PP, PO, DC	V

let us first define several auxiliary relations which we will need throughout the sequel:

$$P = PP \cup 1' \qquad \text{part–of} \qquad (2.9)$$
$$O = P^{\smile} \circ P \qquad \text{overlap} \qquad (2.10)$$
$$\# = -(P \cup P^{\smile}) \qquad \text{incomparable} \qquad (2.11)$$

The relations are generated as follows:

$$PO = O \setminus \#,$$
$$DC = V \setminus (O \cup P \cup P^{\smile}).$$

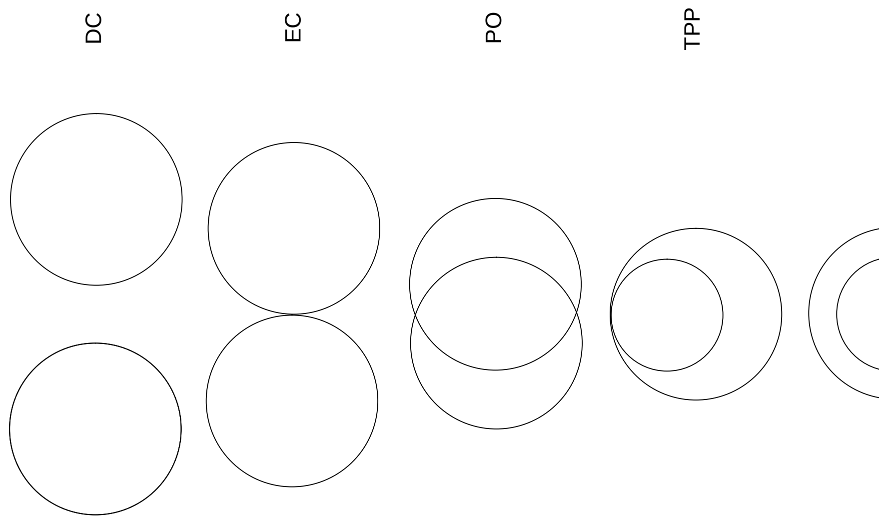
2.1.6 The closed disk algebra

Extending the point algebra to two dimensions led to the containment algebra. If we want to extend the time interval relations to two dimensional space, a natural domain to choose is the set U of closed (or open) disks. In this domain, we do not have the unique directions "left - right" of the real line any more, and thus, for example, we cannot distinguish between the "starts" and the "ends" relations, and between the "before" relation and its converse. In this spirit, we define a binary relation C on U by $xCy \iff x \cap y \neq \emptyset$. The BRA generated by C on U has the eight atoms

$$1', DC, EC, PO, NTPP, NTPP^{\smile}, TPP, TPP^{\smile}, \qquad (2.12)$$

some of which are shown in Figure 6.

Figure 6: Closed disk relations



The relational definition of these and some other relations are given in Table 8, and its relational composition is given in Table 9. It has recently been shown that the collection of simple regions in the Euclidean plane – put forward by Egenhofer [26] as a domain for spatial reasoning – is also a representation of this algebra [61].

The composition table of $\mathcal{D}_c$ has been considered by Egenhofer and Sharma [33] and Smith and Park [91] in the context of binary topological relations, and is also known as the RCC8 table [86], see Section 5. The algebra $\mathcal{D}_c$ (as an abstract relation algebra to be explained below) is isomorphic to the subalgebra of $\mathcal{I}$ generated by the union of the "before" relation and its converse.

Table 8: Relational definition of the atoms of $\mathcal{D}_c$

$P = -(C \circ -C),$	part of	(2.13)
$PP = P \cap -1'.$	proper part of	(2.14)
$O = P^{\smile} \circ P$	overlap	(2.15)
$PO = O \cap -(P \cup P^{\smile})$	partial overlap	(2.16)
$EC = C \cap -O$	external contact	(2.17)
$TPP = PP \cap (EC \circ EC)$	tangential proper part	(2.18)
$NTPP = PP \cap -TPP$	non–tangential proper part	(2.19)
$DC = -C$	disconnected	(2.20)

Table 9: The composition table of $\mathcal{D}_c$

$\circ$		C					
	DR		O				
				PP		$PP^{\smile}$	
	DC	EC	PO	TPP	$NTPP$	$TPP^{\smile}$	$NTPP^{\smile}$
DC	1	DR,PO,PP	DR,PO,PP	DR,PO,PP	DR,PO,PP	DC	DC
EC	DR,PO,$PP^{\smile}$	1',DR,PO, TPP $TPP^{\smile}$	DR,PO,PP	EC,PO,PP	PO,PP	DR	DC
PO	DR,PO,$PP^{\smile}$	DR,PO,$PP^{\smile}$	1	PO,PP	PO,PP	DR,PO,$PP^{\smile}$	DR,PO,$PP^{\smile}$
TPP	DC	DR	DR,PO,PP	PP	NTPP	1',DR,PO, TPP,$TPP^{\smile}$	DR,PO, $PP^{\smile}$
$NTPP$	DC	DC	DR,PO,PP	NTPP	NTPP	DR,PO,PP	1
$TPP^{\smile}$	DR,PO,$PP^{\smile}$	EC,PO,$PP^{\smile}$	PO,$PP^{\smile}$	1',PO, TPP,$TPP^{\smile}$	PO,PP	$PP^{\smile}$	$NTPP^{\smile}$
$NTPP^{\smile}$	DR,PO,$PP^{\smile}$	PO,$PP^{\smile}$	PO,$PP^{\smile}$	PO,$PP^{\smile}$	$O \cup 1'$	$NTPP^{\smile}$	$NTPP^{\smile}$

2.1.7 The complemented closed disk algebra

If we take as a domain all closed disks in the plane along with the closure of their complements, the connection relation C as defined for the closed disk algebra, generates an RA with 11 atoms. Compared to the closed disk algebra, the relation EC splits into ECD and ECN, and PO splits into $PODZ, PODY, PON$. The definitions are given in Table 10. This algebra was first considered in [22](where its composition table can be found), and the representation was discovered by Li et al. [63]. They also point out that these relations can be described by the 9–intersection model of Egenhofer and Herring [30], Egenhofer [34].

2.1.8 The Boolean order algebra

All previous examples disregarded a possible algebraic structure on the universe of the relations. The next example is, in a way, an extension of the open disk model of the containment algebra to domains whose underlying algebraic structure is a Boolean algebra. Suppose that B is an atomless Boolean algebra, $B^+ = B \setminus \{0, 1\}$, and that PP is the relation defined on B^+ by $xPPy \iff x \lneq y$. Furthermore,

Table 10: Complemented closed disk relations

$$\begin{aligned}
ECD &= -(P \circ P^{\smile}) \cap -(P^{\smile} \circ P) \\
ECN &= EC \cap -ECD \\
PODZ &= ECD \circ NTTP \\
PODY &= ECD \circ TPP \\
PON &= PO \cap -(PODZ \cup PODY)
\end{aligned}$$

define the following relations on B^+:

$$\begin{aligned}
T &= (PP \circ PP^{\smile}) \cap -1' && = \{\langle x, z\rangle : x \neq z,\ x + z \neq 1\} \\
PON &= O \cap \# \cap T && = \{\langle x, z\rangle : x\#z,\ x \cdot z \neq 0,\ x + z \neq 1\} \\
POD &= O \cap \# \cap -T && = \{\langle x, z\rangle : x\#z,\ x \cdot z \neq 0,\ x + z = 1\} \\
DN &= -O \cap T && = \{\langle x, z\rangle : x \cdot z = 0,\ x + z \neq 1\} \\
DD &= -(O \cup T \cup 1') && = \{\langle x, z\rangle : x \cdot z = 0,\ x + z = 1\},
\end{aligned}$$

where $x, z \in B^+$. Then, PP generates a BRA $\mathcal{G}$ with atoms

$$1',\ PP,\ PP^{\smile},\ PON,\ POD,\ DN,\ DD$$

whose composition table is given in Table 11.

Table 11: The Boolean order algebra $\mathcal{G}$

$\circ$	DR		O			
	DN	DD	PON	POD	PP	$PP^{\smile}$
DN	$-(POD \cup DD)$	PP	DN, PON, PP	PP	$-(P^{\smile} \cup 1')$	DN
DD	$PP^{\smile}$	$1'$	PON	PP	POD	DN
PON	DN, PON, $PP^{\smile}$	PON	1	PON, POD, PP	PON, POD, PP	DN, PON, $PP^{\smile}$
POD	$PP^{\smile}$	$PP^{\smile}$	PON, POD, $PP^{\smile}$	$O \cup 1'$	POD	$-(PP \cup 1')$
PP	DN	DN	DN, PON, PP	$-(PP \cup 1')$	PP	$-(POD \cup DD)$
$PP^{\smile}$	$-(PP \cup 1')$	POD	PON, POD, $PP^{\smile}$	POD	$O \cup 1'$	$PP^{\smile}$

3 Abstract relation algebras

One of Tarski's aims was to give a formal axiomatization of the calculus of relatives [93]. This led to the definition of the class of *relation algebras.*

A relation algebra (RA)

$$\langle A, +, \cdot, -, 0, 1, \circ, \ ^{\smile}, 1'\rangle$$

is a structure of type $\langle 2, 2, 1, 0, 0, 2, 1, 0\rangle$ which satisfies

(R0). $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra.

(R1). $x \circ (y \circ z) = (x \circ y) \circ z$.

(R2). $(x + y) \circ z = (x \circ z) + (y \circ z)$.

(R3). $x \circ 1' = x$.

(R4). $x^{\smile\smile} = x$.

(R5). $(x + y)^{\smile} = x^{\smile} + y^{\smile}$.

(R6). $(x \circ y)^{\smile} = y^{\smile} \circ x^{\smile}$.

(R7). $(x^{\smile} \circ -(x \circ y)) \leq -y$.

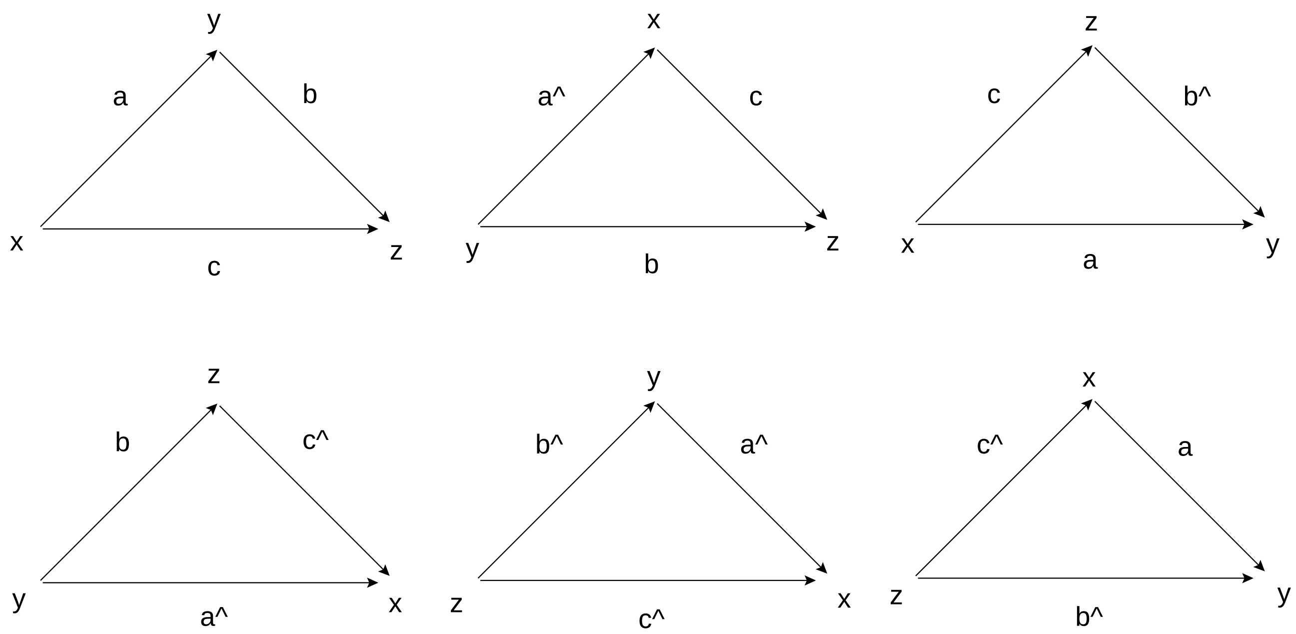
This axiom system is the one given in [40]. With some abuse of language, I will denote the class of relation algebras also by RA.

A decisive property of RA is the following *cycle law*, which is de Morgan's Theorem K [17]:

$$\begin{gathered}\text{The following conditions are equivalent:} \\ (a \circ b) \cdot c \neq 0,\ (a^{\smile} \circ c) \cdot b \neq 0,\ (c \circ b^{\smile}) \cdot a \neq 0.\end{gathered} \tag{3.1}$$

If a, b, c are concrete relations over some domain, (3.1) together with the properties of the converse relation, express the fact that if one of the directed triangles in Figure 7 is satisfable with x, y, z, then so are the others. This simplifies the computation and storage of composition tables, as well as checking

Figure 7: Cycle law for binary relations



for path consistency (as defined below).

The following Proposition makes precise when a composition table is indeed the composition table of a relation algebra:

Proposition 3.1. *[47] Let B be a complete and atomic Boolean algebra with $1' \in B$ a distinguished element, $^{\smile}$ a unary operation on B, and $\circ$ a binary operation, both of which are completely distributive over $+$, and for which $0^{\smile} = 0$ and $(0 \circ x) = (x \circ 0) = 0$. Let $At(B)$ be the set of atoms of B. Then,*

$\langle B, \circ, {}^{\smile}, 1' \rangle$ *is an RA if and only if the following conditions hold for all* $x, y, z \in At(B)$*:*

$$x^{\smile} \in At(B).$$
$$x \circ (y \circ z) \leq (x \circ y) \circ z.$$
$$x \circ 1' = 1'.$$
$$x \leq y \circ z \text{ implies } x^{\smile} \leq z^{\smile} \circ y^{\smile} \text{ and } y \leq x \circ z^{\smile}.$$

These conditions can be used to check whether a given weak composition table is indeed the composition table of a relation algebra. There are several implementations for the arithmetic of relations algebras, e.g. Kahl and Schmidt [50] and Behnke et al. [8].

It is not hard to see that each BRA is an RA; more generally, an RA is called *representable* (RRA) if it is isomorphic to a subalgebra of a direct product of BRAs.

For things to come, I will introduce some more concepts at this stage. In analogy to rings, an RA A is called *integral*, if for all $x, y \in A$,

$$x \circ y = 0 \text{ implies } x = 0 \text{ or } y = 0. \tag{3.2}$$

It is well know that (3.2) is equivalent to

$$1' \text{ is an atom of } A. \tag{3.3}$$

Another concept we require is that of *residuation.* Since $\langle A, \circ, 1' \rangle$ is in general not a group, the equation $a \circ x = b$ does not necessarily have a solution. However, it can be shown that the inequality

$$a \circ x \leq b$$

always has a greatest solution, called the *(right) residual of b by a*, written as $a \diagdown_r b$. The concept of residuation is intimately related to Axiom (3.1) of RAs, cf. Maddux [72], Pratt [84], and also Birkhoff [12].

The residual can be expressed as an RA term in a and b by

$$a \diagdown_r b = -(a^{\smile} \circ -b). \tag{3.4}$$

If $R, S \in Rel(U)$, then the residual is given by the condition

$$x(R \diagdown_r S)y \iff R^{\smile}(x) \subseteq S^{\smile}(y). \tag{3.5}$$

One of the first questions which arose was whether the system (R0) – (R7) captures RRA, i.e. whether each RA is representable. Unfortunately, this is not the case; the first non-representable RA was found by Lyndon [66]. It had 56 atoms and was constructed using projective planes; other examples were subsequently given by [46] and [67]. A non-representable RA A of smallest size was found by McKenzie [74]. It is integral, has four atoms, and is 1-generated; its composition is listed in Table 12.

It is not hard to show that A is not representable: Assume that $a, b, c \in Rel(U)$ for some U; since A is integral, we can assume that $1 = U \times U$. Now,

1. $1' \leq b \circ c$ implies that $c = b^{\smile}$.

2. $1' \cdot b = 0$, and $b \circ b = b$ imply that b is a strict dense partial order. I will sometimes write $\lneq$ for b, and $\leq$ for $b + 1'$.

3. $b \circ c = c \circ b = 1$ imply that for each pair $\langle x, y \rangle$, there are $p, q \in U$ such that $p \leq x, y$ and $x, y \leq q$. .

Table 12: A non-representable RA

$\circ$	b	c	d
b	b	1	$b+d$
c	1	c	$c+d$
d	$b+d$	$c+d$	$b+c+1'$

4. $d \circ d = b + b^{\smile} + 1'$ implies that x, y are comparable if and only if they are incomparable to a third element.

These conditions cannot live together: Suppose that $x, y \in U$ are incomparable, and that $p \leq x, y \leq q$ as provided by 3. above. By 4. there is an $s \in U$ such that s is incomparable to p and x. If s were incomparable to y, then, by the other direction of 4., x would be comparable to y, which is not the case. Hence, s is comparable to y; furthermore, $s \lneq y$, since otherwise, $p \leq y \leq s$. Similarly, there is some $t \in U$, such that $t \lneq x$, and t is incomparable to p and y. Since p is incomparable to both s and t, 4. implies that s and t are comparable. If $s \leq t$, then $s \leq x$, and if $t \leq s$, then $t \leq y$, a contradiction in both cases. □

The following proposition summarizes the structural properties of RRA:

Proposition 3.2. *1. RRA is an equational class [94].*

2. The equational theory of RRA is undecidable [see 95, Section 8.7. for references].

3. RRA is not finitely axiomatisable [75].

4. RRA is not axiomatisable with finitely many variables [48].

As already noted by Tarski, at times the property of associativity of the relational composition is too strong, and weaker properties are considered [69]. A structure similar to RAs is called a

1. *non-associative RA* (NA), if it satisfies (R0) and (R2) – (R7).
2. *weakly associative RA* (WA), if it satisfies (R0) and (R2) – (R7) and
$$((1' \cdot x) \circ 1) \circ 1 = (1' \cdot x) \circ (1 \circ 1). \tag{3.6}$$
3. *semi-associative RA* (SA), if it satisfies (R0) and (R2) – (R7) and
$$(x \circ 1) \circ 1 = x \circ (1 \circ 1). \tag{3.7}$$

It was shown by Maddux [69] that
$$RA \subseteq SA \subseteq WA \subseteq NA.$$

An alternative axiomatization of NA consists of (R0), (R2), (R4), (R5), the identities
$$1' \circ x = x \circ 1' = x,$$
and de Morgan's Theorem K (3.1) [69].

The equational theory of WA is decidable [80]. Moreover, each WA is isomorphic to a subalgebra of an algebra $\langle 2^W, \cup, \cap, -_W, \emptyset, W, \circ_W, {}^{\smile}, 1' \rangle$, where W is a reflexive and symmetric binary relation on a set U, and $x \circ_W y = (x \circ y) \cap W$. It may be intersting to note that the "container–surface" algebra of Egenhofer and Rodríguez [31] is a WA.

For many decidability results for various classes of relation algebras, as well as pointers to earlier work, the reader will find [3] and [53] valuable sources.

4 The expressiveness of BRAs

The question arises what can be expressed by the logic of relation algebras. To answer this question needs some preparation. A first order language consists of predicate symbols, logical connectives $\wedge$, $\neg$, the existential quantifier $\exists$ and equality, and the usual abbreviations. When considering relational structures $\langle U, R\rangle$ as first order models, I tacitly assume that an appropriate first order language L is given. For notational convenience, I shall sometimes identify predicate symbols with the predicates which interpret them, when no confusion is likely to arise.

If $\varphi(x, y)$ is a formula with the free variables x, y, and $\langle U, R\rangle$ is a model of the language L, then the *truth set* of $\varphi(x, y)$ in the model $\langle U, R\rangle$ is the relation

$$\operatorname{def}\varphi(x, y) = \{\langle a, b\rangle \in U^2 : \langle U, R\rangle \models \varphi(x|a, y|b).$$

If $S \subseteq U^2$ and $S = \operatorname{def}\varphi(x, y)$ for some φ, then S is called *definable* in the model $\langle U, R\rangle$. Similarly, we extend this definition to languages with more than one predicate symbol and formulas with other than two free variables. For example, (the result of) relative composition is definable by the formula

$$\varphi(x, y) : \ (\exists z)[xRz \wedge zSy],$$

and the fact that x is R–related to every element is expressed by

$$\varphi(x) : \ (\forall y)xRy.$$

A relation S is *RA definable from* $R_0, \dots, R_k$, if it is in the BRA generated by the R_i. In other words, S is RA definable from the R_i, if it is equal to a relational term constructed from the R_i and the relational operators and constants.

As an example which we will need later, I shall show how extreme elements of an ordered set can be relationally defined. As a consequence of this, when considering relation algebras which contain an order relation, it is enough to look at the base set with the extreme elements removed.

Let $\langle U, P\rangle$ be an ordered set with largest element 1 and smallest element 0; furthermore, set $PP = P \cap 0'$. Let $U_0 = U \setminus \{1\}$, $U_1 = \{1\}$, and $U_{ij} = U_i \times U_j$ for $i, j \leq 1$. We first observe that $\langle 1, 1\rangle \notin PP \circ PP^{\smile}$, since there is no element of U which is strictly greater than 1. On the other hand, $\langle x, x\rangle \in PP \circ PP^{\smile}$ for all $x \neq 1$, since $xPP1PP^{\smile}x$. Thus,

$$U_{11} = 1' \cap -[(PP \circ PP^{\smile}) \cap 1'],$$

defines $\{\langle 1, 1\rangle\}$. Now, set

$$1'_u = 1' \cap -U_{11}.$$

Then,

$$\begin{aligned} U_{00} &= 1'_u \circ U^2 \circ 1'_u, \\ U_{10} &= U_{11} \circ U^2 \circ U_{00}, \\ U_{01} &= U_{00} \circ U^2 \circ U_{11}, \end{aligned}$$

which shows that all U_{ij} and $1'_u$ are RA definable from P. The equation which tells us that 1 is the largest element with respect to P now is

$$U_{01} \subseteq P. \tag{4.1}$$

Similarly, we can RA define $\{0\}$.

The expressiveness of BRAs corresponds to a fragment of first order logic, and the following fundamental result is due to A. Tarski [see 95]:

Proposition 4.1. *If $\mathcal{R} \subseteq Rel(U)$, then $[\mathcal{R}]$ is the set of all binary relations on U which are definable in the (language of the) relational structure $\langle U, \mathcal{R}\rangle$ by first order formulas using at most three variables, two of which are free.*

The question arises: Is this as good as it gets? Let us call a BRA A *first order closed*, if it contains every relation which is first order definable in A, regarded as a relational structure. It is worthy to point out that first order closedness is domain dependent, i.e. it is a property of a concrete relational representations of an (abstract) RA.

For small sets, we have the following

Proposition 4.2. *[2] Every BRA on a set with at most six elements is first order closed.*

Hence, on small sets, RA logic, i.e. the three variable fragment of first order logic, is as powerful as full first order logic. In the sequel, we will meet many other first order closed BRAs.

Look again at the RA of Table 1, and its two representations. The K_3 in the right representation is first order definable by

$$\begin{aligned} \varphi(x,y):\ & xSz \wedge (\forall u)(\forall z)[xRu \wedge yRu \wedge xRz \wedge yRz \\ & \Longrightarrow x = u \vee x = z \vee y = u \vee y = z \vee u = z]. \end{aligned}$$

As a relation, the K_3 is not in the BRA A generated by S_1, and thus, A is not first order closed. On the other hand, the representation of A shown in Figure 2 is first order closed by Proposition 4.2.

In our context, examples of first order closed algebras are

- Any representation of the point algebra over the set of rational numbers.
- The representation of the left linear point algebra [42].
- The interval representation of Allen's algebra $\mathcal{I}$ [42].
- The representation of the Boolean order algebra over all proper nonempty regular closed sets of the Euclidean plane [23].

In all these cases, RA logic is sufficient to describe the first order properties of these structures.

5 RCC algebras

The region connection calculus (RCC) of [86] is one of the widely studied systems of qualitative spatial reasoning. RCC structures are of the form $\langle B, C\rangle$, where B is a Boolean algebra and C is a binary relation on the set $B^+ = B \setminus \{0,1\}$ such that for all $x, y \in B^+$,

RCC 1. C is symmetric and reflexive.

RCC 2. $xC-x$.

RCC 3. $xC(y+z)$ implies xCy or xCz,

RCC 4. xCy and $y \leq z$ imply xCz.

RCC 5. For all $x \in B^+$ there is some $y \in B^+$ such that $\langle x, y\rangle \notin C$.

One prominent model of the RCC is the Boolean algebra of regular open sets $\mathrm{RegOp}(X)$ of a connected regular T_1 space (such as a Euclidean space $\mathbb{R}^n$), with $xCy \iff cl(x) \cap cl(y) \neq \emptyset$ for $x, y \neq \emptyset, X$. A representation Theorem by Düntsch and Winter [25] shows that each RCC model is isomorphic to a dense substructure of $\mathrm{RegOp}(X)$ for a weakly regular connected T_1 space X. Here, a space is weakly regular, if it has a basis of regular open sets, and for each non–empty open u, there is some non–empty open v such that $\mathrm{cl}(v) \subseteq u$. Each regular space is weakly regular, but not vice versa. This result shows two things:

- Algebraic reasoning is equivalent to topological reasoning in the class of weakly regular connected T_1 spaces.
- The RCC calculus, as it stands, is too weak to capture regularity of topological spaces, and therefore not the best instrument for topological reasoning in regular spaces.

Since RCC is a first order theory with infinite models it must have a countable model, and here is one construction of such a model: Suppose that $\langle L, \leq \rangle$ is a countable dense linear order with smallest element m, and that $\infty \notin L$. Let $L^\infty = L \cup \{\infty\}$, and extend the ordering of L to L^∞ by setting $y \leq \infty$ for all $y \in L^\infty$. The set $IntAlg(L)$ of finite unions of left-closed, right-open intervals of L^∞ is an atomless Boolean algebra, called the *interval algebra of L* (see [52], p 10, for details). Each $x \in IntAlg(L)\{m\}$ can be written in the form

$$[x_0^0, x_0^1) \cup [x_1^0, x_1^1) \cup \ldots \cup [x_{t(x)}^0, x_{t(x)}^1),$$

where $x_j^i \in L^+$, $x_j^0 < x_j^1 < x_{j+1}^0$, and the intervals $[x_j^0, x_j^1]$ are pairwise disjoint. These intervals are called *relevant for x*. For each $x \neq \emptyset$, we let

$$CP(x) = \{x_j^0 : j \leq t(x)\} \cup \{x_j^1 : j \leq t(x)\}$$

be the set of *critical points of x*. Now we define for $m \lneq x, y \lneq \infty$,

$$xCy \stackrel{\text{def}}{\Longleftrightarrow} x \cdot y > 0 \text{ or } CP(x) \cap CP(y) \neq \emptyset. \tag{5.1}$$

We now have

Proposition 5.1. *[24] $\langle IntAlg(L), C \rangle$ is an RCC model.*

I will finish the introduction to RCC structures with a construction of RCC models from a given model:

Proposition 5.2. *[24] Let $\langle B, C \rangle$ be an RCC model, F, G be distinct maximal filters of B, and $R = C \cup (F \times G) \cup (G \times F)$. Then, $\langle B, R \rangle$ is an RCC model.*

Seeing that an RCC model is, in particular, a relational structure, the question arises what the relation algebras generated by C look like on various models. It was already noted in [86] that the relations

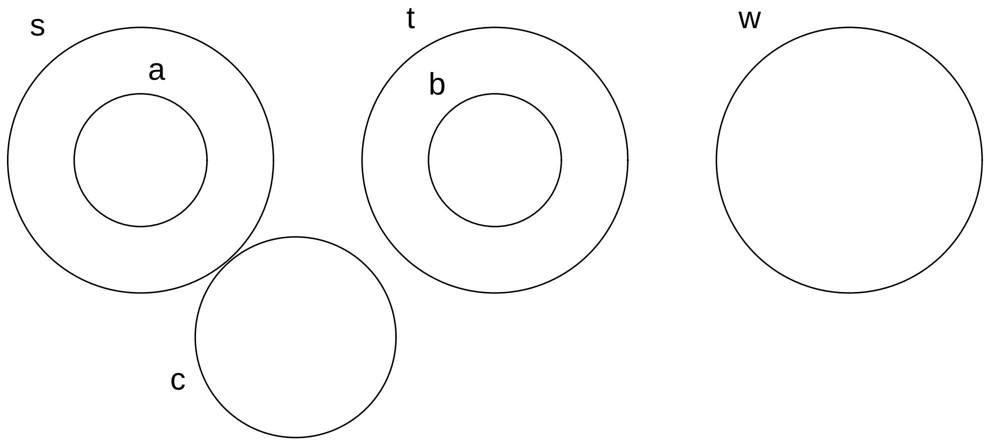
$$TPP, TPP^{\smile}, NTPP, NTPP^{\smile}, DC, EC, PO, 1' \tag{5.2}$$

which have been defined above, are present in the relation algebra generated by C in any RCC model, and they are usually taken as the base relations of the RCC, called *RCC8*.

A weak composition table for the relations of (5.2) was presented in [86]; this table had the same cell entries as the "real" composition table of the closed disk algebra $\mathcal{D}_c$, given in Table 9. It became clear, however, that this table could not be extensional in any RCC model: Suppose that $\langle B, C \rangle$ is such a model, and consider, for example, the entry

$$TPP \circ_w TPP^{\smile} = 1' \cup DC \cup EC \cup PO \cup TPP \cup TPP^{\smile}. \tag{5.3}$$

Figure 8: $sDCt,\ sDCw,\ tDCw,\ s+t+w \lneq 1,\ aNTPPs,\ bNTPPt.$



If $x \in B \setminus \langle 0, 1 \rangle$, then $xEC - x$, but $x[-(TPP \circ TPP^{\smile})] - x$, so that $EC \not\subseteq TPP \circ TPP^{\smile}$, and thus, the table is not extensional for $\langle B, C \rangle$. It was conjectured by Bennett et al. [11] that every example which shows that a cell in the weak RCC8 table is not extensional must involve the universal region. However, it was shown by Li and Ying [62] that this fails in every possible way. For each instance of $R \circ_w S \neq R \circ S$ an example is given which does not involve the universal region.

It is problematic to regard the RCC calculus as a spatial counterpart to Allen's interval algebra, as proposed in [89]:

- The domain of intervals considered by Allen is not assumed to have a particular algebraic structure, while RCC models are Boolean algebras with a contact relation which is not independent of the algebraic structure.
- The relations of Allen's interval domain are the atoms of the interval algebra $\mathcal{I}$, while the RCC8 relations are never the atoms of a relation algebra generated by the contact relation of a model of RCC.

Possible spatial counterparts to $\mathcal{I}$ are the extensional interpretations of the RCC8 table given above – the set of closed disks or the set of planar regions bounded by Jordan curves.

For these reasons, closed disks are not a proper illustration of the atoms of $\mathcal{D}_c$, when interpreted in the RCC calculus. The situations where the relations hold, and the landscape of relations which must exist in an RCC model is much richer than the pictures of closed disks indicate. As a simple example, let x be the union of two disjoint disks y and z. Then, $yTPPx,\ zTPPx$, which, topologically, is a totally different situation from the TPP picture of Figure 6.

Currently, no relation algebra arising from an RCC model is completely known, but partial results are available: Observing that EC splits into two relations ECN and ECD, one could conjecture that the complemented closed disk algebra could provide a composition table which would also work for RCC models. This turned out not to be the case; starting from the relations defined in Table 10, Düntsch et al. [22] give a list of 25 disjoint relations and show that they are not empty in any RCC model; these are shown in Table 13.

To give an impression of some of these relations in the algebra of regular opens sets in the Euclidean plane, consider Figures 8 – 10.

$xTPPAz$: In Figure 9 set $x = a + t,\ z = s + t$.

$xTPPBz$: In Figure 8, set $x = s,\ z = s + t$ or $x = a^* \cdot s, z = s$.

Table 13: Relations present in every RCC model

$1'$	
$TPPA$	$= TPP \cap (ECN \circ TPP)$
$TPPA^{\smile}$	$= TPP^{\smile} \cap (ECN \circ TPP)^{\smile}$
$TPPB$	$= TPP \cap -(ECN \circ TPP)$
$TPPB^{\smile}$	$= TPP^{\smile} \cap -(ECN \circ TPP)^{\smile}$
$NTPP$	
$NTPP^{\smile}$	
$PONYA1$	$= PON \cap (ECN \circ TPP) \cap -(ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap (TPP^{\smile} \circ TPP)$
$PONYA1^{\smile}$	$= PON \cap (ECN \circ TPP)^{\smile} \cap -(ECN \circ TPP) \cap (TPP \circ TPP^{\smile}) \cap (TPP^{\smile} \circ TPP)$
$PONYA2$	$= PON \cap (ECN \circ TPP) \cap -(ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap -(TPP^{\smile} \circ TPP)$
$PONYA2^{\smile}$	$= PON \cap (ECN \circ TPP)^{\smile} \cap -(ECN \circ TPP) \cap (TPP \circ TPP^{\smile}) \cap -(TPP^{\smile} \circ TPP)$
$PONYB$	$= PON \cap (ECN \circ TPP) \cap -(ECN \circ TPP)^{\smile} \cap -(TPP \circ TPP^{\smile})$
$PONYB^{\smile}$	$= PON \cap (ECN \circ TPP)^{\smile} \cap -(ECN \circ TPP) \cap -(TPP \circ TPP^{\smile})$
$PONXA1$	$= PON \cap (ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap (TPP^{\smile} \circ TPP)$
$PONXA2$	$= PON \cap (ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap (TPP \circ TPP^{\smile}) \cap -(TPP^{\smile} \circ TPP)$
$PONXB1$	$= PON \cap (ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap -(TPP \circ TPP^{\smile}) \cap (TPP^{\smile} \circ TPP)$
$PONXB2$	$= PON \cap (ECN \circ TPP) \cap (ECN \circ TPP)^{\smile} \cap -(TPP \circ TPP^{\smile}) \cap -(TPP^{\smile} \circ TPP)$
$PONZ$	$= PON \cap -(ECN \circ TPP) \cap -(ECN \circ TPP)^{\smile}$
$PODYA$	$= ECD \circ (TPP \cap (ECN \circ TPP))$
$PODYB$	$= ECD \circ (TPP \cap -(ECN \circ TPP))$
$PODZ$	$= ECD \circ NTPP$
$ECNA$	$= ECN \cap (TPP \circ TPP^{\smile})$
$ECNB$	$= ECN \cap -(TPP \circ TPP^{\smile})$
ECD	
DC	

$xPONYA1z$: In Figure 8, set $x = a + t + w,\ z = s + w$.
$xPONYA2z$: In Figure 8, set $x = t + a,\ z = s$.
$xPONYBz$: In Figure 9, set $x = b,\ z = s \cdot a^*$.
$xPONXA1z$: In Figure 8, set $x = t + a,\ z = s + b$.
$xPONXA2z$: In Figure 8, set $x = s,\ z = a + c$.
$xPONXB1z$: In Figure 10, set $x = s \cdot (a + c)^*,\ z = s^* + b$.
$xPONXB2z$: In Figure 9, set $x = b,\ z = a + s \cdot b^*$.
$xPONZz$: In Figure 8, set $x = s + t \cdot b^*,\ z = t + s \cdot a^*$.

The topological properties of some of these relations are shown in Table 14. There, $\partial(x)$ denotes the topological boundary of x. From these, the topological characterizations of most of the remaining ones can be determined, since they are intersections, respectively, complements of the given ones. For

Figure 9: $aNTPPbNTPPs \lneq 1$

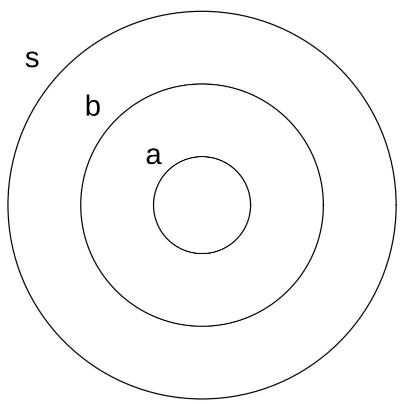
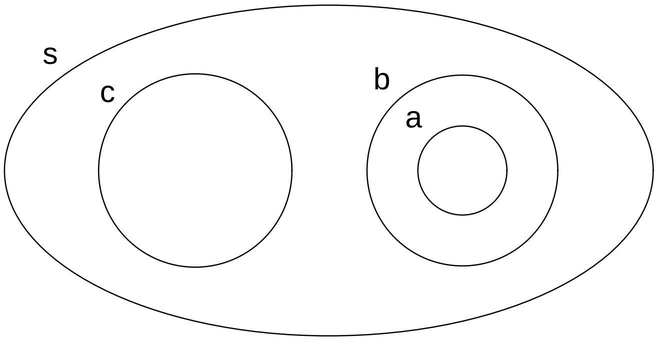


Figure 10: $aNTPPb$, $bNTPPs$, $cNTPPs$, $bDCc$



example,

$$\begin{aligned} xTPPAz \Longleftrightarrow & x(TPP \cap (ECN \circ TPP))z \\ \Longleftrightarrow & x \subsetneq z, \partial(x) \cap \partial(-x \cap z) \neq \emptyset, \\ & \partial(z) \cap \partial(-x \cap z) \neq \emptyset, cl(x) \cup cl(z) \neq X. \end{aligned}$$

Table 14: Topological interpretation of some RCC25 relations

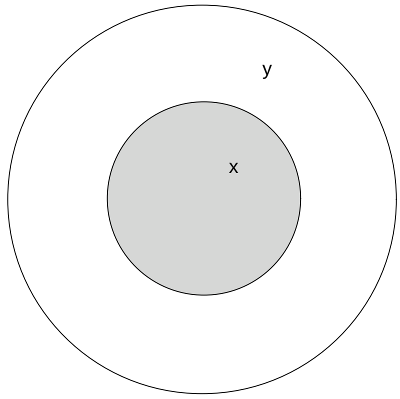
Atom	Name	$x, z \in \mathrm{RegOp}(X) \setminus \{\emptyset, X\}$
		Base relations
	TPP	$x \subsetneq z,\ \partial(x) \cap \partial(z) \neq \emptyset$
*	$NTPP$	$cl(x) \subsetneq z$
	PON	$x \not\subseteq z,\ z \not\subseteq x,\ x \cap z \neq \emptyset,\ cl(x) \cup cl(z) \neq X$
	POD	$x \not\subseteq z,\ z \not\subseteq x,\ x \cap z \neq \emptyset,\ cl(x) \cup cl(z) = X$
	ECN	$x \cap z = \emptyset,\ \partial(x) \cap \partial(z) \neq \emptyset,\ cl(x) \cup cl(z) \neq X$
*	ECD	$x \cap z = \emptyset,\ \partial(x) \cap \partial(z) \neq \emptyset,\ cl(x) \cup cl(z) = X$
*	DC	$cl(x) \cap cl(z) = \emptyset$
		Other relations
	$ECN \circ TPP$	$\partial(x) \cap \partial(-x \cap z) \neq \emptyset, \partial(z) \cap \partial(-x \cap z) \neq \emptyset, cl(x) \cup cl(z) \neq X$
	$TPP \circ TPP^{\smile}$	$\partial(x) \cap \partial(int(cl(x \cup z))) \neq \emptyset, \partial(z) \cap \partial(int(cl(x \cup z))) \neq \emptyset$
	$TPP^{\smile} \circ TPP$	$\partial(x) \cap \partial(x \cap z) \neq \emptyset, \partial(z) \cap \partial(x \cap z) \neq \emptyset$
	$ECD \circ NTPP$	$x \cup z = X$
*	$PODZ$	$x \cup z = X$
*	$ECNA$	$xECNz,\ \partial(x) \cap \partial(x+z) \neq \emptyset, \partial(z) \cap \partial(\ x\ +\ z\ ) \neq \emptyset$
*	$ECNB$	$xECNz,\ cl(x) \subseteq x+z$ or $cl(z) \subseteq x+z$

It turns out that in the RCC model of the regular open (or closed) sets of a connected regular T_1 space X, the relation algebra generated by C is not finite [77]. The key to this result is the construction of a "hole relation" H, which is defined by

$$xHy \Longleftrightarrow xECNy \text{ and } ECN(x) \subseteq O(y). \tag{5.4}$$

A pictorial representation is shown in Figure 11; there, x is the grey disk and y the white "doughnut" around it.

Figure 11: x is a hole of y



It turns out that H can be defined algebraically by $H = ECN \cap (ECN \setminus_r O)$, and that in $RO(X)$, $H^n \neq H^k$ for all $1 \leq k \lneq n$; it is unknown whether this is true for all RCC models. It is worth pointing out that a more restricted definition of a hole relation has been given earlier by Egenhofer et al. [35]. Furthermore, there are models of the RCC theory, realizable as regular closed sets of a regular connected T_1 space, in which $NTPP \circ NTPP \subsetneq NTPP$ [24].

6 Constraint problems and relation algebras

Constraint satisfaction problems have played a significant part in the study of temporal or spatial relations [49, 73, 79, 87, 88], and have also received attention in the relation algebra community [41–44, 54, 55, 57, 58, 73]. In particular, the work of Ladkin and others shows that "the relation algebra of Tarski is an appropriate mathematical context in which to represent and solve binary constraint problems" [58]. This works well for situations in which the RA generated by the constraints is finite (and hence, atomic). If this is not the case, the problem is more intricate and one has to look for approximations to such RA or use only a proper subset of the relational operators for constraint checking. I will return to this topic below.

A (binary) constraint satisfaction problem on n variables consists of a finite set $X = \{x_0, x_1, \dots, x_n\}$ of individuum variables and a formula

$$\varphi = \bigwedge_{i,j \leq n} x_i P_{ij} x_j \tag{6.1}$$

where each P_{ij} is a binary predicate symbol, called a *constraint*. In our context, the individuum variables and predicate symbols are interpreted by a finite set $\mathcal{S}$ of relations over a specified domain $\mathcal{D}$ such that

C1. The relations in $\mathcal{S}$ are nonempty and pairwise disjoint.

C2. The union of the relations in $\mathcal{S}$ is the universal relation of the domain.

If $\mathcal{S}$ has the properties 1.and 2. it is called a *jointly exhaustive and pairwise disjoint* (JEPD) set of relations. The elements of S are usually called *base relations.*

C3. All (interpretations of) constraint relations are unions of members of $\mathcal{S}$.

Following the convention of graph theory, a pair $\mathcal{N} = \langle V, \varphi \rangle$ is called a *binary constraint network* (BCN) or just *network*. Here, we think of the edge $\langle x_i, x_j \rangle$ labeled with P_{ij}, if the constraint $x_i P_{ij} x_j$ appears in φ. If all edges of $\mathcal{N}$ are labeled with elements of $\mathcal{S}$, then the network is called *atomic*. The elements of $\mathcal{S}$ will be called *atomic constraints*. Together with the empty relation, the set of all possible unions of atomic constraints forms a Boolean algebra $B(\mathcal{S})$. If this algebra is closed under composition, converse, and contains the identity, then it is a BRA.

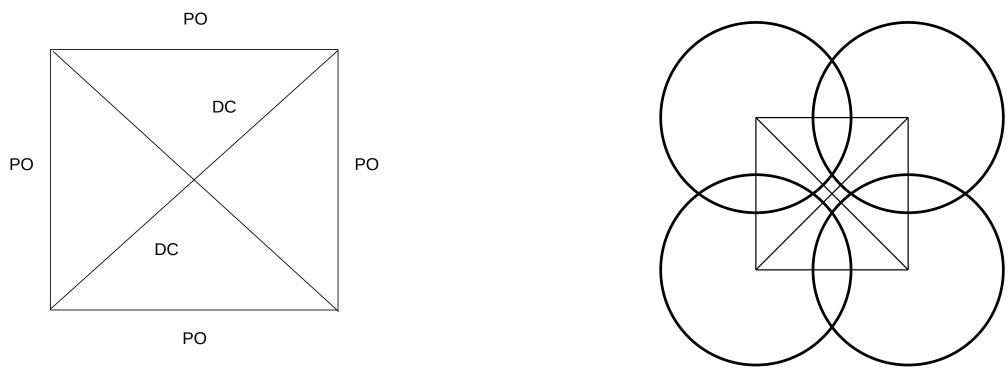
The *network satisfaction problem* $NSP(\mathcal{S})$ over a fixed domain $\mathcal{D}$ is the following:

Instance: An network $\langle V, \varphi \rangle$.

Question: Is there an instantiation $h : V \to \mathcal{D}$ of the variables such that $h(x_i) P_{ij} h(x_j)$ for all $i, j \leq n$?

$\mathcal{N}$ is called *satisfiable* in case the answer is yes. Satisfiability depends, of course, on the domain $\mathcal{D}$: Consider the square and its diagonals in Figure 12, and label the sides of the square with PO and its diagonals with DC. This network cannot be satisfied in any representation of $\mathcal{I}$ as shown in [55], but it can be satisfied in the closed circle algebra by the indicated configuration.

Figure 12: A network satisfiable in $\mathcal{D}_c$ and not in $\mathcal{I}$



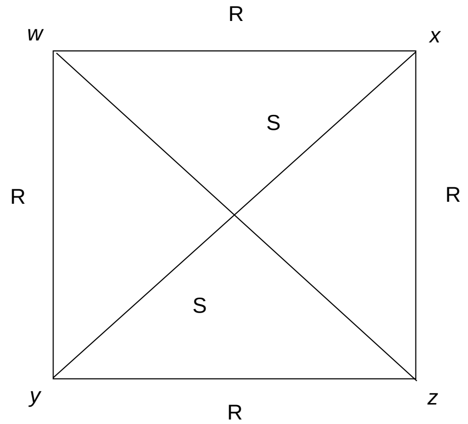
A network is called *path consistent* if for all $a, b \in \mathcal{D}$ for which $a P_{ij} b$ holds, there is, for each $k \leq n$, some $c \in \mathcal{D}$ such that $a P_{ik} c P_{kj} b$ [58, 68]. In relation algebraic parlance, this is equivalent to $P_{ij} \subseteq P_{ik} \circ P_{kj}$ for all $i, j, k \leq n$. If the elements of $\mathcal{S}$ are the atoms of a BRA, then path consistency is necessary for consistency [76], but not sufficient. For the latter, consider the atomic network given in Figure 13 with relations from the pentagonal algebra $\mathcal{P}$, which is path consistent, but not satisfiable.

There are well known algorithms to check whether a network $\mathcal{N}$ contains a path consistent subnetwork. The basic principle underlying many of these can be formulated as follows: For all i, j, k, compute the relation $P_{ij} \circ P_{jk}$ and intersect it with P_{ik}; here is an illustration given in [55]:

$$\begin{tikzcd} & x_1 \arrow[dr, "S"] & \\ x_0 \arrow[ur, "R"] \arrow[rr, "T"'] & & x_2 \end{tikzcd} \qquad \begin{tikzcd} & x_1 \arrow[dr, "S \cap (R^{\smile} \circ T)"] & \\ x_0 \arrow[ur, "R \cap (T \circ S^{\smile})"] \arrow[rr, "T \cap (R \circ S)"'] & & x_2 \end{tikzcd}$$

This results in the possibly smaller relation $P_{ik} \cap (P_{ij} \circ P_{jk})$, which can be used as the new constraint on the edge $\langle x_i, x_k \rangle$; Ladkin and Maddux [55] call this the *triangle operation*.

Figure 13: A path consistent but not satisfiable network over $\mathcal{P}$



A triangle operation *stabilizes* if $P_{ik} \cap (P_{ij} \circ P_{jk}) = P_{ik}$. Clearly $\mathcal{N}$ is path consistent if every triangle operation stabilizes. The complexity of the triangle operation over a finite relation algebra is $O(n^3)$, where n is the number of nodes. For a variety of tractable procedures which improve on the triangle operation, the reader is invited to consult [44] and [58].

Observe that the triangle operation only guarantees success, if the set $\mathcal{S}$ of base relations is not only JEPD, but also the set of atoms of a finite relation algebra A. In this case, $P_{ij} \circ P_{jk}$ is a union of atoms, i.e. elements of $\mathcal{S}$, and $P_{ik} \cap (P_{ij} \circ P_{jk})$ will again be a union of atoms (or empty). The finiteness of A implies that the algorithm terminates. Otherwise, it may happen that a triangle operation $P_{ik} \cap (P_{ij} \circ P_{jk})$ leads to a relation outside the unions of the base relations, and the algorithm need not terminate, even if the set of base relations is finite.

The triangle operation works with all examples given in Section 2.1, but not for the RCC8 base relations, when they are interpreted over an RCC model, since the relation algebra generated by the base relations is infinite and not atomic. This has the unfortunate consequence, that a network may be satisfiable, but not path consistent:

Proposition 6.1. *There is an atomic network $\mathcal{N}$ over the RCC8 base relations such that for all RCC models $\mathfrak{M}$,*

1. *$\mathcal{N}$ is satisfiable in $\mathfrak{M}$.*
2. *$\mathcal{N}$ is not path consistent in $\mathfrak{M}$.*

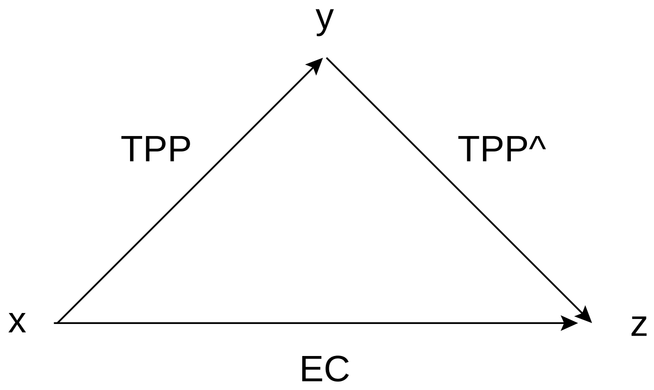
Proof. Consider the network of Figure 14. The network is consistent since $EC \cap (TPP \circ TPP^{\smile}) \neq \emptyset$ which follows from the RCC axioms, and can be obtained from the RCC weak composition table 8; thus, it is model independent. On the other hand, let $\mathfrak{M} = \langle B, C \rangle$ be any model of the RCC. Choose $x, z \in B \setminus \{0, 1\}$, such that x is a hole of z; such elements exist in any RCC model. Then, by definition (5.4) of the hole relation,

$$xECz,\ x + z \neq 1,\ (\forall t)[t + x \neq 1 \text{ and } xECt \Longrightarrow t \cdot z \neq 0].$$

Now, let $xTPPy$. Then, $y \neq 1$, and from Lemma 5.1.2 of [22] we obtain $xECy^*$. Thus, $y^* \cdot z \neq 0$, which implies that $z(-P)y$, i.e. $y(-P)^{\smile}z$. □

The analysis of the RCC8 weak composition table, started by [23] and was continued by Düntsch et al. [22] who already exhibit many instances of triples $\langle T, S, R \rangle$ of RCC8 base relations for which

Figure 14: A consistent but not path consistent network over any RCC model



$T \not\subseteq R \circ S$ in any RCC model. In an exhaustive investigation Li and Ying [62] identify altogether 18 essentially different such triples; on a more positive note, they identify 127 triples $\langle T, S, R \rangle$ for which $T \subseteq R \circ S$ if the universal region is not an element of the domain. Since the universal region can be relationally defined – and thus "excluded" from relational reasoning –, it may be worthwhile (and more realistic) to use the relations of the complemented closed disk algebra as RCC base relations, and their composition table as a weak composition table for RCC models.

One may argue that the operators used in the triangle operation are only $\cap, \circ$ and $\breve{}$, applied to unions of base relations, and that the full power of relation algebras, which includes complementation, is not required. For finite RAs, this is immaterial, since every element is a union of atoms, but for a (finite) JEPD set of base relations which generate an infinite RA, the restriction to these operators need not be the desired remedy. As we have seen above, there are RCC models in which the $\{\cap, \circ, \breve{}\}$ – closure $\hat{\mathcal{S}}$ of a set $\mathcal{S}$ of base relations is infinite, for example a model where $NTTP^{k+1} \subsetneq NTPP^k$ for all $k \geq 1$. This puts some doubt on several results of Renz [87] and Renz and Nebel [89], whose proofs seem to assume that $\hat{\mathcal{S}}$ is always finite.

Given a finite (abstract) representable RA A, and a network $\mathcal{N}$ whose edges are labeled with elements from A, we can ask whether there is a representation $\mathcal{D}$ of A such that the NSP over $\mathcal{D}$ has a solution. This is the *general network satisfaction problem* (GNSP). In practice, one usually works over a particular domain, and therefore the GNSP is of more theoretical interest. The GNSP is investigated in several papers concerned with constraint problems over RCC models, e.g. [49, 78, 89].

The complexity of the (G)NSP for many of our examples is known:

The pentagonal algebra $\mathcal{P}$: Even though this algebra is small and has only one representation, which is finite, all that is known is that the NSP over $\mathcal{P}$ is in NP [42].

The point algebra $\mathcal{P}t$: This algebra is homogenous in the sense that (a copy) of the ordering on the rationals embeds in any representation, and it follows that the GNSP and the NSP coincide. It was shown in [96] that the NSP has cubic complexity.

The left linear point algebra $\mathcal{L}$: In this algebra, path consistency does not imply satisfiability; nevertheless, there is an algorithm with quintic complexity which decides whether a given network is satisfiable over a given representation [42].

Allen's interval algebra $\mathcal{I}$: All countable representations of $\mathcal{I}$ are (base) isomorphic to the representation by Allen's interval relations with rational endpoints [55], and thus, GNSP and NSP are equivalent. Since path consistency implies satisfiability in $\mathcal{I}$, the complexity is cubic for atomic networks; in general, the NSP complexity is NP complete [96]. A maximal tractable set of interval interval relations has been identified by Nebel and Bürckert [79].

Compass algebras : It was shown by Maddux [70] that the GNSP is NP complete for any compass algebra with at least two directions.

The containment algebra $\mathcal{C}$: Here, the NSP and the GNSP differ, as the example of Figure 12 shows. It was shown in [56] that there are unsatisfiable path consistent atomic networks in $\mathcal{C}$, and an unpublished result of Maddux, mentioned in [56], shows the GNSP to be NP hard. Renz and Nebel [89] improved this result by showing that the GNSP is NP complete. A complete classification of tractable networks was given by Jonsson and Drakengren [49].

As we can see, NSP complexity is usually high. But this is not as bad as it gets: Hirsch [43] has exhibited a finite relation algebra $\mathcal{A}$ and a representation $\mathcal{D}$ of $\mathcal{A}$ such that the NSP over $\mathcal{D}$ is undecidable.

7 Conclusions and outlook

We have seen that finite relation algebras are a useful scenario for reasoning about temporal constraints. For spatial reasoning, the containment algebra $\mathcal{C}$, the algebra $\mathcal{D}_c$, interpreted over bounded closed Jordan curves in the plane, and the compass algebra are also positive examples. Topological relationships are explored in the relation algebraic context by Egenhofer and his co–workers, exhibiting many powerful properties [27–31, 34, 36]. Furthermore, many of the examples of temporal or spatial RAs given in Section 2.1 are first order closed, and thus, anything that can be said about them in first order logic can be stated with formulas which use only three variables.

Reasoning about the elements of $\mathcal{D}_c$, when interpreted over an RCC model, is more intricate, since the RA generated by these relations is infinite, and the standard consistency algorithms need not work. The composition table of $\mathcal{D}_c$ has no extensional interpretation over RCC models, and thus, there may be a difference between path consistency and satisfiablity of a triangle as Proposition 6.1 shows. Furthermore, even the $\cap, \circ, \breve{}$ closure of a finite set of RCC base relations need not be finite. It may therefore be useful to investigate, which sets of RCC relations have a finite closure under these operations, and determine their complexity.

The general algebraic framework for constraint problems seem to be inf – semilattices with converse and relative multiplication, and it would interesting to find a suitable set of axioms for such structures; investigations in this direction have already been started in [21] and [20].

Epilogue

I should like to finish this paper with the closing sentences of Tarski's 1941 article, which express a feeling for Mathematics which often is lost in our days, when commercial exploitability is of primary concern, and recognition (and funding) is often given by the criterion of immediate applicability, but which, at least for me, is still a major motivation for engaging in the pursuit of mathematical knowledge:

> "Aside from the fact that the concepts occurring in this calculus possess an objective importance and are in these times almost indispensable in any scientific discussion, the calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it." [93]

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