A relation – algebraic approach to the region connection calculus

Ivo Düntsch,[*] Hui Wang, Steve McCloskey
School of Information and Software Engineering,
University of Ulster,
Newtownabbey, BT 37 0QB, N.Ireland
{I.Duentsch,H.Wang,S.McCloskey}@ulst.ac.uk

Abstract

We explore the relation–algebraic aspects of the region connection calculus (RCC) of Randell *et al.* (1992a). In particular, we present a refinement of the RCC8 table which shows that the axioms provide for more relations than are listed in the present table. We also show that each RCC model leads to a Boolean algebra. Finally, we prove that a refined version of the RCC5 table has as models all atomless Boolean algebras B with the natural ordering as the "part – of" relation, and that the table is closed under first order definable relations iff B is homogeneous.

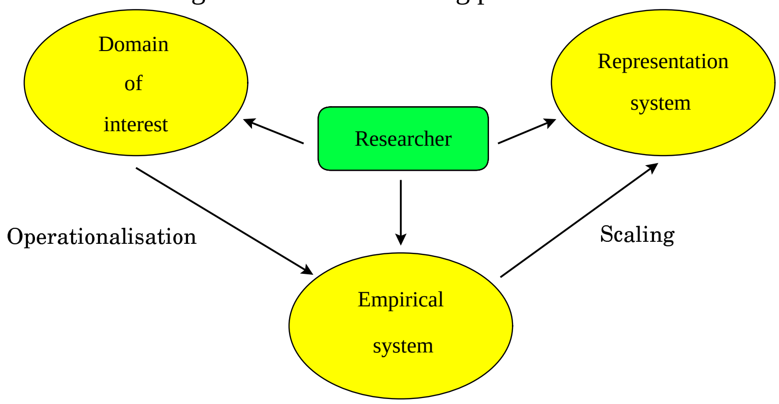
1 Introduction

Qualitative reasoning (QR) has its origins in the exploration of properties of physical systems when numerical information is not sufficient – or not present – to explain the situation at hand (Weld and Kleer, 1990). Furthermore, it is a tool to represent the abstractions of researchers who are constructing numerical systems which model the physical world. Thus, it fills a gap in data modeling which often leaves out the researcher as an active component in the modelling process. If we follow the description of data modelling presented by Gigerenzer (1981) which is pictured in Figure 1, then the two places where QR resides are at the level of the empirical model, and in including the intentions and actions of the researcher as part of the process. Conceptually, QR can be called a form of soft computing, in particular related to the philosophy of rough set data analysis (Pawlak, 1982, 1991) as presented in Düntsch and Gediga (1997) (see also Cohn, 1997, p.1, footnote 1, which points in the same direction).

A special area of QR, *qualitative spatial reasoning* (QSR), has evolved in the last decade which is concerned with the qualitative aspects of representing – and reasoning about – spatial entities as opposed to the earlier emphasis on one–dimensional situations.

[*]Address from May 2002: Department of Computer Science, Brock University, St. Catherines, Ontario, Canada, L2S 3AI, duentsch@cosc.brocku.ca

Fig. 1: The data modeling process



> "The challenge of QSR then is to provide calculi which allow a machine to represent and reason with spatial entities of higher dimension, without resorting to the traditional quantitative techniques prevalent in, for example, the computer graphics or computer vision communities." (Cohn, 1997)

Applications of QSR can be found in geographical information systems (Worboys, 1998), spatial query languages (Clementini *et al.*, 1994), natural languages (Asher and Vieu, 1995) and many other fields. Evidence that QSR is now firmly established in AI are numerous presentations, workshops and tutorials which have been given at important AI conferences, most recently at COSIT'97, KR'98 and ECAI'98. We invite the reader to consult Cohn (1996) and its updated version Cohn (1997) for an introduction and an overview of current trends. In the wider context of formal ontology, the special edition of the *International Journal of Human–Computer Studies* **43** (1995) exhibits the width and depth of the area.

The basis of QSR are "part – of" and "connection", respectively, "contact" relations. The formalization of the "part – of" relationship goes back to the mereology of Leśniewski (1886 – 1939), developed from 1915 onwards. One of the main concerns of Leśniewski was to build a paradox–free foundation of Mathematics, one pillar of which was mereology, or, as it was originally called, the general theory of manifolds or collective sets (Leśniewski, 1928).

Mereology was subsequently taken up by Leonard and Goodman (1940) (though, for a somewhat different purpose). Formally, Leśniewski's mereology and the calculus of Leonard and Goodman – the classical mereology (CM) – are the same.

Based on classical mereology, and the work of Whitehead (1929) on the relation "x is extensionally connected with y", Clarke (1981) presents an axiom system for a "Calculus of individuals" based on a "connected – with" relation C. The intended domain is such that

> " . . . we may interpret the individual variables as ranging over spatial–temporal regions and the two–place primitive 'x is connected with y' as a rendering of 'x and y share a common point." (Clarke, 1981, p.205).

Suppose that M is a nonempty set of regions, and C a binary relation on R; we let $Cx = \{y \in M : xCy\}$. If $X \subseteq M$, and $x \in M$, then x is the *fusion of* X, if

$$Cx = \bigcup_{y \in X} Cy. \tag{1.1}$$

Clarke assumes are two mereological axioms:

A 1. C is reflexive and symmetric,

A 2. If $Cx = Cy$, then $x = y$,

and one axiom concerning the fusion:

A 3. If $X \subseteq M$ is nonempty, then the fusion of X exists in M. .

If a region x is not connected to every other region, then the complement $-x$ of x is defined as the fusion of all regions z which are not connected to x. In other words,

$$C(-x) = \bigcup_{z(-C)x} Cz. \tag{1.2}$$

Biacino and Gerla (1991) show that the domains satisfying A 1 – A 3 are exactly the complete ortho-complemented lattices in which

$$xCy \Longleftrightarrow x \not\leq -y. \tag{1.3}$$

Here, $\leq$ is the lattice ordering; the fusion is just the lattice join. It may be worthy to point out that, although Clarke calls his operations "quasi–Boolean", the models for his calculus are not necessarily (quasi–) Boolean algebras.

Clarke (1985) adds another axiom, the purpose of which is to define a 'point' within his calculus. Unfortunately, the full system collapses to classical mereology, as Biacino and Gerla (1991) show, and they suggest a modification of Clarke's system calculus:

> "The new system should still admit as models the class of the nonempty regular open sets of a topological space[1] ... But in these models the connection relation should be as follows:
>
> $$xCy \Longleftrightarrow \overline{x} \cap \overline{y} \neq \emptyset.\text{"} \tag{1.4}$$

Here, $\overline{x}$ is the topological closure of x. Such a system, the "region connection calculus" (RCC), was presented by Randell *et al.* (1992a,b), and has since received prominence in spatial reasoning (see Cohn *et al.*, 1997, for an overview). The differences to Clarke's system are that only the existence of the fusion of finite sets is postulated, and different notion of "complement".

Already in the early RCC presentation of Randell *et al.* (1992a), the importance of relational transitivity tables for qualitative reasoning about regions was recognized; recently, Bennett *et al.* (1997)

[1] These are the models of classical mereology.

have raised several questions regarding the expressiveness of relational reasoning, in particular with respect to the RCC.

Relational reasoning as algebraic manipulation of relations has a long-standing tradition, going back to A. De Morgan, C.S. Peirce, and E. Schröder (cf. Anellis and Houser, 1991). From the 1940s onwards, A. Tarski (who, incidentally, was Leśniewski's only doctoral student) and his colleagues have continued the work on the calculus of relations which eventually led to an algebraization of first order logic via cylindric algebras (Henkin *et al.*, 1971, 1985), and its finite fragments, in particular, first order logic with three variables via relation algebras (cf. Tarski and Givant, 1987). In this paper we shall explore the relation – algebraic aspects of the RCC relations, and suggest some modifications. We also hope to answer some of the questions raised in Bennett *et al.* (1997).

The paper is structured as follows: We first introduce the necessary machinery of relation algebras; based on these, we will then discuss some aspects of Bennett *et al.* (1997) from a relation – algebraic point of view. Section 4 introduces the RCC and lists some of its properties. We show that the algebraic part of the RCC leads to quasi – Boolean operations, and present a refined (weak) composition table which contains additional definable relations which do not appear in the original RCC. Finally, we investigate a reduced set of RCC relations (RCC5).

2 Relations and their algebras

A relation algebra (RA)

$$\langle A, +, \cdot, -, 0, 1, \circ, \breve{}, 1' \rangle$$

is a structure of type $\langle 2, 2, 1, 0, 0, 2, 1, 0 \rangle$ which satisfies for all $a, b, c \in A$,

1. $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra (BA).
2. $\langle A, \circ, \breve{}, 1' \rangle$ is an involuted monoid, i.e.
   (a) $\langle A, \circ, 1' \rangle$ is a semigroup with identity $1'$,
   (b) $a^{\breve{}\breve{}} = a,\ (a \circ b)^{\breve{}} = b^{\breve{}} \circ a^{\breve{}}$.
3. The following conditions are equivalent:

$$(a \circ b) \cdot c = 0,\ (a^{\breve{}} \circ c) \cdot b = 0,\ (c \circ b^{\breve{}}) \cdot a = 0. \tag{2.1}$$

In the sequel, we will usually identify algebras with their base set.

The *full algebra of binary relations* is a structure $\langle Rel(U), \cup, \cap, -, \emptyset, U \times U, \circ, \breve{}, 1' \rangle$ on a set U, where $Rel(U)$ is the set of all binary relations on U, $\cap, \cup, -$ are the usual set theoretic operations, $\emptyset,\ U \times U$ are, respectively, the empty and the universal relation, $\circ$ is relational composition, $\breve{}$ the relational converse (i.e. $P^{\breve{}} = \{\langle x, y \rangle : \langle y, x \rangle \in P\}$), and $1'$ is the identity relation on U. A subset A of $Rel(U)$ which is closed under the distinguished operations of $Rel(U)$ and contains the distinguished constants is called an *algebra of binary relations* (BRA) on U. It is a subalgebra of $Rel(U)$, a fact which we denote by $A \leq Rel(U)$. If $\{R_i : i \in I\} \subseteq Rel(U)$, we denote the BRA generated by $\{R_i : i \in I\}$ by $\langle R_i : i \in I\rangle$. If $S \in \langle R_i : i \in I\rangle$, we say that S *is RA – definable by* $\{R_i : i \in I\}$. If the set of generators is understood, we shall usually omit mentioning it, and just say that S is RA definable.

A relation algebra A is called *representable* if it is isomorphic to a subalgebra of a product of full algebras of binary relations.

The logic of RAs is a fragment of first order logic, and the following fundamental result is due to A. Tarski (see Tarski and Givant, 1987):

Proposition 2.1. *If $R_0, \dots, R_k \in Rel(U)$, then $\langle R_0, \dots, R_k\rangle$ is the set of all binary relations on U which are definable in the (language of the) relational structure $\langle U, R_0, \dots, R_k\rangle$ by first order formulas using at most three variables.*

An RA A is called *integral*, if $1'$ is an atom of A. If $A = \langle U, R_i\rangle_{i\in I}$ is a BRA, then A is integral if and only if no proper nonempty subset of U is definable in the model $\langle U, R_i\rangle_{i\in I}$ by a formula with at most three variables (see Andréka *et al.*, 1995b).

Suppose that $A \leq Rel(U)$. For $P, Q \in A$ and $x, y, z \in U$ we usually write xPy if $\langle x, y\rangle \in P$, and $xPyQz$ means xPy and yQz. With some abuse of notation, we let $Rx \ := \{y \in U : xRy\}$ be the *range of x w.r.t. R.*

Let Σ_U be the symmetric group of U, and $\varphi \in \Sigma_U$; we will write $\varphi\langle x, y\rangle$ instead of $\langle\varphi(x), \varphi(y)\rangle$. The image of $R \in A$ under φ is denoted by R^φ, i.e.

(2.2) $$R^\varphi = \{\varphi\langle x, y\rangle : \langle x, y\rangle \in R\}.$$

If $R^\varphi = R$, we call R *invariant under* φ. The permutation φ is called a *base automorphism of* A, if every $R \in A$ is invariant under φ. The set of all base automorphisms of A is denoted by A^ρ; it is easy to see that A^ρ is a subgroup of Σ_U. We call A *permutational*, if A^ρ is transitive, i.e. for all $x, y \in U$ there is some $\varphi \in A^\rho$ such that $\varphi(x) = y$. If A is permutational, then no proper nonempty subset of U is first order definable in the model $\langle U, R\rangle_{R\in A}$.

Conversely, if G is a subgroup of Σ_U and $x, y \in U$, we set

$$G_{x,y} = \{\varphi(x, y) : \varphi \in G\},$$

and let G^σ be the BRA on U generated by $\{G_{x,y} : x, y \in U\}$. Observe that the sets $G_{x,y}$ are just the orbits of the action of G on U^2, and hence a partition of U^2. Indeed, each $G_{x,y}$ is an atom of G^σ, and every atom of G^σ has this form. The assignments ρ and σ form a Galois connection, and A is called *Galois closed* if $A^{\rho\sigma} = A$ (see Jónsson, 1984, Börner and Pöschel, 1991, Andréka *et al.*, 1995a).

A BRA A on U is called *first order closed* if every first order definable binary relation in the language of A is an element of A. The next result is most likely known, and its easy proof is left to the reader:

Lemma 2.2. *If A is integral and first order closed, then A^ρ is transitive.*

The following result gives a connection between Galois closure and first order closure (Jónsson, 1991, Andréka and Németi, 1991):

Proposition 2.3. *If A is finite, then*

$A = A^{\rho\sigma}$ *iff A is closed under every permutation invariant operation on binary relations.*

We shall need this in our discussion of RCC5 in Section 4.4.

The concept of *residuation* will be of importance in our later considerations. It will turn out that many theorems of the mereological part of spatial relations are consequences of the residual operators, since the "part of" relation turns out to be the right residual of the "connected to" relation.

Suppose that A is an RA, and that $a, b \in A$. Even though the equations $a \circ x = b$ and $x \circ a = b$ do not always have a solution, there are always elements $a \setminus b$ and $b \mathbin{/} a$, called, respectively, the *right* and *left residual* of b by a such that

$$a \circ x \leq b \iff x \leq a \setminus b,$$
$$x \circ a \leq b \iff x \leq b \mathbin{/} a.$$

The residuals can be expressed as RA terms in a and b by

$$a \setminus b = -(a^{\smile} \circ -b), \tag{2.3}$$
$$b \mathbin{/} a = -(-b \circ a^{\smile}). \tag{2.4}$$

If $a = b$, we shall only speak of the right (left) residual of a. These residuals have the following properties:

Lemma 2.4. *1. $a \setminus a$ and $a \mathbin{/} a$ are reflexive and transitive.*

2. If a is reflexive, then $a \setminus a \leq a$.

3. If a is symmetric, then $a \setminus a \leq a$ if and only if $(a \setminus a)^{\smile} \circ (a \setminus a) \leq a$.

Proof. A proof of 1. can be found in Pratt (1990). For 2., the monotony of $\circ$ and the reflexivity of a imply that $-a = 1' \circ -a \leq a \circ -a$. Thus, $a \setminus a = -(a \circ -a) \leq a$.

Suppose that a is symmetric. Then, one line implies the next:

$$a \setminus a \leq a$$
$$(a \setminus a) \circ -a \leq a \circ -a$$
$$((a \setminus a) \circ -a) \cdot -(a \circ -a) = 0$$
$$((a \setminus a)^{\smile} \circ (a \setminus a)) \cdot -a = 0.$$

Conversely, suppose that $(a \setminus a)^{\smile} \circ (a \setminus a) \leq a$, and assume that $(a \setminus a)^{\smile} \cdot -a \neq 0$. Then, $((a \setminus a)^{\smile} \circ 1') \cdot -a \neq 0$, and hence, $(a \setminus a \circ -a) \cdot 1' \neq 0$. On the other hand,

$$(a \setminus a)^{\smile} \circ a \setminus a \leq a \iff ((a \setminus a)^{\smile} \circ a \setminus a) \cdot a = 0 \iff (a \setminus a \circ -a) \cdot a \setminus a = 0. \tag{2.5}$$

Reflexivity of $a \setminus a$ implies that

$$(a \setminus a \circ -a) \cdot 1' \leq (a \setminus a \circ -a) \cdot a = 0, \tag{2.6}$$

which proves our claim. □

If $R, S \in Rel(U)$, then the residuals are given by the conditions

$$x(R \setminus S)y \iff (\forall z)(zRx \to zSy),$$
$$x(S \,/\, R)y \iff (\forall z)(yRz \to xSz),$$

(see e.g. Jónsson, 1991). We also use the following conditions (Jipsen, 1992):

Lemma 2.5. *Let A be an RA and $a, b, c \in A \setminus \{0\}$. Then,*

1. *If $1' \leq a^{\smile} \circ a$, and $a \circ b \leq c$, then $b \leq a^{\smile} \circ c$. (Integral lemma)*
2. *If b is an atom and $a \leq b \circ c$, then $b \leq a \circ c^{\smile}$. (Atom lemma)*

Suppose that A is a complete and atomic RA with atoms $At(A) = \{a_1, \ldots, a_n\}$. Then, relational composition can be interpreted as a mapping $\tau : A \times A \to \mathfrak{P}(A)$ with $\tau(a, b) = \{c \in At(A) : c \leq a \circ b\}$. Indeed, it is sufficient to look at the restriction of τ to $At(A) \times At(A)$.

The composition table of A is an (n, n)–matrix where entry (i, j) contains a list of all atoms below $a_i \circ a_j$; an example is shown in Table 1. A complete and atomic RA A is completely determined by

Table 1: A composition table

$\circ$	p	$p^{\smile}$	o	d
p	p	$-o$	o	d
$p^{\smile}$	$p, p^{\smile}, 1'$	$p^{\smile}$	o	$p^{\smile}, d$
o	o	o	$-o$	o
d	p, d	d	o	$-o$

the relational composition table of its set of atoms $At(A)$. When writing such a table, we will omit column and row $1'$, if $1'$ is an atom of A.

In our construction of RAs we have been aided by the RA Scratchpad, designed and written by Peter Jipsen (1992). For other properties of relations and their algebras see Jónsson (1982, 1991) and Andréka *et al.* (1998); we recommend Grätzer (1978) as a reference text for lattice theory, and Koppelberg (1989) for Boolean algebras.

3 Weak composition

A more general view of composition is taken in Randell *et al.* (1992a) and Bennett *et al.* (1997); there, a composition table (CT) is just a mapping $\tau : \mathsf{R} \times \mathsf{R} \to \mathfrak{P}(\mathsf{R})$, where R is a set of relational symbols. A model of $\langle \mathsf{R}, \tau \rangle$ is a pair $\langle U, v \rangle$, where U is a set and $v : \mathsf{R} \to Rel(U)$ is a mapping such that $\{v(a) : a \in \mathsf{R}\}$ is a partition of $U \times U$. The model is called *consistent* if

(3.1) $$c \in \tau(a, b) \iff (v(a) \circ v(b)) \cap v(c) \neq \emptyset.$$

for all $a, b, c \in \mathsf{R}$. In the sequel, we will call such a table a *weak composition* (table) to distinguish it from the usual relational composition.

Bennett *et al.* (1997) call a weak composition table *extensional*, if

$$v(a) \circ v(b) = \bigcup_{c \in \tau(a,b)} v(c). \tag{3.2}$$

Then, they write

> "One might ... conjecture that by refining relations in a set Rels one can always arrive at a set Rels' which is more expressive than Rels and whose CT can be interpreted extensionally."

If $\langle \mathsf{R}, \tau \rangle$ has a model $\langle U, v \rangle$ at all, then, in case $\{v(c) : c \in \mathsf{R}\}$ is closed under converse, and the identity is a union of elements of $\{v(c) : c \in \mathsf{R}\}$, its elements are the atoms of a relation algebra iff the table is extensional (see e.g. Jónsson, 1984). Given a partition $\mathcal{P}$ of $U \times U$, the relations in $\mathcal{P}$ will always generate a relation algebra which has an extensional table in case it is atomic and closed under arbitrary unions.

A weak composition table is called *complete* w.r.t. a theory Θ whose language contains Rels if,

> "... whenever a set of (ground) facts involving only relations in Rels and constants is inconsistent, this can be detected by reference to the table." (Bennett *et al.*, 1997)

Bennett *et al.* then conjecture that

> "... a CT is complete w.r.t. a theory Θ iff Θ implies all formulae corresponding to the extensional interpretation of composition."

The following is a simple counterexample: Consider the RA with the two atoms $1'$ (identity) and $0'$ (diversity), and let A be its representation on a three element set. Let Θ say that there are four elements, e.g. $\Theta = \{x_i 0' x_j : i, j \leq 3,\ i \neq j\}$. Then, Θ is not satisfiable in A, but each triangle is.

4 The region connection calculus

The Region Connection Calculus (RCC) introduced in Randell *et al.* (1992a) is a system similar to the mereological and quasi–Boolean part of Clarke's calculus of individuals, the difference being the definition of complementation. The base relation is a connection C, and further relations are obtained from C by the relational operators as in Clarke's system:

$P = C \setminus C$	part of
$PP = P \cap -1'$	proper part of
$O = P^{\smile} \circ P$	overlap
$PO = O \cap -(P \cup P^{\smile})$	partially overlap
$TPP = PP \cap (EC \circ EC)$	tangential proper part
$NTPP = PP \cap -TPP$	non–tangential proper part
$EC = C \cap -O$	externally connected
$DC = -C$	disconnected.

4.1 RCC axioms

A model for the RCC consists of a base set $U = R \cup N$, where R, N are disjoint, a distinguished $u \in R$, a unary operation $- : R_0 \to R_0$, where $R_0 := R \setminus \{u\}$, a binary operation $+ : R \times R \to R$, another binary operation $\cdot : R \times R \to R \cup N$, and a binary relation C on R.

There are 8 axioms for the RCC:

RCC 1. $(\forall x \in R)xCx$

RCC 2. $(\forall x, y \in R)[xCy \Rightarrow yCx]$

RCC 3. $(\forall x \in R)xCu$

RCC 4. $(\forall x \in R, y \in R_0)$,

(a) $\langle x, -y\rangle \in C \iff \neg\, xNTPPy$

(b) $\langle x, -y\rangle \in O \iff \neg\, xPy$

RCC 5. $(\forall x, y, z \in R)[\langle x, y + z\rangle \in C \iff xCy$ or xCz

RCC 6. $(\forall x, y, z \in R)[\langle x, y \cdot z\rangle \in C \iff (\exists w \in R)(wPy$ and wPz and $xCw)]$

RCC 7. $(\forall x, y \in R)[x \cdot y \in R \iff xOy]$

RCC 8. If xPy and yPx, then $x = y$.

We have added RCC 8, since this is what is intended, but it does not seem to follow from the other axioms. By the definition of P, Lemma 2.4.1, and RCC 8, we see that P is a partial order. The original RCC system contains another axiom:

> "...A rather deep theorem of the theory is given by the formula $\forall x \exists y[NTPP(y, x)]$ which was demonstrated by informal argument in Randell, Cohn, and Cui (1992a). Because we have so far not been able to give a fully formal proof of this theorem we often regard the formula as an additional axiom of the theory" (Cohn *et al.*, 1997).

Below follows a simple proof of this property:

Lemma 4.1.

$$(\forall x \in R)(\exists y \in R) y NTPP x \tag{4.1}$$

Proof. Assume that there is some $x \in R$ such that for all $y \in R$, $\neg y NTPP x$; By RCC 4a, this implies that $yC - x$ for all $y \in R$. Since $P = C \setminus C$– i.e. P is the largest relation S on R with $C \circ S \leq C$–, and $\langle x, -x \rangle \notin P$, we obtain that $C \circ \{\langle x, -x \rangle\} \not\leq C$. Hence, there is some $t \in R$ such $\langle t, -x \rangle \notin C$, a contradiction. □

According to Randell *et al.* (1992a), the weak composition table has the form given in Tab. 2. Since there are eight base relations, the system is called RCC8.

Table 2: RCC8 weak composition table

$\circ_w$		C					
	DR		O				
				PP		PP˘	
	DC	EC	PO	TPP	NTPP	TPP˘	NTPP˘
DC	1	DR,PO,PP	DR,PO,PP	DR,PO,PP	DR,PO,PP	DC	DC
EC	DR,PO,PP˘	1’,DR,PO, TPP TPP˘	DR,PO,PP	EC,PO,PP	PO,PP	DR	DC
PO	DR,PO,PP˘	DR,PO,PP˘	1	PO,PP	PO,PP	DR,PO,PP˘	DR,PO,PP˘
TPP	DC	DR	DR,PO,PP	PP	NTPP	1’,DR,PO, TPP,TPP˘	DR,PO, PP˘
NTPP	DC	DC	DR,PO,PP	NTPP	NTPP	DR,PO,PP	1
TPP˘	DR,PO,PP˘	EC,PO,PP˘	PO,PP˘	1’,PO, TPP,TPP˘	PO,PP	PP˘	NTPP˘
NTPP˘	DR,PO,PP˘	PO,PP˘	PO,PP˘	PO,PP˘	O ∪1’	NTPP˘	NTPP˘

Recall that the weak composition $\circ_w$ is to be interpreted as minimal inclusion, e.g. $DC \circ_w EC = DR \cup PO \cup PP$ means that $DC \circ EC \subseteq DR \cup PO \cup PP$, and $DC \circ EC$ intersects each relation on the right hand side. The table is, in a way, a minimal requirement: It follows from the RCC axioms that each RCC model must satisfy the condition given in the table.

Michael Winter has pointed out, that Table 2 has an extensional interpretation, namely, the *closed circle algebra* introduced in Düntsch *et al.* (1999b). There, the domain of regions is the collection of closed circles in the Euclidean plane, and $xCy \iff x \cap y \neq \emptyset$.

4.2 RCC models are Boolean algebras

As in Clarke’s system, the operations of the RCC axioms are called “quasi–Boolean”. In contrast to Clarke’s operations – which define the more general orthocomplemented lattices –, our next result shows that RCC operations indeed define a Boolean algebra, if we extend them and the relation P over the set $N = \{0\}$ in a natural way[2].

Proposition 4.2. *Suppose that* $N = \{0\}$*, and let* $R^+ = R \cup \{0\}$*. Then,*

$$B = \langle R^+, +, \cdot, -, 0, u \rangle$$

[2] The same result has been independently obtained by John Stell (1997).

is an atomless Boolean algebra with natural order P.

Proof. We extend P in such a way that $0Px$ for all $x \in R^+$; furthermore, we set $-u = 0,\ -0 = u$, and also, $x \cdot y = 0 \overset{\text{def}}{\iff} \neg xOy$. Finally, we extend $+$ in the obvious way. It is clear that all these extensions can be reversed uniquely, so that we can always return to the original structure.

Note that $\langle R, P \rangle$ is atomless by Lemma 4.1. The claim follows now from the statements below:

1. $(\forall x \in R)xPu$, and u is the only element with this property.
2. $x + y$ is the supremum of x and y w.r.t. P.
3. $x \cdot y$ is the infimum of x and y w.r.t. P.
4. $-x$ is the unique complement of x.
5. The lattice $\langle R^+, +, \cdot, -, 0, u \rangle$ is modular.

1. The first part follows immediately from RCC 3 and the definition of P. Suppose that xPv for all $x \in R$; then, in particular, uPv. Since vPu, we have $u = v$ by RCC 8.

2. Assume that $\langle x, x+y \rangle \notin P$. Then, there is some $z \in R$ such that zCx and $\langle z, x+y \rangle \notin C$. This contradicts RCC 5. Now, suppose that xPz and yPz, and that $\langle w, x+y \rangle \in C$. By RCC 5, we can assume w.l.o.g. that wCx. Now, $wCxPz$, and $C \circ P \subseteq C$ implies wCz. The definition of P now gives us $\langle x+y, z \rangle \in P$.

3. Suppose w.l.o.g that $x \cdot y \in R$. Let $\langle z, x \cdot y \rangle \in C$. By RCC 6, there is some $w \in R$ such that $wPx,\ wPy$, and zCw, and thus, $\langle z, x \rangle,\ \langle z, y \rangle \in C \circ P \subseteq C$. This shows $\langle x \cdot y, x \rangle \in P$, $\langle x \cdot y, y \rangle \in P$. Now, let $z \in R, zPx,\ zPy$, and wCz; we need to show that $\langle w, x \cdot y \rangle \in C$. This follows immediately from RCC 6.

4. If $\langle x, -x \rangle \in O$, then $\neg xPx$ by RCC 4, a contradiction. Thus, $x \cdot -x = 0$ by RCC 7. Let $z \in R$, and assume that $\neg\ zC(x + -x)$. Then, by RCC 5, $\neg\ zCx$, $\neg\ zC - x$, and RCC 4 implies that $zNTPPx$ and $zNTPP - x$. Now, $NTPP^{\smile} \circ NTPP \subseteq O$, and it follows that $\langle x, -x \rangle \in O$, a contradiction.

Next, we show that $-(-x) = x$ which will be needed for the uniqueness proof. Since $\langle -(-x), -x \rangle \notin O$, we obtain $-(-x) \leq x$. For the converse, assume that $x \not\leq -(-x)$. By definition of P, there is some $w \in R$ such that wCx and $\neg wC - (-x)$, i.e. $wNTPP - x$. Now, $xEC - x$, and thus $xEC \circ NTPP^{\smile} z$; however, $EC \circ NTPP \subseteq DC$, contradicting xCw.

It remains to show that $-x$ is the unique complement of x. Suppose that $x + y = u,\ x \cdot y = 0$. Then, $\neg xOy$ by RCC 7, and RCC 4b implies that $y \leq -x$. Assume that $-x \not\leq y$; then, again by RCC 4b, $-xO - y$, i.e. $-x \cdot -y > 0$. With (4.1) choose some $w \in R$ such that $wNTPP - x,\ wNTPP - y$. Since $x + y = u$, let w.l.o.g wCx. By RCC 4a we have $\langle w, -x \rangle \notin NTPP$, a contradiction.

We also note that $x \leq y$ implies $-y \leq -x$: Assume not; then, there is some $z \in R$ such that $zC - y$. i.e. $\neg zNTPPy$, and $\neg zC - x$, i.e. $zNTPPx$. Since $x \leq y$ and $NTPP \circ P \subseteq NTPP$, we obtain $zNTPPy$, a contradiction.

It follows that $-$ also fulfills the De Morgan condition

$$-(x+y) = -x \cdot -y :$$

"$\leq$": Since $x \leq x+y$ and $y \leq x+y$, we have

$$-(x+y) \leq -x \text{ and } -(x+y) \leq -y.$$

"$\geq$": We show $x+y \leq -(-x \cdot -y)$. Assume w.l.o.g. that $x \not\leq -(-x \cdot -y)$; then, $xO(-x \cdot -y)$, contradicting $\langle x, -x \rangle \notin R$.

5. To show modularity, it is enough to prove

(4.2) $$\text{If } zPx, \text{ then } z + (x \cdot y) = x \cdot (y+z),$$

see Grätzer (1978), Lemma I.4.12.

"$\leq$": In any lattice we have

(4.3) $$z + x \cdot y \leq (z+x) \cdot (z+y),$$

see Grätzer (1978), Lemma I.4.9. $z \leq x$ implies $z + x = x$, and the claim follows.

"$\geq$": We first show

(4.4) $$\text{If } t \leq -z \text{ and } t \leq y+z, \text{ then } t \leq y.$$

Proof. Assume not; then, $tO-y$ by RCC 4b. By (4.1) and the definition of O there is some $w \in R$ such that $wNTPPt$ and $wNTPP-y$. It follows that wCt, wDy, and wDz, the latter because $wNTTPt \leq -z$. On the other hand, since $t \leq y+z$, we have sCy or sCz for any s with sCr, a contradiction. □

Next, we need

(4.5) $$x \cdot y + x \cdot -y = x.$$

Proof. Since $x \cdot y \leq x$ and $x \cdot -y \leq x$, we have $x \cdot y + x \cdot -y \leq x$.

Conversely,

$$\begin{aligned} x \not\leq x \cdot y + x \cdot -y &\iff xO-(x \cdot y + x \cdot -y) \\ &\iff xO(-x + -y) \cdot (-x + y) \\ &\iff xO(-x + -y) \text{ and } xO(-x+y). \end{aligned}$$

Thus, if this is true, there some w such that $wNTPPx$ and $wNTPP(-x + -y)$, and from (4.4) we obtain that $w \leq -y$; it follows that $xO-y$. Similarly, we see that xOy, a contradiction. □

Now, assume that $x \cdot (y+z) \not\leq z + x \cdot y$, and w.l.o.g. $x \cdot (y+z) \in R$. Then, by definition of P, there is some $w \in R$ such that

1. $wC(x \cdot (y+z))$, and
2. $\neg\, wC(z + x \cdot y)$.

The first condition says with RCC 6 that there is a $t \in R$ such that

(4.6) $$t \leq x \text{ and } t \leq y \text{ and } tCw.$$

The second condition tells us with RCC 5, RCC 4a, and RCC 6 that $wNTTP - z$, and

(4.7) $$(\forall s)[s \leq x \text{ and } s \leq y \text{ imply } \neg\, wCs].$$

Set $s = t \cdot -z$. If $s = 0$, then $\neg tCw$, since $wNTPP - z$, a contradiction. Otherwise, $s \in R,\ s \leq x$, and $s \leq y$, the latter by (4.4). Now,

$$\begin{aligned} wCt &\iff wC(t \cdot z + t \cdot -z) && \text{by (4.5)} \\ &\iff wCt \cdot z \text{ or } wCt \cdot -z && \text{by RCC 5} \\ &\iff wCt \cdot -z && \text{since } wNTPP - z, \\ &\iff wCs \end{aligned}$$

which contradicts (4.7). □

4.3 Refining the RCC table

It is pointed out in Bennett *et al.* (1997) that the RCC axioms do not take into account that the largest region u is definable. Our next task will be to refine the RCC table to take care of this fact. Set $U = R$, $U_0 = \{u\},\ U_1 = R \setminus \{u\}$, and $U_{ij} = U_i \times U_j$ for $i, j \leq 1$; it is easy to check that for all base relations S of the RCC (listed on p. 8), and $i, j \leq 1$,

$$U_{ij} \subseteq S \text{ or } U_{ij} \cap S = \emptyset.$$

Now,

$$\begin{aligned} U_{00} &= 1' \cap -[(PP \circ PP^{\smile}) \cap 1'], \\ 1'_u &= 1' \cap -U_{00}, \\ U_{11} &= 1'_u \circ U^2 \circ 1'_u, \\ U_{01} &= U_{00} \circ U^2 \circ U_{11}, \\ U_{10} &= U_{11} \circ U^2 \circ U_{00}, \end{aligned}$$

which shows that all U_{ij} and $1'_u$ are RA definable. The equation which tells us that u is the largest element with respect to P now is

(4.8) $$U_{10} \subseteq P.$$

Thus, in the sequel, we shall restrict the relations to $R \setminus \{u\}$. In order to show that the defining equations on p. 8 and the axioms still hold, it suffices to prove it for O, TPP, and the axiom RCC 6, since all other definitions, respectively axioms, are universal, and thus carry over to substructures. This is straightforward, and is left to the reader; note that complementation of relations is restricted to $R \setminus \{u\} \times R \setminus \{u\}$.

Let $\#$ be the incomparability relation, i.e. $\# = -(P \cup P^{\smile})$. We extend the original RCC8 by replacing EC by

$$ECD = -(PP \circ PP^{\smile} \cup PP^{\smile} \circ PP),$$
$$ECN = EC \cap -ECD,$$

and PO by

$$PON = \# \cap (PP^{\smile} \circ PP) \cap (PP \circ PP^{\smile}),$$
$$POD = \# \cap (PP^{\smile} \circ PP) \cap -(PP \circ PP^{\smile}).$$

Then,

$$xECDy \iff x = -y,$$
$$xECNy \iff xECy \text{ and } x + y \neq u,$$
$$xPONy \iff x\#y,\ x \cdot y \neq 0,\ x + y \neq u,$$
$$xPODy \iff x\#y,\ x \cdot y \neq 0,\ x + y = u.$$

This gives us 10 base relations, and we call the resulting system RCC10. The extended weak composition can be found in Table 3 on the following page. For cells containing $=$, the RCC axioms together with general RA properties such as Lemma 2.4 or the equations (2.1) imply that strict composition (i.e. equality) holds; for cells containing $\neq$, there is a model in which the composition is strictly smaller than the cell entry. For cell entries which can be shown to be below the weak composition we use the superscript $^{\leq}$. In this way, we indicate in which cells the composition is extensional, and when it need not be.

In computing the table, we have used the RA scratchpad, which in turn uses Lemma 2.5 and the equations (2.1); we are grateful to Michael Winter who spotted and corrected several inaccuracies. We have also used the following properties, which may be interesting in their own right.

Lemma 4.3. *1.* $xECNy \iff xTPP - y$.

2. *If* $xDCz$, *then* $xTPP(x + z)$.

3. $xNTPPz$ *and* $yNTPPz \iff (x + y)NTPPz$.

4. *If* $xNTPPz$, *then* $-x \cdot zTPPz$.

Table 3: The RCC 10 weak composition table

$\circ_w$	TPP	TPP$^\smile$	NTPP	NTPP$^\smile$	PON	POD	ECN	ECD	DC
TPP	TPP, NTPP	1', TPP, TPP$^\smile$, PON, ECN, DC, $\neq$	NTPP, =	TPP$^\smile$, NTPP$^\smile$($\leq$), PON, ECN, DC, $\neq$	TPP, NTPP ($\leq$), PON ($\leq$), ECN, DC ($<$)	TPP, NTPP, PON, POD ($\leq$), ECN, ECD ($<$)	ECN, DC	ECN, =	DC, =
TPP$^\smile$	1', TPP, TPP$^\smile$, PON, POD	TPP$^\smile$, NTPP$^\smile$	TPP, NTPP, PON, POD	NTPP$^\smile$, =	TPP$^\smile$, NTPP$^\smile$ ($\leq$), PON ($\leq$), POD	POD	TPP$^\smile$, NTPP$^\smile$, PON, POD, ECN, ECD, $\neq$	POD	TPP$^\smile$, NTPP$^\smile$, PON, ECN, DC
NTPP	NTPP, =	TPP, NTPP, PON, ECN, DC, $\neq$	NTPP	1', TPP, TPP$^\smile$, NTPP, NTPP$^\smile$, PON, ECN, DC, =	TPP, NTPP, PON, ECN, DC, =	TPP, NTPP, PON, POD, ECN, ECD, DC,=	DC,=	DC, =	DC
NTPP$^\smile$	TPP$^\smile$, NTPP$^\smile$, PON, POD	NTPP$^\smile$, =	1', TPP, TPP$^\smile$, NTPP, NTPP$^\smile$, PON, POD, =	NTPP$^\smile$	TPP$^\smile$, NTPP$^\smile$, PON, POD, =	POD	TPP$^\smile$, NTPP$^\smile$, PON, POD	POD	TPP$^\smile$, NTPP$^\smile$, PON, POD, ECN, ECD, DC, =
PON	TPP, NTPP($\leq$), PON($\leq$), POD	TPP$^\smile$, NTPP$^\smile$, PON, ECN, DC	TPP($\leq$), NTPP, PON, POD($\leq$)	TPP$^\smile$, NTPP$^\smile$, PON, ECN, DC,=	1', TPP, TPP$^\smile$, NTPP, NTPP$^\smile$, PON, POD, ECN, ECD, DC,=	TPP, NTPP, PON, POD, =	TPP$^\smile$, NTPP$^\smile$, PON, ECN, DC, $\neq$	PON, =	TPP$^\smile$, NTPP$^\smile$, PON, ECN, DC, =
POD	POD	TPP$^\smile$, NTPP$^\smile$, PON, POD($\leq$), ECN, ECD($<$)	POD	TPP$^\smile$, NTPP$^\smile$, PON, POD, ECN, ECD, DC,=	TPP$^\smile$, NTPP$^\smile$, PON, POD,=	1', TPP, TPP$^\smile$, NTPP, NTPP$^\smile$, PON, POD,=	TPP$^\smile$, NTPP$^\smile$($\leq$)	TPP$^\smile$, NTPP$^\smile$	NTPP$^\smile$,=
ECN	TPP, NTPP, PON, POD, ECN, ECD ($<$), $\neq$	ECN, DC	TPP, NTPP, PON, POD	DC, =	TPP, NTPP, PON ($\leq$), ECN, DC ($<$), $\neq$	TPP, NTPP ($\leq$)	1', TPP, TPP$^\smile$, PON, ECN, DC	TPP, =	TPP$^\smile$, NTPP$^\smile$ ($\leq$), PON, ECN, DC
ECD	POD	ECN,=	POD	DC	PON,=	TPP, NTPP,=	TPP$^\smile$,=	1',=	NTPP$^\smile$,=
DC	TPP, NTPP, PON, ECN, DC	DC =	TPP, NTPP, PON, POD, ECN, ECD, DC,=	DC	TPP, NTPP, PON, ECN, DC,=	NTPP, =	TPP, NTPP, PON, ECN, DC, =	NTPP, =	1', TPP, TPP$^\smile$, NTPP, NTPP$^\smile$, PON, ECN, DC

Proof. 1: Suppose that $x \neq -y$, so that $x + y \neq u$. Then,

$$\begin{aligned} xECNy &\iff x \cdot y = 0,\ xCy \\ &\iff x \lneq -y,\ xCy \\ &\iff x \lneq -y,\ x(-NTPP) - y \\ &\iff xTPP - y. \end{aligned}$$

2: Let $xDCz$; then, $z \lneq -x$ and $x(-C)z$. Since $xC - x$, and $-x = z + -(x + z)$, we have $xC - (x + z)$ by RCC 5. Thus, $x(-NTPP)(x + z)$ by RCC 4a.

3: "⇒": Let $xNTPPz,\ yNTPPz$ and assume $\neg\ (x + y)NTPPz$. Then, by RCC 4a, $(x + y)Cz^*$, and RCC 5 implies that w.l.o.g. xCz^*. RCC 4a now implies $\neg xNTPPz$, a contradiction.

"⇐": Let $(x + y)NTPPz$, and assume that $\neg xNTPPz$. Then, xCz^*, which, together with $\neg(x + y)Cz^*$ contradicts RCC 5.

4: This follows from $x(-NTPP)x$ and 3 by setting $y = -x \cdot z$. □

There is no relation algebra which is a model of the RCC10 table. In the standard model of nonempty proper regular open sets of a regular connected topological space, there are at least 25 atoms Düntsch *et al.* (1999a).

4.4 A reduced set of RCC relations

The subset $\{1', DR, PO, PP, PP^{\smile}\}$ of RCC relations has received some attention, and is usually called RCC5. It arises from disregarding the split of C into O and EC, and PP into TPP and $NTPP$; in other words, one adds the additional axiom $C = O$. If one takes the weak composition induced by the RCC8 table, one arrives at Table 4.

Table 4: RCC5 weak composition table

$\circ_w$	*DR*	*C = O*		
		PO	*PP*	*PP*$^{\smile}$
DR	*1*	*DR, PO, PP*	*DR, PO, PP*	*DR*
PO	*DR, PO, PP*$^{\smile}$	*1*	*PO, PP*	*DR, PO, PP*$^{\smile}$
PP	*DR*	*DR, PO, PP*	*PP*	*1*
PP$^{\smile}$	*DR, PO, PP*$^{\smile}$	*PO, PP*$^{\smile}$	$-DR$	*PP*$^{\smile}$

If we take into consideration that the largest region u is RA definable and work within $R \setminus \{u\}$, then, similar to the RCC10 table, PO splits into PON and POD, and DR splits into DN and DD, where DD is the complement, and $DN = DR \cap -DD$. The composition induced by the RCC10 table is given in Table 5.

The next proposition shows that the RCC7 table – and thus RCC5 – has a very simple interpretation (Düntsch *et al.*, 1998). Suppose that B is an atomless Boolean algebra, and that $B_0 = B \setminus \{0, 1\}$;

Table 5: RCC7 table

$\circ$	*DR*		*O*			
	DN	*DD*	PON	POD	*PP*	*PP*$^{\smile}$
DN	$-(POD \cup DD)$	*PP*	*DN,PON,PP*	*PP*	$-(P^{\smile} \cup 1')$	*DN*
DD	*PP*$^{\smile}$	*1'*	*PON*	*PP*	*POD*	*DN*
PON	*DN, PON, PP*$^{\smile}$	*PON*	*1*	*PON,POD,PP*	*PON,POD,PP*	*DN,PON, PP*$^{\smile}$
POD	*PP*$^{\smile}$	*PP*$^{\smile}$	*PON,POD, PP*$^{\smile}$	$O \cup 1'$	*POD*	$-(PP \cup 1')$
PP	*DN*	*DN*	*DN, PON, PP*	$-(PP \cup 1')$	*PP*	$-(POD \cup DD)$
PP$^{\smile}$	$-(PP \cup 1')$	*POD*	*PON,POD, PP*$^{\smile}$	*POD*	$O \cup 1'$	*PP*$^{\smile}$

also, let $P = \leq$ be the natural order on B, and $PP = \lneq$. Furthermore, define the following relations on B_0:

$$\begin{aligned}
O &= (PP^{\smile} \circ PP) \cap -1' && = \{\langle x, z\rangle : x \neq z,\ x \cdot z \neq 0\} \\
T &= (PP \circ PP^{\smile}) \cap -1' && = \{\langle x, z\rangle : x \neq z,\ x + z \neq 1\} \\
PON &= O \cap \# \cap T && = \{\langle x, z\rangle : x\#z,\ x \cdot z \neq 0,\ x + z \neq 1\} \\
POD &= O \cap \# \cap -T && = \{\langle x, z\rangle : x\#z,\ x \cdot z \neq 0,\ x + z = 1\} \\
DN &= -O \cap T && = \{\langle x, z\rangle : x \cdot z = 0,\ x + z \neq 1\} \\
DD &= -(O \cup T \cup 1') && = \{\langle x, z\rangle : x \cdot z = 0,\ x + z = 1\},
\end{aligned}$$

where $x, z \in B_0$. We now have

Proposition 4.4. *Let B be an atomless Boolean algebra. Then, the relations*

$$1',\ PP,\ PP^{\smile},\ PON,\ POD,\ DN,\ DD$$

as defined above are the atoms of the algebra $\mathcal{G}$ on B_0 generated by P whose composition table is given in Tab. 5.

Proof. Clearly, these relations partition $B_0 \times B_0$. The computations are straightforward, if somewhat tedious, and are left to the reader. $\square$

In the algebra $\mathcal{G}$, there are two possibilities to define a relation C which satisfies (A 1) and (A 2): We can take either $C = O \cup 1'$ or $C = O \cup DD \cup 1'$. In both cases, $P = C \setminus C$. If the BA is complete, then, the first case, we have a model of classical mereology (if we remove 0).

This seems a very general result: Whenever a relational model for spatial reasoning assumes an underlying atomless Boolean algebra with the Boolean ordering as the "part – of" relation, then the relations of $\mathcal{G}$ must be present. Indeed, every relation ov on an atomless Boolean algebra which satisfies Clarke's axioms A 1 and A 2 on page 3 with $\leq = P$, must satisfy $O \subseteq$ ov by Lemma 2.4.

It may be interesting to note that Table 4 as an extensional interpretation by the algebra generated by the relation

(4.9) $$xCy \iff x \cap y \neq \emptyset,$$

defined on the collection of all nonempty proper regular open sets of a regular connected topological space.

Our final result characterizes those Boolean models of $\mathcal{G}$ which are Galois closed. For this, we need some preparation. If B is a Boolean algebra and $x \in B$, then $B \upharpoonright x$ is the Boolean algebra with base set $\{y \in B : y \leq x\}$, meet and join inherited from B, and complementation relative to x. B is called *homogeneous*, if $B \upharpoonright x \cong B$ for every $x \in B, x > 0$. The following characterisation of homogeneous BAs can be found in Koppelberg (1989), 9.13:

Lemma 4.5. *Let $|B| \neq 4$. Then, B is homogeneous iff for all $x, y \in B,\ 0 < x, y < 1$ there is an automorphism of B taking x to y.*

In other words, an infinite B is homogeneous iff the restrictions of its automorphisms to $B \setminus \{0, 1\}$ form a transitive group.

Lemma 4.6. *Let $\langle B, \leq \rangle$ be an atomless BA and $\mathcal{G}$ the RA generated by $\leq$. Then, $\mathcal{G}^\rho$ is the automorphism group of B.*

Proof. Since $\mathcal{G}$ is generated by $\leq$, it suffices to consider all permutations φ of B_0 for which

$$x \leq y \iff \varphi(x) \leq \varphi(y). \tag{4.10}$$

i.e. the order isomorphisms of $\langle B_0, \leq \rangle$; it is well known that these permutations are exactly the automorphisms of B. □

Proposition 4.7. *$\mathcal{G}$ is Galois closed if and only if B is homogeneous.*

Proof. "⇒": If $\mathcal{G}$ is Galois closed, then, in particular, its group of base automorphisms is transitive. By the preceding lemma, the automorphism group of B is transitive, which is the case just when B is homogeneous by Lemma 4.5.

"⇐": Let $G = \mathcal{G}^\rho$; we show that the orbits of G are just the atoms of $\mathcal{G}$. Since G is transitive, $1'$ is an atom, and since every automorphism of B preserves complements, and G is transitive, we see that DD is an orbit of G.

If $0 \lneq a \lneq b \lneq 1$ and $0 \lneq c \lneq d \lneq 1$, then homogeneity implies that

$$B \upharpoonright b \cong B \upharpoonright d \cong B \upharpoonright -b \cong B \upharpoonright -b \cong B.$$

Thus, let $g : B \upharpoonright -b \to B \upharpoonright -d,\ h : B \upharpoonright b \to B \upharpoonright d$ be isomorphisms. Furthermore, let $p : B \upharpoonright d \to B \upharpoonright d$ be an isomorphism such that $p(h(a)) = c$, and set $q = p \circ h$. Then, by 9.13. of Koppelberg (1989), $f : B \to B$ defined by $f(x) = q(x \cdot b) + g(x \cdot -b)$ is an automorphism of B such that $f(a, b) = \langle c, d \rangle$, and it follows that P and $P^{\smile}$ are orbits of G.

Next, let $a \cdot b = 0,\ c \cdot d = 0,\ a + b \lneq 1,\ c + d \lneq 1$. If $\varphi \in G$, then φ preserves these properties, and hence, $G_{a,b} \subseteq DN$. Noting that $a \lneq -b$, and $c \lneq -d$, we can use the previous construction to find an isomorphism f of B with $f(a, b) = \langle c, d \rangle$; hence, $G_{a,b} = DN$.

Let a, b and c, d be incomparable, $a \cdot b \neq 0,\ c \cdot d \neq 0,\ a + b \lneq 1,\ c + d \lneq 1$. As above, these properties are preserved by isomorphisms. Since $0 \neq a \cdot b \lneq a \lneq a + b$, and $0 \neq c \cdot d \lneq c \lneq c + d$, there is an isomorphism $f : B \upharpoonright (a + b) \to B \upharpoonright (c + d)$ such that $f(a \cdot b) = c \cdot d$ and $f(a) = c$; now, $f(-a \cdot (a + b)) = -c \cdot (c + d))$, since f preserves complements. Furthermore,

$$\begin{aligned} f(b) &= f(a \cdot b + -a \cdot b) \\ &= f(a \cdot b + -a \cdot (a + b)) \\ &= f(a \cdot b) + f(-a \cdot (a + b)) \\ &= c \cdot d + -c \cdot (c + d) \\ &= d. \end{aligned}$$

If $g : B \upharpoonright -(a + b) \to B \upharpoonright -(c + d)$ is an isomorphism, then $h : B \to B$, defined by $h(x) = f(x \cdot (a + b)) + g(x \cdot -(a + b))$ is the desired isomorphism. If $a + b = 1,\ c + d = 1$, then we can just use f as defined above. In the first case we see that $G_{a,b} = PON$, and in the second that $G_{a,b} = POD$. □

To show that $\mathcal{G}$ is Galois closed over the standard model of Euclidean regions, we first need

Lemma 4.8. *(Birkhoff, 1948, p.177)*
If X is a subspace of the Euclidean space without isolated points, then the Boolean algebra of regular open sets of X is (isomorphic to) the completion of the free countable Boolean algebra.

Proposition 4.9. *$\mathcal{G}$ is Galois closed (and hence first order closed) over the BA of regular open sets of a Euclidean space.*

Proof. This follows from the previous lemma and the facts that any infinite free BA and its completion are homogeneous. □

We should like to close with the following observation: If A is an integral BRA on the Boolean algebra B, and A is obtained from a model of RCC10, then B must be homogeneous in order for A to be first order closed. This can be seem as follows: Since $\mathcal{G} \leq A$, we have $A^\rho \leq \mathcal{G}^\rho$. If A is first order closed, then A^ρ is transitive by Lemma 2.2, and thus, $\mathcal{G}^\rho$ is transitive as well. By Lemmas 4.6 and 4.5, B is homogeneous.

5 Summary and outlook

We have explored the properties of the connection relation in the RCC with the tools of relation algebras. The fact that the part – of relation is the right residual of the connection relation led to an easy proof that the density axiom of the RCC is redundant. We also show that the RCC operations indeed define a Boolean algebra. Taking into account that the largest region is RA definable, we have refined the RCC8 table to arrive at 10 base relations, one of which is complementation. The RCC5 relations in their refined form RCC7 lead to models of classical mereology, but also to a notion of connectedness, where a region is connected to its complement. We have also shown that a representation of RCC7 over a Boolean algebra is Galois closed if and only if B is homogeneous.

Another area which deserves to be looked at is the complexity of mereological RAs. The complexity of Allen's interval algebra has been studied by Ladkin and Maddux (1994), and more general results, which may serve as a starting point, can be found in Hirsch (1997).

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