

# Chapter 16

## Beyond Modalities: Sufficiency and Mixed Algebras

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**Abstract.** In [24] a generalisation of relation algebras to Boolean algebras with normal and additive operators is introduced. These operators are the counterparts to the modal operators of possibility. In this paper we introduce a class of Boolean algebras with co-normal and co-additive operators referred to as sufficiency operators. They are the algebraic counterpart to the logical sufficiency operators introduced in [17] for an extension of modal logics. Next, we define a class of mixed algebras i.e., Boolean algebras with an additional modal operator and a sufficiency operator. We study representation and duality theory for these new classes of algebras. The motivation for those algebras comes from the problems of reasoning with incomplete information and spatial reasoning.

**Keywords:** Boolean algebras with operators, modal operator, sufficiency operator, Kripke frame, incomplete information, spatial reasoning

### 1 Introduction

Relational systems (Kripke frames) are widely used as semantics for traditional modal logics. Vice versa, correspondence theory looks for modal expressions which describe relational properties [41]. Several simple properties of binary relations, however, cannot be expressed by modal sentences, a case in point being co-reflexivity (irreflexivity). Noting that a relation is co-reflexive if and only if its complement is reflexive – and reflexivity is modally expressible – [20] introduced an “inaccessibility” operator, which was determined by the complement of a frame relation; a similar idea was put forward in [17] where a “sufficiency” operator is used. We invite the reader to consult this paper for a discussion on the merits or otherwise of Kripke semantics and its “sufficiency” extension.

Just as Kripke frames are dual to a class of Boolean algebras with modal operators [18,24], one can build a duality for frames and Boolean algebras with sufficiency operators. Mixed structures occur when modal and sufficiency operators arise from the same accessibility relation.

In this paper we introduce the classes of sufficiency algebras and that of mixed algebras which include both a modal and a sufficiency operator, and study representation and duality theory for these classes of algebras. We also give examples for classes of first-order definable frames, where such operators are required for a “modal-style” axiomatisation.

## 2 Why sufficiency and mixed algebras?

One primary area, where sufficiency and mixed operators are required is the treatment of relations arising from information systems. In its general form, such a system consists of a set  $OB$  of objects and a set  $A$  of functions  $a : OB \rightarrow 2^{V_a}$ , each of which assigns to an object  $x$  a set of attribute values  $a(x)$ ; such a system  $\mathcal{I} = \langle OB, A, \{V_a : a \in A\} \rangle$  is called an *information system*.

Suppose that  $P$  is a set of attributes, and that  $R_a$  is a binary relation on  $OB$  for each attribute  $a$ .  $P$  determines two relations (strong and weak relation) on the object set  $OB$  with respect to the family  $\langle R_a \rangle_{a \in A}$ , namely,

$$\begin{aligned} xR_P^s y &\Leftrightarrow a(x)R_a a(y) \text{ for all } a \in P, \\ xR_P^w y &\Leftrightarrow a(x)R_a a(y) \text{ for some } a \in P. \end{aligned}$$

Let  $P, Q \subseteq A$ . In an algebraic setting, strong relations are characterised by the condition

$$\begin{aligned} R_{P \cup Q}^s &= R_P^s \cap R_Q^s, \\ R_\emptyset^s &= OB^2, \end{aligned}$$

and weak relations by

$$\begin{aligned} R_{P \cup Q}^w &= R_P^w \cup R_Q^w, \\ R_\emptyset^w &= \emptyset. \end{aligned}$$

These relations are commonly called *information relations*, and an overview can be found in [29,32]. There are two types of information relations: Those which express similarity of objects, and those which describe some form of distinctness. The most prominent example of the first type is that of *indiscernibility*:

$$x \text{indy} y \Leftrightarrow a(x) = a(y) \text{ for all } a \in P.$$

While relations of similarity have been frequently studied and are well understood [31], the situation of the distinctness relations is much less clear.

The relations of complementarity and incomplementarity in an information system are defined as follows:

$$x \text{ com}_P^s y \Leftrightarrow a(x) = -a(y) \text{ for all } a \in P, \tag{1}$$

$$x \text{ com}_P^w y \Leftrightarrow a(x) = -a(y) \text{ for some } a \in P, \tag{2}$$

$$x \text{ icom}_P^s y \Leftrightarrow a(x) \neq -a(y) \text{ for all } a \in P, \tag{3}$$

$$x \text{ icom}_P^w y \Leftrightarrow a(x) \neq -a(y) \text{ for some } a \in P. \tag{4}$$

Logical aspects of these relations are studied in [8,10,28]; some applications are outlined in [7].

In order to define the frames which these relations generate, we recall some definitions: A binary relation  $R$  on  $U$  is called

$$\begin{aligned} 3\text{-transitive} &\Leftrightarrow R; R; R \subseteq R, \\ \text{co-3-transitive} &\Leftrightarrow -R; -R; -R \subseteq -R, \\ \text{reflexive} &\Leftrightarrow I \subseteq R, \\ \text{co-reflexive} &\Leftrightarrow I \subseteq -R, \\ \text{symmetric} &\Leftrightarrow R^\sim \subseteq R. \end{aligned}$$

Here,  $;$  is relational composition,  $\sim$  is relational converse, and  $I$  is the identity relation.

The relations defined by (1)-(4) give rise to the following classes of frames of the form  $\langle U, \{R_P : P \subseteq A\} \rangle$ , where  $U$  and  $A$  are nonempty sets, and  $A$  is finite:

- COM** Strong complementarity frames; the relations are strong, symmetric, 3-transitive, and for each  $a \in A$ ,  $R_{\{a\}}$  is co-reflexive.
- WCOM** Weak complementarity frames; the relations are weak, symmetric, and for each  $a \in A$ ,  $R_{\{a\}}$  is co-reflexive and 3-transitive.
- ICOM** Strong incomplementarity relations are strong, symmetric, and for each  $a \in A$ ,  $R_{\{a\}}$  is reflexive and co-3-transitive.
- WICOM** Weak incomplementarity frames; the relations are weak, symmetric, co-3-transitive, and for each  $a \in A$ ,  $R_{\{a\}}$  is reflexive.

The algebraic study of the operators arising from the parameterised frames defined above requires mixed algebras.

However, in the present paper we consider frames with a single relation, and therefore we do not (need to) distinguish between strong and weak relations.

Our second example for mixed structures are *contact relations*. These arise in the context of qualitative geometry and spatial reasoning, going back to the work of [6,27,42], and, more recently, of [3,4,11,13,34] and others. They are a generalisation of the “part of” relation which for the first time was formalised by [26] in his mereology. We shall show that the class of frames  $\langle U, C \rangle$ , where  $C$  is a contact relation, can be captured by a mixed modal – sufficiency system, but not by equations of either system alone.

In this paper, we investigate sufficiency algebra with a single sufficiency operator, and mixed algebras with a single modal operator and a single sufficiency operator. In the forthcoming [12], we will present sufficiency and mixed algebras with multiple operators, arising from the parametrised frames mentioned above.

Some of the results below have been announced in [9].

### 3 Definitions and notation

In this Section we will recall the basic relationships between frames and modal algebras. We assume a basic knowledge of the theory of Boolean algebras and modal logic, and invite the reader to consult [25] for the first topic and [2] for the second one.

A *frame* is a pair  $\langle U, R \rangle$ , where  $R$  is a binary relation on  $U$ , called an *accessibility relation*. If  $x, y \in U$ , we usually write  $xRy$  for  $\langle x, y \rangle \in R$ , and set  $R(x) = \{y \in U : xRy\}$ . The *converse* of  $R$ , denoted by  $R^\sim$ , is the relation  $\{\langle x, y \rangle : yRx\}$ .

Suppose that  $\langle B, +, \cdot, -, 0, 1 \rangle$  is a Boolean algebra (BA). The set of atoms of a BA  $B$  will be denoted by  $At(B)$ . If  $f : B \rightarrow B$  is a mapping, then its *dual* is the mapping  $f^\partial : B \rightarrow B$  defined by

$$f^\partial(x) = -f(-x). \tag{5}$$

The *canonical extension* of  $B$  is a complete and atomic BA  $B^\sigma$  containing an isomorphic copy of  $B$  as a subalgebra with the properties

$$\text{Every atom of } B^\sigma \text{ is the meet of elements of } B. \tag{6}$$

$$\text{If } A \subseteq B \text{ such that } \sum_{B^\sigma} A = 1, \text{ then there is a finite subset } A_0 \text{ of } A \text{ whose join is } 1. \tag{7}$$

It is well known, that each BA has a canonical extension which is unique up to isomorphism. One such construction is given by Stone’s representation theorem for Boolean algebras: Let  $B^\sigma$  be the powerset algebra of the set of ultrafilters  $X$  of  $B$ , and embed  $B$  into  $B^\sigma$  by  $b \mapsto \{U \in X : b \in U\}$ . For more details and discussions we refer the reader to [24] and [21–23].

An operator  $f : B \rightarrow B$  is called *completely additive*, if

$$\text{If } \sum_{i \in I} b_i \text{ exists, then } \sum_{i \in I} f(b_i) \text{ exists, and is equal to } f(\sum_{i \in I} b_i). \tag{8}$$

A *modal operator* on  $B$  is a mapping  $f : B \rightarrow B$  for which

$$f(0) = 0, \quad \text{Normal} \tag{9}$$

$$f(a + b) = f(a) + f(b) \quad \text{Additive} \tag{10}$$

for all  $a, b \in B$ .

A *modal algebra* is a Boolean algebra with additional modal operators. Modal algebras are normal Boolean algebras with operators in the sense of [24]. The class of all modal algebras will be denoted by MOA. If  $B$  is a complete Boolean algebra, and  $f$  is a completely additive normal operator, then  $\langle B, f \rangle$  is a *complete modal algebra*.

If  $f$  is a modal operator on  $B$ , then the mapping  $f^\sigma : B^\sigma \rightarrow B^\sigma$  defined by

$$f^\sigma(x) = \sum \{ \prod \{ f(z) : z \in B, p \leq z \} : p \in At(B^\sigma), p \leq x \} \quad (11)$$

is called the *canonical extension of  $f$* .  $\langle B^\sigma, f^\sigma \rangle$  is complete modal algebra, called the *canonical extension of  $\langle B, f \rangle$* .

A *necessity operator* on  $B$  is a function  $g : B \rightarrow B$ , for which

$$g(1) = 1, \quad (12)$$

$$g(a \cdot b) = g(a) \cdot g(b) \quad \text{Multiplicative} \quad (13)$$

for all  $a, b \in B$ .

Modal and necessity operators are dual to each other: If  $g$  is a necessity (modal) operator, then,  $g^\partial$  defined by (5) is a modal (necessity) operator. Thus, a *dually modal algebra* is a Boolean algebra with an additional necessity operator.

If  $\langle U, R \rangle$  is a frame, then we define two mappings on the powerset algebra  $2^U$  by

$$\langle R \rangle(X) = \{x \in U : R(x) \cap X \neq \emptyset\}, \quad (14)$$

$$[R](X) = \{x \in U : R(x) \subseteq X\}. \quad (15)$$

In other words,

$$\langle R \rangle(X) = \{x \in U : (\exists y \in X) xRy\}, \quad (16)$$

$$[R](X) = \{x \in U : (\forall y \in U) [xRy \Rightarrow y \in X]\}, \quad (17)$$

and, in particular,

$$\langle R \rangle(\{x\}) = R^\sim(x). \quad (18)$$

The following result is fundamental:

**Proposition 1.** [24, Theorem 3.3.]

- (a) If  $K = \langle U, R \rangle$  is a frame, then  $\langle R \rangle$  is a complete modal operator on  $2^U$ ,  $[R]$  is a necessity operator, and both are dual to each other.
- (b) If  $f$  is a modal operator on  $2^U$ , and  $f^\partial$  its dual, then there is exactly one binary relation  $S_f$  on  $U$  such that  $\langle S_f \rangle = f$ , and  $[S_f] = f^\partial$ . This relation is defined by

$$xS_f y \Leftrightarrow x \in f(\{y\}). \quad (19)$$

The algebra  $\langle 2^U, \langle R \rangle \rangle$  is called the *full complex algebra of  $K$* . There is the following representation theorem:

**Proposition 2.** [24, Theorem 3.10]

If  $\langle B, f \rangle$  is a modal algebra, then there is, up to isomorphism, a unique frame  $\langle U, R \rangle$ , such that  $\langle 2^U, \langle R \rangle \rangle \cong \langle B^\sigma, f^\sigma \rangle$ .

$\langle U, R \rangle$  as above is called the *atomic structure of  $\langle B, f \rangle$* .

### 4 Representation theory for sufficiency algebras

In this section we introduce sufficiency algebras and prove representation theorems in analogy to those for modal algebras in the spirit of [24].

An operator  $g : B \rightarrow B$  is called *completely co-additive*, if

$$\text{If } \sum_{i \in I} b_i \text{ exists, then } \prod_{i \in I} g(b_i) \text{ exists, and is equal to } g\left(\sum_{i \in I} b_i\right). \quad (20)$$

A *sufficiency operator* on  $B$  is a function  $g : B \rightarrow B$  which satisfies

$$g(0) = 1, \quad \text{Co-normal} \quad (21)$$

$$g(a + b) = g(a) \cdot g(b) \quad \text{Co-additive} \quad (22)$$

for all  $a, b \in B$ . This is called a “strong permission operator” in [40]. A sufficiency operator which is completely co-additive is a *complete sufficiency operator*.

A *sufficiency algebra* is a Boolean algebra with an additional sufficiency operator; the class of sufficiency algebras will be denoted by SUA. With some abuse of language we will use MOA and SUA also for the respective algebras. A SUA  $\langle B, g \rangle$  is *atomic*, if  $B$  is atomic, and *complete*, if  $B$  is complete, and  $g$  is completely co-additive.

The next result is recorded for later use:

**Lemma 1.** *Suppose that  $g$  is a sufficiency operator on  $B$ . Then,  $g$  is anti-tone.*

*Proof.* Let  $x \leq y$ . Then,  $g(y) = g(x + -x \cdot y) = g(x) \cdot g(-x \cdot y) \leq g(x)$ .  $\square$

If  $g : B \rightarrow B$  is a mapping, we let  $g^c : B \rightarrow B$  be defined by  $g^c(x) = g(-x)$ . We call the mapping  $g^c$  the *complementary mapping of  $g$* , and two mappings  $f, g$  on  $B$  are *complementary*, if  $f = g^c$ .

Our first result is an algebraic version of the “correspondence theorem” for modal and sufficiency logic of [38], quoted in [17].

**Proposition 3.** (a) *If  $\langle B, g \rangle$  is a dually modal algebra, then  $\langle B, g^c \rangle$  is a sufficiency algebra.*

(b) *If  $\langle B, g \rangle$  is a sufficiency algebra, then  $\langle B, g^c \rangle$  is a dually modal algebra.*

*Proof.* We only show the first part, and leave the second part to the reader. Let  $\langle B, g \rangle$  be a dually modal algebra. First,

$$g^c(0) = g(1) = 1.$$

Let  $a, b \in B$ . Then,

$$g^c(a + b) = g(-a \cdot -b) = g(-a) \cdot g(-b) = g^c(a) \cdot g^c(b),$$

which completes the proof.  $\square$

Thus, necessity and sufficiency operators are mutually term definable, and the classes MOA and SUA are equipollent in the sense of [37].

If  $\langle B, g \rangle \in \text{SUA}$ , we let  $g^\sigma : B^\sigma \rightarrow B^\sigma$  be defined by

$$g^\sigma(x) = \prod \left\{ \sum \{g(z) : p \leq z, z \in B\} : p \in \text{At}(B^\sigma), p \leq x \right\}. \quad (23)$$

The pair  $\langle B^\sigma, g^\sigma \rangle$  is called the *canonical extension* of  $\langle B, g \rangle$ . The mapping  $g^\sigma$  does what we would expect it to do:

**Proposition 4.**  $g^\sigma$  is a complete sufficiency operator, and  $g^\sigma \upharpoonright B = g$ .

*Proof.* First,  $g(0) = \prod \emptyset = 1$ . Next, let  $\{b_i : i \in I\} \subseteq B^\sigma$ . Then,

$$\begin{aligned} g^\sigma \left( \sum \{b_i : i \in I\} \right) &= \\ &= \prod \left\{ \sum \{g(z) : p \leq z, z \in B\} : p \leq \sum \{b_i : i \in I\}, p \in U \right\}, \\ &= \prod \left\{ \sum \{g(z) : p \leq z, z \in B\} : p \leq b_i, i \in I, p \in U \right\}, \\ &= \prod \left\{ \prod \left\{ \sum \{g(z) : p \leq z, z \in B\} : p \leq b_i, p \in U \right\} : i \in I \right\}, \\ &= \prod \{g^\sigma(b_i) : i \in I\}. \end{aligned}$$

Finally, we show that  $g^\sigma(x) = g(x)$  for  $x \in B$ :

“ $\leq$ ”: Let  $q \in U$ ,  $q \leq g^\sigma(x)$ . Then,  $q \leq g^\sigma(p)$  for each  $p \in U$ ,  $p \leq x$ , and thus,  $q \leq \sum \{g(z) : z \in B, p \leq z\}$ . Therefore, for each such  $p$ , there is some  $z_p \in B$ , such that  $p \leq z_p$  and  $q \leq g(z_p)$ . We can choose  $z_p \leq x$  because of the following: Since  $p \leq z_p$  and  $p \leq x$ , we have  $p \leq z_p \cdot x$ ; furthermore,  $q \leq g(z_p) \leq g(z_p \cdot x)$  by Lemma 1. Now,  $x = \sum_{p \leq x} z_p$ , and we may choose  $\{z_p : p \leq x\}$  to be finite because of (7). Therefore,

$$q \leq \prod g(z_p) = g \left( \sum z_p \right) = g(x).$$

“ $\geq$ ”:  $g(x) \leq \sum \{g(z) : p \leq z, z \in B\} = g^\sigma(p)$  for each  $p \leq x$ ,  $p \in U$ , and thus,  $g(x) \leq \prod \{g^\sigma(p) : p \leq x, p \in U\} = g^\sigma(x)$ .  $\square$

In the rest of this Section, we will establish a representation theorem between frames and sufficiency algebras in analogy to Proposition 1.

**Proposition 5.** *Suppose that  $K = \langle U, R \rangle$  is a frame. The mapping  $[[R]] : 2^U \rightarrow 2^U$  with*

$$[[R]](X) = \{x \in U : X \subseteq R(x)\} \quad (24)$$

*is a complete sufficiency operator.*

*Proof.* 1.  $[[R]](\emptyset) = \{x \in U : \emptyset \subseteq R(x)\} = U$ .

2. Let  $X = \bigcup_{i \in I} X_i$ . Then,

$$\begin{aligned} x \in [[R]](X) &\Leftrightarrow X \subseteq R(x), \\ &\Leftrightarrow \bigcup_{i \in I} X_i \subseteq R(x), \\ &\Leftrightarrow (\forall i \in I) X_i \subseteq R(x), \\ &\Leftrightarrow (\forall i \in I) x \in [[R]](X_i), \\ &\Leftrightarrow x \in \bigcap_{i \in I} [[R]](X_i). \end{aligned}$$

This completes the proof.  $\square$

Observe that

$$\begin{aligned} x \in [R](X) &\Leftrightarrow (\forall y)[xRy \Rightarrow y \in X] \quad (y \in X \text{ is necessary for } xRy) \\ x \in [[R]](X) &\Leftrightarrow (\forall y)[y \in X \Rightarrow xRy]. \quad (y \in X \text{ is sufficient for } xRy) \end{aligned}$$

which explains the names of the operators. Furthermore,

$$[R](X) = [[-R]](-X), \quad (25)$$

$$[[R]](X) = [-R](-X), \quad (26)$$

$$[[R]](\{x\}) = \langle R \rangle(\{x\}). \quad (27)$$

The last equation reflects the fact that on a one element set,  $\exists$  and  $\forall$  are the same operation. This is also present in the equality of weak and strong information operators of the same type on one element attribute sets [29].

The *full co-complex algebra*  $[[K]]$  of a frame  $K = \langle U, R \rangle$  is the Boolean powerset algebra of  $U$  with the additional sufficiency operator  $[[R]]$  defined by (24).

Conversely, suppose that  $B = \langle 2^W, g \rangle$  is a complete and atomic SUA, and set

$$R_g = \{\langle x, y \rangle \in W \times W : x \in g(y)\}. \quad (28)$$

**Proposition 6.** *Let  $K = \langle U, R \rangle$  be a frame, and  $B = \langle 2^W, g \rangle$  be a complete and atomic SUA. Then,*

$$R_{[[R]]} = R. \quad (29)$$

$$[[R_g]] = g. \quad (30)$$

*Furthermore, if  $S$  is a binary relation on  $W$  with  $[[S]] = g$ , then  $S = R_g$ .*

*Proof.* First,

$$\begin{aligned} xR_{[[R]]}y &\Leftrightarrow x \in [[R]](\{y\}) \\ &\Leftrightarrow y \in R(x) \\ &\Leftrightarrow xRy. \end{aligned}$$

Let  $X \subseteq U$ . Then,

$$\begin{aligned} x \in [[R_g]](X) &\Leftrightarrow x \in [[R_g]]\left(\bigcup_{y \in X} \{y\}\right) \\ &\Leftrightarrow x \in \bigcap_{y \in X} [[R_g]](\{y\}) \\ &\Leftrightarrow (\forall y \in X) y \in R_g(x) \\ &\Leftrightarrow (\forall y \in X) xR_gy \\ &\Leftrightarrow (\forall y \in X) x \in g(y) \\ &\Leftrightarrow x \in \bigcap_{y \in X} g(y) \\ &\Leftrightarrow x \in g\left(\bigcup_{y \in X} \{y\}\right) \\ &\Leftrightarrow x \in g(X). \end{aligned}$$

Finally, let  $[[S]] = g$ . Then,

$$\begin{aligned} xSy &\Leftrightarrow \{y\} \subseteq S(x), \\ &\Leftrightarrow x \in [[S]](\{y\}), \\ &\Leftrightarrow x \in g(\{y\}), \\ &\Leftrightarrow xR_gy. \end{aligned}$$

This completes the proof.  $\square$

We now have the following representation theorem for SUAs, corresponding to Proposition 2:

**Proposition 7.** *If  $\langle B, g \rangle$  is a sufficiency algebra, then there is (up to isomorphism) a unique frame  $\langle U, R \rangle$ , such that  $\langle 2^U, [[R]] \rangle \cong \langle B^\sigma, g^\sigma \rangle$ .*

*Proof.* This follows from Propositions 4 and 6.  $\square$

$\langle U, R \rangle$  as above is called the *atomic structure* of  $\langle B, g \rangle$ .

**Proposition 8.** *Let  $\langle B, f \rangle \in \text{MOA}$ , and set  $g = (f^\partial)^c$ . If  $\langle U, R \rangle$  is the atomic structure of  $\langle B, f \rangle$ , and  $\langle U, S \rangle$  is the atomic structure of  $\langle B, g \rangle$ , then  $R = -S$ .*

*Proof.*

$$\begin{aligned}
xRy &\Leftrightarrow x \in f^\sigma(\{y\}) \\
&\Leftrightarrow x \notin (f^\sigma)^\partial(U \setminus \{y\}) \\
&\Leftrightarrow x \notin g^\sigma(\{y\}) \\
&\Leftrightarrow x(-S)y.
\end{aligned}$$

□

## 5 Duality between frames and sufficiency algebras

We will now develop the machinery for the duality theory in analogy to the duality for modal algebras [19].

A *co-bounded morphism* from a frame  $K = \langle W, S \rangle$  to a frame  $L = \langle U, R \rangle$  is a mapping  $h : W \rightarrow U$  such that for all  $x, y \in W$ ,  $t \in U$ ,

$$x(-S)y \Rightarrow h(x)(-R)h(y) \quad (31)$$

$$t(-R)h(y) \Rightarrow (\exists w \in W)[h(w) = t \text{ and } w(-S)y]. \quad (32)$$

**Proposition 9.** *Let  $K = \langle W, S \rangle$ ,  $L = \langle U, R \rangle$  be frames.*

(a) *If  $h : W \rightarrow U$  is a co-bounded morphism, then, the mapping  $h^+ : [[L]] \rightarrow [[K]]$  defined by*

$$h^+(X) = \{y \in W : h(y) \in X\}$$

*is a complete SUA homomorphism.*

(b) *Let  $p : [[L]] \rightarrow [[K]]$  be a complete SUA homomorphism, and  $B \stackrel{\text{def}}{=} \{p(X) : X \subseteq U\}$ . Then, the mapping  $p_+ : W \rightarrow U$  with*

$$p_+(w) = u, \text{ where } u \in U \text{ and } p(u) \text{ is the atom of } B \text{ above } \{w\}$$

*is a co-bounded morphism.*

*Proof.* 1. It is well known that  $h^+$  is a complete Boolean homomorphism, so, all that is left to show is that

$$h^+([[R]](X)) = [[S]](h^+(X)).$$

First, observe that

$$z \in h^+([[R]](X)) \Leftrightarrow (\forall u \in U)[h(z)(-R)u \Rightarrow u \notin X], \quad (33)$$

$$z \in [[S]](h^+(X)) \Leftrightarrow (\forall w \in W)[z(-S)w \Rightarrow h(w) \notin X]. \quad (34)$$

“ $\subseteq$ ”: Let  $z \in h^+([[R]](X))$  and  $z(-S)w$ . By (31), we have  $h(z)(-R)h(w)$ , and (33) implies  $h(w) \notin X$ .

“ $\supseteq$ ”: Let  $z \in [[S]](h^+(X))$ , and  $h(z)(-R)u$ . By (32), there is some  $w \in W$  such that  $z(-S)w$  and  $h(w) = u$ . Then,  $h(z)(-R)h(w)$  by (31), and by (34), we have  $u = h(w) \notin X$ .

2. Since  $p$  is a complete homomorphism,  $B$  is complete and atomic. Let  $w \in W$ , and  $M_w$  be the atom of  $B$  containing  $w$ . If  $F_w = \{X \subseteq U : w \in p(X)\}$ , the completeness of  $p$  implies that  $\bigcap F_w = \{u\}$  for some  $u \in U$ , and  $p(u) = M_w$ . Thus,  $p_+$  is well defined.

“(31)”: Let  $x, y \in W$ , and  $M_x, M_y$  be the atoms of  $B$  above them. We will prove the contrapositive:

$$\begin{aligned} p_+(x)Rp_+(y) &\Rightarrow p_+(x) \in [[R]](p_+(y)), \text{ by (25)} \\ &\Rightarrow p(p_+(x)) \subseteq p([[R]](p_+(y))), \\ &\Rightarrow M_x \subseteq [[S]](M_y), \text{ by definition of } p \\ &\Rightarrow xSy, \text{ since } x \in M_x, y \in M_y, \text{ and (25)}. \end{aligned}$$

“(32)”: Let  $x, y \in W$ , and  $t \in U$ . Then,

$$\begin{aligned} p_+(x)(-R)t &\Rightarrow p_+(x) \notin [[R]](t), \text{ by (25)} \\ &\Rightarrow x \notin [[S]]p(t), \text{ by definition of } p \\ &\Rightarrow (\exists w)(w \in p(t) \text{ and } x(-S)w), \text{ by (25)} \\ &\Rightarrow (\exists w)(p_+(w) = t \text{ and } x(-S)w), \text{ by definition of } p. \end{aligned}$$

This finishes the proof.  $\square$

**Corollary 1.** *Let  $\mathcal{K}_1$  be the category of power set SUAs with complete homomorphisms, and  $\mathcal{K}_2$  be the category of frames with co-bounded morphism.*

(a) *The assignments  $r, s$*

$$\begin{aligned} \langle 2^W, g \rangle &\xrightarrow{r} \langle W, R_g \rangle, \quad p \xrightarrow{r} p_+, \\ \langle U, R \rangle &\xrightarrow{s} \langle 2^U, [[R]] \rangle, \quad h \xrightarrow{s} h^+ \end{aligned}$$

*are mutually inverse covariant functors.*

- (b) *If the homomorphism  $p : \langle 2^U, f \rangle \rightarrow \langle 2^W, g \rangle$  is injective (surjective), then  $p_+ : \langle W, R_g \rangle \rightarrow \langle U, R_f \rangle$  is surjective (injective).*  
 (c) *If  $K = \langle W, S \rangle$ ,  $L = \langle U, R \rangle$ , and the co-bounded morphism  $h : K \rightarrow L$  is injective (surjective), then  $h^+ : [[L]] \rightarrow [[K]]$  is surjective (injective).*

*Proof.* This follows from the Propositions 6 and 9.  $\square$

Suppose that  $\mathcal{L}$  is a logic for sufficiency structures with formula set  $Fml$ . In the usual set formulation of semantics, originating in [16] and commonly used in [1,14], for a frame  $K = \langle W, S \rangle$ , the meaning function  $m : Fml \rightarrow 2^W$  for formulas with the sufficiency operator is defined by

$$m([[S]](\psi)) = \{z \in W : m(\psi) \subseteq S(z)\}. \quad (35)$$

As usual, a formula  $\varphi$  is true in  $K$ , if  $m(\varphi) = W$  for all meaning functions  $m : Fml \rightarrow 2^W$ .

**Corollary 2.** *Let  $K = \langle W, S \rangle$ ,  $L = \langle U, R \rangle$  be frames,  $h : W \rightarrow U$  be an onto co-bounded morphism, and  $\varphi$  be a formula of  $\mathcal{L}$ , such that  $K \models \varphi$ . Then,  $L \models \varphi$ .*

*Proof.* Suppose that for all meaning functions  $m : Fml \rightarrow 2^W$  we have  $m(\varphi) = W$ , and let  $v : Fml \rightarrow 2^U$  be a meaning function. Using the notation and the result of Proposition 9, the function  $m : Fml \rightarrow 2^W$  defined by  $m(\chi) = h^+(v(\chi))$  is a meaning function. Now,

$$m(\varphi) = W \Rightarrow h^+(v(\varphi)) = W, \quad (36)$$

$$\Rightarrow h(h^+(v(\varphi))) = h(W), \quad (37)$$

$$\Rightarrow v(\varphi) = U, \quad (38)$$

which completes the proof.  $\square$

A *sufficiency substructure* of a frame  $\langle U, R \rangle$  is a frame  $\langle W, S \rangle$  such that

$$W \subseteq U, \quad (39)$$

$$S = R \upharpoonright (W \times W), \quad (40)$$

$$(-R)(x) \subseteq W \text{ for all } x \in W. \quad (41)$$

If  $\langle W, S \rangle$  is a sufficiency substructure of  $\langle U, R \rangle$ , we simply write  $W \Subset U$ , assuming that the relations involved are understood from the context.

Observing that

$$S = R \upharpoonright (W \times W) \Leftrightarrow (-S) = (-R) \upharpoonright (W \times W),$$

and in view of Corollary 1, we immediately arrive at

**Proposition 10.** *Let  $\langle W, S \rangle$  and  $\langle U, R \rangle$  be frames, and  $g : W \rightarrow U$  be a mapping. Then,*

$$g(W) \Subset U \Leftrightarrow g \text{ is a co-bounded morphism.}$$

## 6 An example of a sufficiency algebra

Consider a binary relation  $R$  which is co-reflexive, symmetric, and co-3-transitive.

**Proposition 11.**

$$R \text{ is co-reflexive} \Leftrightarrow [[R]](X) \subseteq -X, \quad (42)$$

$$R \text{ is symmetric} \Leftrightarrow X \subseteq [[R]][[R]](X), \quad (43)$$

$$R \text{ is 3-transitive} \Leftrightarrow \langle R \rangle \langle R \rangle \langle R \rangle (X) \subseteq \langle R \rangle (X), \quad (44)$$

$$R \text{ is co-3-transitive} \Leftrightarrow [[R]](X) \subseteq [[R]](-[[R]](-[[R]](X))) \quad (45)$$

for all  $X \subseteq U$ .

*Proof.* “(42)”: If  $x \in [[R]](X) \cap X$ , then  $xRx$  by (25).

Conversely, suppose  $[[R]](X) \subseteq -X$ ; then, in particular,  $x \notin [[R]](\{x\})$ , and it follows that  $x(-R)x$ .

“(43)”: “ $\Rightarrow$ ”: Suppose that  $R$  is symmetric, and assume there is some  $x \in X \setminus [[R]]([R]](X))$ . Then, there is some  $z$  such that  $z \in [[R]](X)$  and  $x(-R)z$ . The first condition implies that

$$(\forall y)[y \in X \Rightarrow zRy].$$

Since  $x \in X$ , it follows that  $x(-R)z$ , contradicting that  $R$  is symmetric.

“ $\Leftarrow$ ”: Assume that  $aRb$ ,  $b(-R)a$  for some  $a, b \in U$ . Consider a model  $M = \langle U, R, m \rangle$  such that  $m(p) = \{x \in U : x(-R)a\}$  for some propositional variable  $p$ ; then,  $b \in m(p)$ , and hence,  $M, b \models p$ . By our assumption we have  $m(p) \subseteq [[R]]([R]]m(p))$ , and it follows that  $M, b \models [[R]]([R]]p)$ , in other words,

$$(\forall y)(\forall z)[(M, z \models p \Rightarrow yRz) \Rightarrow bRy].$$

If  $y = a$ ,  $z = b$ , then the hypothesis of the implication is true, while its conclusion is not, since  $b(-R)a$ .

We leave the proof of (44) and (45) to the reader.  $\square$

Thus, an algebra appropriate for an abstract characterization of operator  $[[R]]$  with  $R$  satisfying (42), (43), (45) is a sufficiency algebra  $\langle B, g \rangle$  characterised by

$$g(x) \leq -x,$$

$$x \leq g(g(x)),$$

$$g(x) \leq g(-g(-g(x))).$$

Since it is well known that co-reflexivity cannot be expressed by a modal operator, this class of algebras cannot be captured by modal operators alone.

## 7 Mixed algebras

In this Section we shall look at algebras  $\langle B, f, g \rangle$ , where  $f$  is a modal operator,  $g$  a sufficiency operator, and the corresponding atomic structures  $\langle U, R \rangle$  and  $\langle U, S \rangle$  satisfy  $R = S$ .

By Proposition 3, the classes MOA and SUA are mutually term-definable, and Proposition 8 tells us that, in terms of atomic structures, a necessity operator  $f$  talks about  $R$ , while its complementary sufficiency operator  $f^c$  talks about  $-R$ . It follows that terms built from the necessity operator  $[R]$  express properties of the relation  $R$ , while terms built with  $[[R]]$  express properties of  $-R$ . We read in [17] (modified for our notation),

“Necessity and sufficiency split the modal realm into two dual branches each of which spreads over less than half the Boolean realm. The complement  $-R$  remaining outside the scope of both branches cannot be framed before uniting them:  $[-R](X) = [[R]](-X)$  and  $[[[-R]](X) = [R](-X)$ .”

The definition of  $[R]$  and  $[[R]]$  is such that these conditions are fulfilled. Since both  $R$  and  $-R$  are used, we have to find suitable condition for an algebraic characterisation which involves only the operators without any reference to the accessibility relations. A necessary condition was given in (27), namely,

$$\langle R \rangle(\{x\}) = [[R]](\{x\}).$$

We shall show below that this condition is sufficient to guarantee the desired interplay between a modal and a sufficiency operator. A *mixed modal sufficiency algebra* (MIA) is a BA  $B$  with two additional operators  $f, g$  such that

$$f \text{ is a modal operator.} \tag{46}$$

$$g \text{ is a sufficiency operator.} \tag{47}$$

$$f^\sigma(p) = g^\sigma(p) \text{ for each atom } p \text{ of } B^\sigma. \tag{48}$$

Next, we show that a common canonical extension exists:

**Proposition 12.** *For each MIA  $\langle B, f, g \rangle$  there is (up to isomorphism) a unique frame  $\langle U, R \rangle$  such that  $\langle 2^U, \langle R \rangle, [[R]] \rangle \cong \langle B^\sigma, f^\sigma, g^\sigma \rangle$ .*

*Proof.* The construction of the atomic structures  $\langle U, R \rangle$  of  $\langle B^\sigma, f^\sigma \rangle$  and  $\langle U, S \rangle$  of  $\langle B^\sigma, g^\sigma \rangle$  was such that

$$\begin{aligned} xRy &\Leftrightarrow x \in f^\sigma(\{y\}), \\ xSy &\Leftrightarrow x \in g^\sigma(\{y\}). \end{aligned}$$

Condition (48) now assures that  $R = S$ . □

**Proposition 13.** *If  $\langle B, f, g \rangle \in \text{MIA}$  and  $x, y \in B$ , then*

$$x \cdot y \neq 0 \text{ implies } g(x) \leq f(y). \tag{49}$$

*Proof.* Let  $B^\sigma$  be the canonical extension of  $B$ , and suppose that  $p$  is an atom of  $B^\sigma$ . Then, by (11) and (23), we have

$$\begin{aligned} f^\sigma(p) &= \prod \{f(x) : p \leq x, x \in B\}, \\ g^\sigma(p) &= \sum \{g(x) : p \leq x, x \in B\}. \end{aligned}$$

If  $p \leq x, y$ , then  $x \cdot y \neq 0$ , and thus,  $g(x) \leq f(y)$  for all such  $x, y \in B$ . It follows that  $g^\sigma(p) \leq f^\sigma(p)$ . □

We do not know whether the class MIA is first order axiomatisable, but we doubt very much that it is. The following result may (weakly) point into this direction:

**Proposition 14.** *Let  $B$  be an atomless Boolean algebra, and  $f$  be the identity mapping on  $B$ . Then, there is no sufficiency operator on  $B$  such that  $\langle B, f, g \rangle \in \text{MIA}$ .*

*Proof.* Let  $p \in \text{At}(B^\sigma)$ ; then, by (11),

$$\begin{aligned} f^\sigma(p) &= \prod \{f(b) : p \leq b, b \in B\}, \\ &= \prod \{b : p \leq b, b \in B\}, \\ &= p, \end{aligned}$$

the latter by (6). Assume that  $g$  is a sufficiency operator on  $B$  such that  $\langle B, f, g \rangle \in \text{MIA}$ . By (48), we have  $g^\sigma(p) = f^\sigma(p) = p$ . Since

$$g^\sigma(p) = \sum \{g(b) : p \leq b, b \in B\},$$

and  $p$  is an atom, there is some  $b \in B$ ,  $p \leq b$ , such that  $p = g^\sigma(p) = g(b) \in B$ . Since  $p$  is an atom in  $B^\sigma$ , it is an atom in  $B$ . This contradicts that  $B$  is atomless.  $\square$

Suppose that  $\langle B, f, g \rangle$  is a MIA, and define  $e : B \times B \rightarrow B$  by

$$e(x, y) = f^\partial(x) \cdot g(y). \quad (50)$$

Then,  $e$  is in the clone generated by the operations of  $\langle B, f, g \rangle$ . Conversely,

$$e(x, 0) = f^\partial(x) \cdot g(0) = f^\partial(x), \quad (51)$$

$$e(1, x) = f^\partial(1) \cdot g(x) = g(x) \quad (52)$$

show that  $f$  and  $g$  are definable from  $e$  and the Boolean operations. Let  $m : B \rightarrow B$  be defined by

$$m(x) = e(x, -x). \quad (53)$$

**Lemma 2.**

$$m(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* First, note that

$$m(1) = e(1, 0) = f^\partial(1) \cdot g(0) = -f(0) \cdot g(0) = 1.$$

Next, let  $x \not\leq 1$ . Then,  $-x \geq 0$ , and

$$g(-x) \leq f(-x) \text{ by (49)}$$

$$\begin{aligned}
-f(-x) \cdot g(-x) &= 0 \\
f^\partial \cdot g(-x) &= 0 \\
m(x) &= 0,
\end{aligned}$$

which completes the proof.  $\square$

If  $B$  is the full co-complex algebra of a frame  $\langle U, R \rangle$  and  $X, Y \subseteq U$ , then

$$z \in e(X, Y) \Leftrightarrow Y \subseteq R^\sim(z) \subseteq X. \quad (54)$$

In particular,

$$z \in e(X, X) \Leftrightarrow R^\sim(z) = X. \quad (55)$$

$$(56)$$

[17] presented a sound and complete system for mixed structures with operators  $[R], [[-R]]$  which, translated into our terminology, is as follows:

$$e(a, -b) \cdot e(-a + a', b \cdot -b') \leq e(a', -b'), \quad (57)$$

$$m(1) = 1, \quad (58)$$

$$m(a) \leq a, \quad (59)$$

$$m(a) \leq m(m(a)), \quad (60)$$

$$a \leq m(m^\partial(a)). \quad (61)$$

There is one derivation rule:

$$a \leq a' \text{ and } b \leq b' \text{ imply } e(a, -b) \leq e(a', -b'). \quad (62)$$

**Proposition 15.** (a) *Every MIA satisfies (57) – (62).*

(b) *There is an algebra  $\langle B, f, g \rangle$  such that  $f$  is a modal operator,  $g$  a sufficiency operator,  $B$  satisfies (57) – (62), but is not a MIA.*

*Proof.* 1. (58) – (61) follow immediately from Lemma 2, and because of the monotony of  $f$  and the anti-monotony of  $g$ , it is easy to see that the rule holds.

To show (57), we first note that

$$-f(a \cdot -a' + -a) \leq -f(-a'), \quad (63)$$

since  $-a' = -a \cdot -a' + a \cdot -a' \leq a \cdot -a' + -a \cdot -a'$ , and  $f$  is monotone. Similarly, since  $g$  is anti-monotone by Lemma 1,

$$g(b \cdot -b' + -b) \leq g(-b'). \quad (64)$$

The following statements are now equivalent:

$$\begin{aligned}
 e(a, -b) \cdot e(-a + a', b \cdot -b') &\leq e(a', -b') \\
 f^\partial(a) \cdot g(-b) \cdot f^\partial(-a + a') \cdot g(b \cdot -b') &\leq f^\partial(a') \cdot g(-b') \\
 -f(-a) \cdot g(-b) \cdot -f(a \cdot -a') \cdot g(b \cdot -b') &\leq -f(-a') \cdot g(-b') \\
 -f(a \cdot -a' + -a) \cdot g(b \cdot -b' + -b) &\leq -f(-a') \cdot g(-b'),
 \end{aligned}$$

and the last line is true because of (63) and (64).

2. Let  $f, g : B \rightarrow B$  be defined as follows:

$$\begin{aligned}
 f(a) &= \begin{cases} 1, & \text{if } a \neq 0, \\ 0, & \text{otherwise,} \end{cases} \\
 g(a) &= \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

First, note that

$$\begin{aligned}
 g(a) &= f^\partial(-a), \\
 e(a, -b) &= f^\partial(a) \cdot g(-b) = f^\partial(a) \cdot f^\partial(b) = f^\partial(a \cdot b),
 \end{aligned}$$

and thus,

$$\begin{aligned}
 e(a, -b) &= \begin{cases} 1, & \text{if } a = b = 1, \\ 0, & \text{otherwise,} \end{cases} \\
 m(a) &= f^\partial(a).
 \end{aligned}$$

It is now straightforward to show that  $f$  and  $g$  satisfy the axioms as well as the rule. If  $f^\sigma$  and  $g^\sigma$  are the canonical extensions of  $f$ , resp.  $g$ , and if  $a \in B^\sigma$  such that  $0 \leq a \leq 1$ , then, by (11) and (23), we have

$$f^\sigma(a) = 1, \quad g^\sigma(a) = 0.$$

Condition (48) now assures that  $\langle B, f, g \rangle \notin \text{MIA}$ . □

The logic of [17] enables us to explicitly express properties of relations in frames of the form  $\langle U, R, S \rangle$  where the relations  $R$  and  $S$  satisfy  $R \cup S = U \times U$ . The specific operators in these logics are of the form  $[R]$  and  $[[S]]$ . The completeness theorem for the standard frames such that the above condition and  $R \cap S = \emptyset$  are satisfied, is obtained indirectly by the copying method of [39]. The class MIA of mixed algebras provides a framework for directly expressing and reasoning about the interplay between a modal operator  $\langle R \rangle$  and a sufficiency operator  $[[R]]$  which are determined by the same frame relation.

## 8 Examples of mixed algebras

### 8.1 Complementarity algebras

Recall that a relation  $R$  on  $U$  is complementarity relation, if it is co-reflexive, symmetric, and 3-transitive. It is shown in [7] that these are the defining properties of the  $\text{comp}_{\{a\}}^s$  – relation obtained from an attribute  $a$  of an information system as defined in (1). We will show that complementarity can be expressed by mixed algebra, but not by purely modal or sufficiency operators.

It is well known, that co-reflexivity is not modally expressible. Furthermore, unlike symmetry and co-3-transitivity, the property of 3-transitivity cannot be defined by the sufficiency operator. To show this, we first quote (part of) a result from [41]:

**Proposition 16.** *If a first order definable class of frames is modally definable, then it is closed under disjoint unions.*  $\square$

Here, the  $\langle L, T \rangle$  is the disjoint union of the frames  $\langle U, R \rangle, \langle W, S \rangle$ , if  $U \cap W = \emptyset$ ,  $L = U \cup W$ , and  $T = R \cup S$ .

**Proposition 17.** *3-transitivity is not definable by a sufficiency formula.*

*Proof.* Let  $\langle U, R \rangle$  be 3-transitive,  $S = -R$ , and observe that

$$xSt \text{ implies } xSy \text{ or } ySz \text{ or } zSt. \quad (65)$$

Then,

$$R \text{ is 3-transitive iff } S \text{ satisfies (65).}$$

If we show that (65) is not definable in the modal language with the necessity operator  $[S]$ , then 3-transitivity is not definable with the sufficiency operator  $[[R]]$ , since

$$[S]F = [-R]F = [[R]]\neg F.$$

Let  $U = \{a, b\}$ ,  $Q = \{a, b\}$ ,  $V = \{c, d\}$ ,  $T = \{c, d\}$ ,  $W = U \cup V$ ,  $S = Q \cup T$ . Then, both frames  $\langle U, Q \rangle$  and  $\langle V, T \rangle$  satisfy (65), but  $\langle W, S \rangle$  does not: Just let  $x = a, t = b, y = d, z = c$ .  $\square$

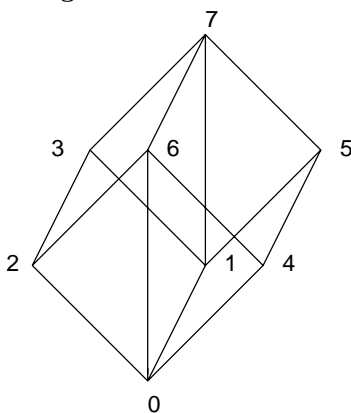
We conclude that a complementarity algebra is a mixed modal/sufficiency algebra  $(B, f, g)$ , characterised by

$$\begin{aligned} g(x) &\leq -x, \\ x &\leq g(g(x)), \\ f(f(f(x))) &\leq f(x). \end{aligned}$$

Similarly, since reflexivity is not expressible with a sufficiency operator, the algebras of incomplementarity are mixed algebras characterised by

$$\begin{aligned} x &\leq f(x), \\ x &\leq g(g(x)), \\ g(x) &\leq -g(-g(-g(x))). \end{aligned}$$

**Fig. 1.** A contact structure



**8.2 Contact relations**

A *contact relation*  $C$  on a set  $U$  satisfies the following properties:

$$C \text{ is reflexive.} \tag{66}$$

$$C \text{ is symmetric.} \tag{67}$$

$$C(x) = C(y) \text{ implies } x = y. \tag{68}$$

Contact relations go back to the qualitative geometry of [6,27], and they nowadays play a prominent role in spatial reasoning [3,35,34]. Our next result shows that to describe the class of contact frames we indeed need a “truly” mixed modal – sufficiency logic, where both operators are needed in one equation:

**Proposition 18.** *The class of contact structures is not closed under onto bounded or co-bounded morphisms.*

*Proof.* We have shown in [11] that the extensionality condition (68) cannot be expressed by modal operators, and we repeat the construction for the convenience of the reader.

Consider  $W = \{0, 1, \dots, 7\}$ ,  $U = \{1, 3, 5, 7\}$  and  $C \in Rel(W)$  as depicted in Figure 1; there  $aCb$  iff  $a = b$  or  $a$  and  $b$  are direct neighbours. It is not hard to check that  $C$  is a contact relation. Let  $S$  be the restriction of  $C$  to  $U \times U$ ; then,

$$S(1) = \{1, 3, 5, 7\} = S(7),$$

and thus,  $S$  does not satisfy (68). On the other hand, the mapping  $f : W \rightarrow U$  defined for all  $a \leq 7$  by

$$f(a) = \begin{cases} a + 1, & \text{if } a \text{ is even,} \\ a, & \text{otherwise,} \end{cases}$$

is a bounded morphism.

For the second part, consider two frames  $\langle W, S \rangle$ ,  $\langle U, R \rangle$ , where  $W = \{x, y\}$ ,  $S$  is the identity on  $W$ ,  $U = \{a\}$ , and  $R = \emptyset$ . Then,  $S$  is a contact relation, while  $R$  is not. Since  $x(-S)y$ , it is easy to see that the mapping defined by  $h(x) = h(y) = a$  is a co-bounded morphism.  $\square$

**Corollary 3.** *The class of contact frames cannot be axiomatized by equations which contain only modal or only sufficiency operators.*

*Proof.* A necessary condition for a first order property to be expressible by a modal formula is invariance under bounded morphisms, see [41]. Furthermore, we have shown in Corollary 2 that co-bounded morphisms preserve truths of formulas of a logic with a sufficiency operator.  $\square$

On the other hand, contact frames can be described by mixed structures:

**Proposition 19.** *Let  $\langle U, C \rangle$  be a frame, and  $\langle B, f, g \rangle$  its mixed complex algebra, i.e.  $B = 2^U$ ,  $f = \langle C \rangle$ , and  $g = [[C]]$ . Then,  $C$  is a contact relation iff*

$$[C](X) \subseteq X, \tag{69}$$

$$X \subseteq [[C]][[C]](X), \tag{70}$$

$$m(-(e(X, X) \cap -Y)) \cup m(-(e(X, X) \cap Y)) = U. \tag{71}$$

Here, the mappings  $m$  and  $e$  are as defined by (50) and (53).

*Proof.* It is well known that (69) expresses reflexivity, and we have shown above that (70) expresses symmetry. So, all that is left to show is that  $C$  satisfies (71) if and only if it satisfies (68). Suppose that  $C$  is reflexive and symmetric, and recall that

$$z \in e(X, X) \Leftrightarrow C(z) = X, \tag{72}$$

and that

$$m(-X) = \begin{cases} U, & \text{if } X = \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{73}$$

“ $\Rightarrow$ ”: Suppose that  $x \in U$ , and let  $X = C(x)$ . (71) tells us that for each  $Y \subseteq U$  we have  $e(X, X) \cap -Y = \emptyset$  or  $e(X, X) \cap Y = \emptyset$ , in other words,  $e(X, X)$  can have at most one element. By reflexivity of  $C$  we have  $x \in X$ , and thus, if  $C(y) = X$ , then  $x = y$ .

“ $\Leftarrow$ ”: If  $e(X, X) = \emptyset$ , there is nothing more to show; thus, suppose that  $x \in e(X, X)$ . By (68) we know that  $e(X, X) = \{x\}$  for some  $x \in U$ . Thus, (71) is fulfilled for every  $Y \subseteq U$ .  $\square$

## 9 Concluding remarks

In this paper we presented the classes SUA and MIA of algebras that emerged from an algebraic analysis of information systems and spatial reasoning. In the algebras derived from information systems the modal and sufficiency operators are determined by the relations that reflect either similarity of objects or their distinctness. The classes of algebras that model similarity have been studied in [5,36]. The need for the algebras for the relations from the second group is one of the motivations for the present paper; some of these algebras have been suggested in [29,36] and in the paper by SanJuan and Iturrioz in this volume. The extensive presentation of a broad class of these algebras will be the subject of a separate paper.

In spatial reasoning the need for considering the class MIA comes from the problem of characterisation of the contact relation between regions. We have shown that the properties of this relation require both the modal and the sufficiency operator in one equation.

The present paper is but a starting point for developing correspondence theories for linking expressibility of relational properties in first order logic and logics based on the classes SUA and MIA, respectively. A correspondence theory for SUA-expressibility could possibly be obtained following the methodology of the standard correspondence theory for modal expressibility. A correspondence theory for mixed logics with both a modal and a sufficiency operator determined by the same relation would probably require its own concepts and methods in order to characterise the properties that need both operators in an essential way. The other direction for further work is a development of Sahlqvist-like results for the SUA and MIA algebras and/or the underlying logics [22,23].

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