Relations algebras in qualitative spatial reasoning

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Abstract

The formalization of the "part – of" relationship goes back to the mereology of S. Leśniewski, subsequently taken up by Leonard & Goodman (1940), and Clarke (1981). In this paper we investigate relation algebras obtained from different notions of "part–of", respectively, "connectedness" in various domains. We obtain minimal models for the relational part of mereology in a general setting, and when the underlying set is an atomless Boolean algebra.

1 Introduction

Qualitative reasoning

> " ... is reasoning about physical systems without pursuing quantitative descriptions of system states." (Cunningham & Brady, 1987)

Qualitative spatial reasoning (QSR) aims to express non–numerical relationships among spatial objects; for an introduction to QSR and further references we invite the reader to consult Cohn (1997).

The basis of QSR are "part – of" and "contact" relations. The formalization of the "part – of" relationship, together with the notion of "fusion", goes back to the mereological systems of Leśniewski (1886 – 1939), developed from 1915 onwards (see Leśniewski, 1927 – 1931, 1983, Luschei, 1962, Surma et al., 1992). One of Leśniewski's main concerns was to build a paradox–free foundation of Mathematics, one pillar of which was mereology[1]or, as it was originally called, the general theory of manifolds or collective sets; the axioms of mereology were simplified by Tarski (1937, 1929). Mereology was later taken up by Leonard & Goodman (1940); formally, Leśniewski's mereology and the calculus of Leonard & Goodman – the classical mereology (CM) – are the same.

Based on classical mereology, and the work of Whitehead (1929) on the relation "x is extensionally connected with y", Clarke (1981) presents an axiom system for a "Calculus of individuals" whose

[1] Τὸ μέρεος = The part

Table 1: Interval relations

before:	$\{\langle [q,r],[q',r']\rangle : q < r < q' < r', q,r,q',r' \in \mathbb{R}\}$
meets:	$\{\langle [q,r'],[q',r]\rangle : q < r = q' < r', q,r,q',r' \in \mathbb{R}\}$
overlaps:	$\{\langle [q,r],[q',r']\rangle : q < q' < r < r', q,r,q',r' \in \mathbb{R}\}$
starts:	$\{\langle [q,r],[q',r']\rangle : q = q' < r < r', q,r,q',r' \in \mathbb{R}\}$
ends:	$\{\langle [q,r],[q',r']\rangle : q' < q < r = r', q,r,q',r' \in \mathbb{R}\}$
contains:	$\{\langle [q,r],[q',r']\rangle : q < q' < r' < r, q,r,q',r' \in \mathbb{R}\}$

single primitive notion is an "in contact with" relation. Nowadays, "mereology" has almost become synonymous in the QSR community with the study of relations such as "part of", or "contact" in appropriate domains. If topological aspects such as "connected" or "convex" are also considered, one speaks of "mereotopology" (see e.g. Asher & Vieu, 1995, Pratt & Schoop, 1998, 1999).

Most authors consider *region* as a primitive notion of QSR; thus, the object domain of an ontology for QSR does not consists of points which make up space. A similar stand is taken by the well known calculus based on time intervals (instead of time points) which was given independently by van Benthem (1983) and Allen (1983); its basic relations are defined in Table 1, and pictured in Figure 1 on page 7. Standard domains of regions are collections of regular sets of a suitable topological space. For a discussion of the ontological issues we refer the reader to Cohn et al. (1997) and to the special edition on ontology of the *International Journal of Human–Computer Studies* **43** (1995).

In this paper, we shall investigate relational structures obtained from the notions of "part of" and "contact" in various domains. It may be of interest to note that Tarski, who pioneered the study of relation algebras, was Leśniewski's only doctoral student, and worked closely with him on the foundations of Mathematics.

2 Relations and their algebras

Relations and their algebras have been studied since the latter half of the last century, e.g. by de Morgan (1864), Peirce (1870) and Schröder (1890 - 1905). Tarski (1941) gave a first formal introduction to the algebra of relations; his aim was to give an algebraic semantics to first order logic – just as Boolean algebras were an adequate algebraization of classical propositional logic.

Besides the Boolean set–theoretic connectives, natural operations on binary relations on a set U are *composition* and *converse*, defined, respectively, as

(2.1) $$R \circ S = \{\langle x,y\rangle \in U \times U : (\exists z \in U) xRzSy\},$$

(2.2) $$R^{\smile} = \{\langle y,x\rangle : xRy\}.$$

We also set $Rx = \{y \in U : xRy\}$.

The *full algebra of relations on* U is the structure $Rel(U) = \langle 2^V, \cup, \cap, -, \emptyset, V, \circ, \breve{}, 1' \rangle$, where $V = U \times U$, and $1'$ is the identity relation. A subset of $Rel(U)$ which is closed under the distinguished operations and contains the constants $\emptyset, V, 1'$ is called an *algebra of binary relations* (BRA). If $\{R_i : i \in I\} \subseteq Rel(U)$, then $\langle R_i \rangle_{i \in I}$ is the subalgebra of $Rel(U)$ generated by $\{R_i : i \in I\}$.

An (abstract) relation algebra (RA) is a structure

$$\langle A, +, \cdot, -, 0, 1, \circ, \breve{}, 1' \rangle$$

of type $\langle 2, 2, 1, 0, 0, 2, 1, 0 \rangle$ which satisfies for all $a, b, c \in A$,

1. $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra (BA).
2. $\langle A, \circ, \breve{}, 1' \rangle$ is an involuted monoid, i.e.
   (a) $\langle A, \circ, 1' \rangle$ is a semigroup with identity $1'$,
   (b) $a^{\breve{}\breve{}} = a, \ (a \circ b)^{\breve{}} = b^{\breve{}} \circ a^{\breve{}}$.
3. The following conditions are equivalent:

(2.3) $$(a \circ b) \cdot c = 0, \ (a^{\breve{}} \circ c) \cdot b = 0, \ (c \circ b^{\breve{}}) \cdot a = 0.$$

Each BRA is an RA, but not vice versa (Lyndon, 1950). We will usually use lower case letters for elements of an abstract RA, and capital letters for concrete relations. Where binary relations are the motivation for a construction which also can be done in RA, we will usually use upper case letters and the RA operators instead of the set operators. For example, a contact relation will always be denoted by C, and the relations derived from it by abstract RA operations will also be denoted by upper case letters such as in (3.6) – (3.11).

In the sequel, we will usually identify algebras with their base set. A is called *integral* if the identity $1'$ is an atom of A. If $A \leq Rel(U)$ is a BRA generated by $\langle R_i \rangle_{i \in I}$, then A is integral if and only if no proper nonempty subset of U is definable in the first order structure $\langle U, R_i \rangle_{i \in I}$ by a formula with at most three variables (see Andréka et al., 1995).

A finite RA is completely determined by the action of composition $\circ$ on its atoms. Observe that the converse can be recovered as follows: the converse $a^{\breve{}}$ of each atom a is an atom, and an atom b is the converse of a, if and only if $(a \circ b) \cdot 1' \neq 0$ (see Jónsson, 1984). We describe such algebras by exhibiting their composition as a matrix such as in Table 2 on page 7. There, for example, the entry in cell $\langle EC, TPP \rangle$ means that

$$EC \circ TPP = PP \cup PO \cup EC.$$

If the algebra is integral, we omit row and column $1'$.

The interval relations of Table 1 on the preceding page generate an integral RA $\mathcal{I}$ with 13 atoms on the set of all closed intervals of the real line, namely, the six relations in the table, their converses, and the identity; its composition table can be found in Allen (1983).

The logic of RAs is a fragment of first order logic, and the following fundamental result is due to A. Tarski (see Tarski & Givant, 1987):

Proposition 2.1. *If $R_0, \dots, R_k \in Rel(U)$, then $\langle R_0, \dots, R_k\rangle$ is the set of all binary relations on U which are definable in the (language of the) relational structure $\langle U, R_0, \dots, R_k\rangle$ by first order formulas using at most three variables, two of which are free.*

In certain BRAs, RA logic may be more expressive. Let $A \leq Rel(U)$, Σ_U be the symmetric group of U, and $\varphi \in \Sigma_U$; we will write $\varphi\langle x, y\rangle$ instead of $\langle\varphi(x), \varphi(y)\rangle$. The image of $R \in A$ under φ is denoted by R^φ, i.e.

$$R^\varphi = \{\varphi\langle x, y\rangle : \langle x, y\rangle \in R\}. \tag{2.4}$$

If $R^\varphi = R$, we call R *invariant under* φ. The permutation φ is called a *base automorphism of* A, if every $R \in A$ is invariant under φ. The set of all base automorphisms of A is denoted by A^ρ; it is easy to see that A^ρ is a subgroup of Σ_U.

Conversely, if G is a subgroup of Σ_U and $x, y \in U$, we set

$$G_{x,y} = \{\varphi(x, y) : \varphi \in G\},$$

and let G^σ be the BRA on U generated by $\{G_{x,y} : x, y \in U\}$. Observe that the sets $G_{x,y}$ are the orbits of the action of G on U^2 (see e.g. Wielandt, 1964, for the definitions), and hence a partition of U^2. Indeed, each $G_{x,y}$ is an atom of G^σ, and every atom of G^σ has this form (see Jónsson, 1984). The assignments ρ and σ form a Galois connection, and A is called *Galois closed* if $A^{\rho\sigma} = A$. It is well known that Galois closure implies closure under every permutation invariant operation on binary relations; in particular, every first order definable relation in the language of A is an element of A (see e.g. Jónsson, 1991).

Let A be an RA, and suppose that $a, b \in A$. Even though the equation $a \circ x = b$ does not always have a solution, there is an element $a \setminus b$, called the *residual*[2] of b by a, such that

$$a \circ x \leq b \iff x \leq a \setminus b.$$

The residual can be expressed as an RA term in a and b by

$$a \setminus b = -(a^\smile \circ -b). \tag{2.5}$$

If $R, S \in Rel(U)$, then the residual is given by the condition

$$x(R \setminus S)y \iff R^\smile x \subseteq S^\smile y. \tag{2.6}$$

The proof of the following lemma is straightforward and left to the reader:

[2] What we call “residual” is called “left residual” in Birkhoff (1948) and “right residual” in Jónsson (1982).

Lemma 2.2. *Suppose that A is an RA and $a \in A$.*

1. *$a \setminus a$ is reflexive and transitive, i.e.*
   (a) $1' \leq a$.
   (b) $a \circ a \leq a$.
2. *If a is reflexive and symmetric, then $(a \setminus a)^{\smile} \circ (a \setminus a) \leq a$.*

We will also need the following lemma:

Lemma 2.3. *Suppose that $R, S \in Rel(U)$, and that $xTy \overset{\text{def}}{\iff} Rx = Sy$. Then,*

$$T = -(R \circ -S^{\smile}) \cap -(-R \circ S^{\smile}). \tag{2.7}$$

Proof. Using the hypothesis and the fact that in any RA we have $(-a)^{\smile} = -(a^{\smile})$ (Chin & Tarski, 1951, Theorem 1.10), we have

$$\begin{aligned} xTy &\iff Rx = Sy, \\ &\iff Rx \subseteq Sy \text{ and } Sy \subseteq Rx, \\ &\iff x(R^{\smile} \setminus S^{\smile})y \text{ and } y(S^{\smile} \setminus R^{\smile})x, && \text{by (2.6)} \\ &\iff x[-(R \circ -S^{\smile})]y \text{ and } y[-(S \circ -R^{\smile})]x, && \text{by (2.5)} \\ &\iff x[-(R \circ -S^{\smile}) \cap -(-R \circ S^{\smile})]y, \end{aligned}$$

which proves the claim. □

In our construction of RAs we have been aided by the RA Scratchpad, designed and written by Peter Jipsen (1992). For other properties of relations and their algebras see Chin & Tarski (1951), Jónsson (1982, 1991), and Andréka et al. (1998).

3 Contact relations

To avoid trivialities, we always assume that the structures under consideration have at least two elements. Suppose that U is a nonempty set of regions, and that C is a binary relation on U which satisfies

$$C \text{ is reflexive and symmetric,} \tag{3.1}$$

$$Cx = Cy \text{ implies } x = y. \tag{3.2}$$

These are the axioms A0.1 and A0.2 given by Clarke (1981) for the mereological part of his calculus of individuals. (3.2) is an extensionality axiom, which says that each region is completely determined by those regions to which it is C – related. We call a binary relation C which satisfies (3.1) and (3.2) a *contact relation*; an RA generated by a contact relation will be called a *contact RA* (CRA).

Proposition 3.1. *C is a contact relation iff C is reflexive and symmetric, and*

$$C \setminus C \text{ is antisymmetric.} \tag{3.3}$$

Proof. "$\Rightarrow$": Let $xC \setminus Cy$ and $yC \setminus Cx$. Then, by the symmetry of C and (2.6), we have $Cx = Cy$, and hence, $x = y$ by (3.2).

"$\Leftarrow$": Suppose that $Cx = Cy$, i.e. $Cx \subseteq Cy$ and $Cy \subseteq Cx$. By (2.6), we have $xC \setminus Cy$ and $yC \setminus Cx$, and the antisymmetry of $C \setminus C$ implies $x = y$. ☐

Since antisymmetry is RA – expressible by

$$b \cdot b^{\smile} \leq 1',$$

we can speak of abstract CRAs, i.e. those RAs generated by an element which satisfies (3.1) and (3.3). We now set

(3.4)	$P = C \setminus C,$	part of
(3.5)	$PP = P \cdot -1'.$	proper part of

Lemma 2.2 and (3.3) tell us that P is a partial order. Two elements x, y are called *comparable*, if xPy or $xP^{\smile}y$, otherwise, *incomparable*.

We follow Clarke (1981) in defining other relations as follows:

(3.6)	$O = P^{\smile} \circ P$	overlap
(3.7)	$PO = O \cdot -(P + P^{\smile})$	partially overlap
(3.8)	$EC = C \cdot -O$	external contact
(3.9)	$TPP = PP \cdot (EC \circ EC)$	tangential proper part
(3.10)	$NTPP = PP \cdot -TPP$	non–tangential proper part
(3.11)	$DC = -C$	disconnected

Note that $PO, TPP, TPP^{\smile}, NTPP, NTPP^{\smile}, EC, DC, 1'$ are pairwise disjoint, and their sum is 1.

Given a contact relation C, we will use the definitions of the relations (3.4) – (3.11) throughout the remainder of the paper. Observe that the relations (3.7) – (3.11) arise from the time interval relations by "forgetting the direction", e.g. the union of "before" and its converse becomes DC, and the union of "starts", "ends" becomes TPP, see Figure 1.

If we think of the closed circles of the Euclidean plane as a domain of regions, and $xCy \iff x \cap y \neq \emptyset$, we can picture these relations as shown in Figure 2. The CRA generated by C on this domain, the *closed circle algebra*, is given in Table 2; the relation P is just set inclusion. Considered as an abstract RA, the closed circle algebra is isomorphic to the subalgebra of the interval algebra $\mathcal{I}$ generated by the union of the "before" relation and its converse. However, the representation of $\mathcal{C}_c$ on the domain of

Figure 1: Interval relations

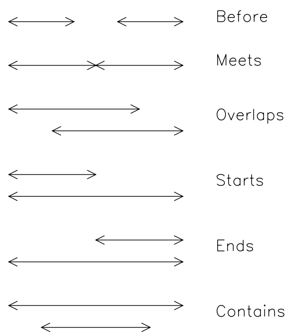
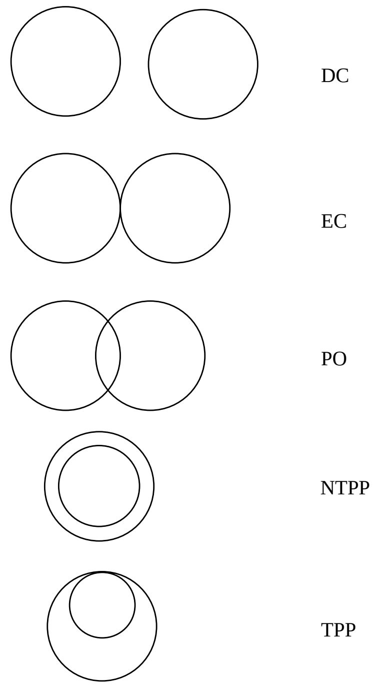


Figure 2: Circle relations



closed circles cannot be embedded into any representation of $\mathcal{I}$: Consider the square and its diagonals in Figure 3 on the next page, and label the square with PO and the diagonals with DC. This network cannot be satisfied in any representation of $\mathcal{I}$ (Ladkin & Maddux, 1994), but it can be satisfied in the closed circle algebra.

Table 2: Closed circle algebra $\mathcal{C}_c$

$\circ$	TPP	$TPP^{\smile}$	$NTPP$	$NTPP^{\smile}$	PO	EC	DC
TPP	PP	$-(NTPP \cup NTPP^{\smile})$	$NTPP$	$-P$	$-P^{\smile}$	EC, DC	DC
$TPP^{\smile}$	$1', TPP, TPP^{\smile}, PO$	$PP^{\smile}$	$PP^{\smile}, PO$	$NTPP^{\smile}$	$PP^{\smile}, PO$	$PP^{\smile}, PO, EC$	$-P$
$NTPP$	$NTPP$	$-P^{\smile}$	$NTPP$	1	$-P^{\smile}$	DC	DC
$NTPP^{\smile}$	$PP^{\smile}, PO$	$NTPP^{\smile}$	$-(EC \cup DC)$	$NTPP^{\smile}$	$PP^{\smile}, PO$	$PP^{\smile}, PO$	$-P$
PO	PP, PO	$-P$	PP, PO	$-P$	1	$-P$	$-P$
EC	PP, PO, EC	$EC \cup DC$	PP, PO	DC	$-P^{\smile}$	$-(NTPP \cup NTPP^{\smile})$	$-P$
DC	$-P^{\smile}$	DC	$-P^{\smile}$	DC	$-P^{\smile}$	$-P^{\smile}$	1

The subalgebra $\mathcal{C}_o$ of $\mathcal{C}_c$ generated by P has five atoms, and its composition is given in Table 3; this algebra is also called the *containment algebra* (Ladkin & Maddux, 1994). It is also a CRA, since $C' = P \cup P^{\smile} \cup PO$ is a contact relation with $C' \setminus C' = P$. Another representation of $\mathcal{C}_o$ arises from the set of all open circles in the Euclidean plane, and $xCy \iff x \cap y \neq \emptyset$. Note that in both algebras $\mathcal{C}_c$ and $\mathcal{C}_o$, contact is defined by nonempty intersection.

Figure 3: Circle network

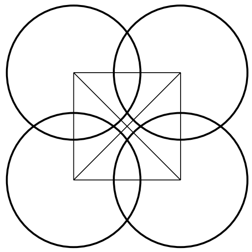


Table 3: Open circle algebra

$\circ$	PP	$PP^{\smile}$	PO	DC
PP	PP	1	$-P^{\smile}$	DC
$PP^{\smile}$	$-DC$	$PP^{\smile}$	$PP^{\smile}, PO$	$-P$
PO	PP, PO	$-P$	1	$-P$
DC	$-P^{\smile}$	DC	$-P^{\smile}$	1

3.1 Small models of CRAs

The smallest CRA with $C \neq 1'$ is the algebra known as $\mathcal{N}_1$ (Comer, 1983); it has four atoms, and its composition is given in Table 4. Clearly, $C = P^{\smile} \circ P = P + P^{\smile}$ is symmetric and reflexive, and P

Figure 4: An ordering for $\mathcal{N}_1$

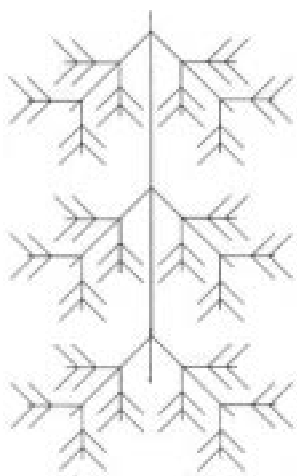


Table 4: The algebra $\mathcal{N}_1$

$\circ$	PP	$PP^{\smile}$	DC
PP	PP	1	DC
$PP^{\smile}$	$-DC$	$PP^{\smile}$	$PP^{\smile}, DC$
DC	PP, DC	DC	1

is a partial order. Finally,

$$C \setminus C = -(C \circ DC) = -((P + PP^{\smile}) \circ DC) = -(DC + P^{\smile}) = P.$$

A representation of $\mathcal{N}_1$ is obtained as follows (Düntsch, 1991): Suppose that ω is the set of natural numbers, and $\mathbb{Q}$ the set of rational numbers with their usual ordering $\leq$. Let U be the set of all functions

$$f : \mathrm{dom}(f) \subseteq \omega \to \mathbb{Q},$$

where $\mathrm{dom}(f)$ is a finite nonempty initial segment of ω. For each $f \in U$, let $m_f \stackrel{\mathrm{def}}{=} \max \mathrm{dom}(f)$. We define a strict partial ordering $PP \stackrel{\mathrm{def}}{=} \prec$ on U as follows: If $f, g \in U$, then $f \prec g$ if and only if

1. $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and $f(k) = g(k)$ for all $k < m_g$.
2. $m_g \leq m_f$, and
   (a) If $m_g < m_f$, then $f(m_g) \leq g(m_g)$,
   (b) If $m_g = m_f$, then $f(m_g) < g(m_g)$.

The relation $\prec$ is a strict dense partial order without endpoints and densely branching. It induces a $\vee$–semilattice, and is linearly ordered to the right, i.e. for each $f \in U$, the set $\{g \in U : f \preceq g\}$ is linearly ordered. These two properties can be interpreted in the sense that two events have a common point in the past, and that the past is uniquely determined. Two points are connected iff they are comparable, and $\mathcal{N}_1$ satisfies $C = O$.

It can be shown that this representation is Galois closed (Hirsch, 1997), and hence, it has the property that every relation which is first order definable from the base relation P is one of the relations in the algebra. Thus, according to Proposition 2.1, everything that can be said in first order logic about these relations can be said with three variables.

A picture of the order derived from a slightly different representation of $\mathcal{N}_1$, given in Andréka et al. (1994), is shown in Figure 4.

For subsequent use, we introduce the algebras $\mathcal{R}$ and $\mathcal{S}$, whose composition is given in Tables 5 and 6. It is shown in Düntsch (1991) that in any representation of $\mathcal{R}$, P is the disjoint union of two dense linear orders, while in any representation of $\mathcal{S}$, P is the disjoint union of three or more dense linear orders; note that neither $\mathcal{R}$ nor $\mathcal{S}$ is a CRA.

Table 5: The algebra $\mathcal{R}$

$\circ$	PP	$PP^{\smile}$	R
PP	PP	$-R$	R
$PP^{\smile}$	$-R$	$PP^{\smile}$	R
R	R	R	$-R$

Table 6: The algebra $\mathcal{S}$

$\circ$	PP	$PP^{\smile}$	R
PP	PP	$-R$	R
$PP^{\smile}$	$-R$	$PP^{\smile}$	R
R	R	R	1

As a next step, we look for a CRA where $O \neq C$, and hence, $EC \neq 0$; thus, our algebra should have the five atoms $1'$, PP, $PP^{\smile}$, EC, and DC. The RA scratchpad tell us that there are 14 isomorphism types of such algebras. As an example, we present $\mathcal{S}_0$ in Table 7.

We note that $\mathcal{R} \leq \mathcal{S}_0$, and therefore, P must be the disjoint union of two dense chains. A representation of $\mathcal{S}_0$ is as follows: Let

$$S = \{\frac{a}{3^k} : a \lneq 3^k, a \text{ odd}, k = 1, 2, 3, \ldots\},$$
$$T = \{\frac{a}{3^k} : 0 \lneq a \lneq 3^k, a \text{ even}, k = 1, 2, 3, \ldots\}.$$

Figure 5: An ordering for $\mathcal{S}_0$

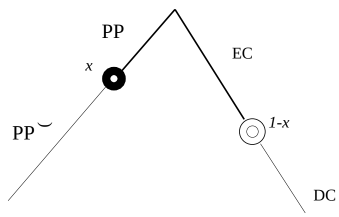


Table 7: The algebra $\mathcal{S}_0$

$\circ$	PP	$PP^{\smile}$	EC	DC
PP	PP	$PP, PP^{\smile}, 1'$	EC, DC	DC
$PP^{\smile}$	$PP, PP^{\smile}, 1'$	$PP^{\smile}$	EC	EC, DC
EC	EC	EC, DC	$PP, PP^{\smile}, 1'$	$PP^{\smile}$
DC	EC, DC	DC	PP	$PP, PP^{\smile}, 1'$

It is not hard to see that

(3.12) $$S \cap T = \emptyset,\ S, T \cong \mathbb{Q},$$

(3.13) S and T are dense in each other,

(3.14) $$x \in S \Rightarrow x = \inf\{y \in T : x \lneq y\} = \sup\{y \in T : y \lneq x\},$$

(3.15) $$x \in T \Rightarrow x = \inf\{y \in S : x \lneq y\} = \sup\{y \in S : y \lneq x\},,$$

(3.16) $$x \in S \iff 1 - x \in T.$$

Now, we let $\langle S_0, \leq\rangle, \langle S_1, \leq\rangle$ be two disjoint copies of $\langle S, \leq\rangle$, $U = S_0 \cup S_1$, and let P be extension of the orders on the S_i to U. Furthermore,

$$xECy \iff x \in S_i, y \in S_{i+1} \text{ and } 1 - m(x) \lneq y,$$
$$xDCy \iff x \in S_i, y \in S_{i+1} \text{ and } 1 - m(x) \gneq y.$$

Here, $i \in \{0, 1\}$, addition is $\mod 2$, and for $x \in S_i$, $m(x)$ is the "twin" of x in S_{i+1}. The RA generated by $C = P \cup P^{\smile} \cup EC$ is just $\mathcal{S}_0$.

The non–identity atoms of this representation example for P are shown in Figure 5. The lines represent the two copies of S, and, for an x, the labels on the various section of the lines indicate the relation which a point in this section has to x. Note that the white circle labeled $1 - x$ is the "border point" between EC and DC, but it is not an element of S.

4 Models of mereology

If X is a collection of objects and C a contact relation, then

(4.1) $$x = \sum X \overset{\text{def}}{\iff} (\forall y)[xCy \iff (\exists z \in X)yCz].$$

This is read as *x is the sum (or fusion) of* X. The following axiom guarantees the existence of the fusion:

(4.2) For each nonempty $X \subseteq U$ the fusion exists.

Assuming (4.2), we follow Clarke (1981) in defining

(4.3) $1 = \sum\{x : xCx\}$ Universal element

(4.4) $x^* = \sum\{y : y(-C)x\}$ Complement

(4.5) $\prod X = \sum\{z : zPx \text{ for all } x \in X\}$ Product

Observe that * and $\prod$ are partial operations. A *model of mereology* is a structure $\langle U, C, \sum\rangle$ which satisfies (3.1), (3.2), and (4.2). The models of mereology have been characterized algebraically by Biacino & Gerla (1991):

Proposition 4.1. *If $\langle L, +, \cdot, -\rangle$ is a complete orthocomplemented lattice, then*

(4.6) $$xCy \iff x \not\leq -y$$

defines a contact relation, and the fusion is just the lattice join. Conversely, if $\langle U, C, \sum\rangle$ is a model of mereology, we let $U' = U \cup \{0\}$, where $0 \notin U$. Then, $\langle U', C, \sum\rangle$ is a complete orthocomplemented lattice with the lattice join being the fusion, and the other operations given by (4.4) *and* (4.5)*, extended by $\prod X = 0$ whenever $\prod X$ does not exist in U, and $0^* = 1,\ 1^* = 0$.*

Each model $\langle L, +, \cdot, -\rangle$ of mereology defines a CRA in a natural way, namely, the BRA generated by C as defined in (4.6) on L.

Since extreme elements are RA definable, we assume from now on that the CRAs from models of mereology are defined on the base set of the model with the extreme elements $0, 1$ removed. This does not mean, however, that we may disregard these elements altogether: If

(4.7) $$R \stackrel{\text{def}}{=} -(P \circ P^{\smile}),$$

then, R has the property that

(4.8) $$xRy \iff x + y = 1.$$

Indeed, xRy if and only if there is no element in $L \setminus \{1\}$ above both x and y. Since $x + y$ exists, we must have $x + y = 1$. Conversely, if $x + y$ = 1, then the smallest element above both x and y is 1, and it follows that xRy.

Complement will always be an element of a CRA which comes from a model of mereology:

Proposition 4.2. *In a model of mereology, complement as defined in* (4.4) *is RA expressible from C, i.e. there is a relation DD which is RA – definable from C such that $xDDy \iff y = x^*$.*

Proof. We have

$$\begin{aligned}
y = x^* &\iff y = \sum\{z : zDCx\} && \text{by (4.4)}\\
&\iff (\forall t)[yCt \iff (\exists z)(zDCx \text{ and } tCz)], && \text{by (4.1)}\\
&\iff (\forall t)[yCt \iff xDC \circ Ct],\\
&\iff Cy = (DC \circ C)x.
\end{aligned}$$

Setting $R = (DC \circ C)$, $S = C$ and using Lemma 2.3, we now have

$$y = x^* \iff \langle x, y\rangle \in -[(DC \circ C \circ DC) \cup (-(DC \circ C) \circ C)], \tag{4.9}$$

which proves our claim. □

Since $P = -(C \circ DC)$, we have $DC \circ C = -P^{\smile}$, and $(P \circ DC) \cap C = \emptyset$ implies $(P^{\smile} \circ C) \subseteq C$. Clearly, $C \subseteq P^{\smile} \circ C$, and thus we can simplify (4.9) to

$$xDDy \iff \langle x, y\rangle \notin [(-P^{\smile} \circ DC) \cup C] \tag{4.10}$$

In the sequel, we will write DN for $DC \cap -DD$. A CRA arising from a model $\langle L, +, \cdot, ^*\rangle$ of mereology satisfies the conditions listed below; there, the right hand side is the property of L corresponding to the relational property:

(4.11) $DD \circ DD = 1'$, $(a^{**} = a)$

(4.12) $DD = -(P \circ P^{\smile}) \cdot -(P^{\smile} \circ P)$, $(a + a^* = 1,\ a \cdot a^* = 0)$

(4.13) $DD \circ PP = (DD \circ PP)^{\smile}$ $(a \leq b \iff b^* \leq a^*)$

Any CRA for mereology must split DC into DD and DN. If $\mathcal{M}$ is a model of classical mereology, i.e. when $C = O$, then the underlying set is a quasi – Boolean algebra, i.e. a Boolean algebra with the smallest element removed; we will look at this case in Section 5. Otherwise $EC = C \cap -O \neq \emptyset$, and a model, similar to the previous one, is as follows: Let E_0, E_1 be two copies of the real interval $(0, 1)$ ordered as usual by $\leq$, and set $E = E_0 \cup E_1,\ E^+ = E \cup \{1\}$. Order E^+ by

$$xPy \iff x, y \in E_i \text{ and } x \leq y, \text{ or y = 1.}$$

In the following, addition is modulo 2. We let $m : E \to E$ be defined in such a way that, if $x \in E_i$, then $m(x)$ is the value of x in E_{i+1}. Now, the relation C defined on E by

$$\langle x, y\rangle \in C \iff y \not\leq m(1 - x) \tag{4.14}$$

defines a contact relation, and

$$\begin{aligned} PP &= \lneq \\ O &= P^{\smile} \circ P = P + P^{\smile} + 1', \\ EC &= C \setminus O = \{\langle x, y\rangle : y \gneq m(1 - x)\}, \\ DD &= -[(-P^{\smile} \circ DC) \cup (P^{\smile} \circ C)] = \{\langle x, y\rangle : y = m(1 - x)\}, \\ DN &= DC \cap -DD = \{\langle x, y\rangle : y \lneq m(1 - x)\} \end{aligned}$$

The composition of the RA $\mathcal{S}_1$ generated by C is given in Table 8. We call $\mathcal{S}_1$ a *scale algebra*, since x is related to its complement like a scale, as indicated in Figure 6. The algebra $\mathcal{S}_1$ shows that in a model of mereology, P and EC do not necessarily split.

Figure 6: An ordering for $\mathcal{S}_1$

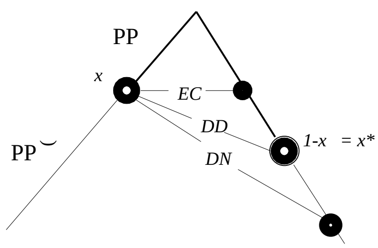


Table 8: The scale algebra $\mathcal{S}_1$

$\circ$	PP	$PP^{\smile}$	EC	DN	DD
PP	PP	CP	$-CP$	DN	DN
$PP^{\smile}$	CP	$PP^{\smile}$	EC	EC, DC	EC
EC	EC	$-CP$	CP	$PP^{\smile}$	$PP^{\smile}$
DN	EC, DC	DN	PP	CP	PP
DD	EC	DN	PP	$PP^{\smile}$	$1'$

Table 9: Algebra $\mathcal{S}_2$ with complement and split EC

$\circ$	PP	$PP^{\smile}$	EN	ED	DN	DD
PP	PP	$PP, PP^{\smile}, 1'$	EN, DC	ED	DN	DN
$PP^{\smile}$	$PP, PP^{\smile}, 1'$	$PP^{\smile}$	EN	ED	EN, DC	EN
EN	EN	EN, DC	$PP, PP^{\smile}, 1'$	ED	$P^{\smile}$	$P^{\smile}$
ED	ED	ED	ED	$-ED$	ED	ED
DN	EN, DC	DN	PP	ED	$PP, PP^{\smile}, 1'$	PP
DD	EN	DN	PP	ED	$PP^{\smile}$	$1'$

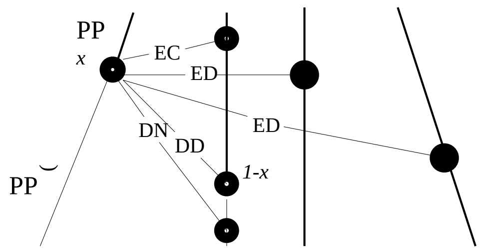
An algebra where EC splits into two atoms EN and ED, and DC splits into DN and DD is given in Table 9. We see that the algebra $\mathcal{S}$ with $R = EC \cup DC$ whose composition is given in Table 6 is a subalgebra of $\mathcal{S}_2$. Now, since DD is a one-one function disjoint from $P \cup P^{\smile}$, there must be an even number of components of P. Furthermore, the Table tells us that, if $xENy$ or $xDCy$, then y is in the same component as $DD(x)$, and, if $xEDy$, then y is in a component different from those of x or $DD(x)$. Let $S_i,\ i < 4$ be disjoint copies of the rational interval $(0, 1)$. The mapping m is defined from

$$m : \begin{cases} S_0 & \rightarrow S_1, \\ S_1 & \rightarrow S_0, \\ S_2 & \rightarrow S_3, \\ S_3 & \rightarrow S_2. \end{cases}$$

and m puts $x \in (0, 1)$ onto its twin in the other component. We now define

$$\begin{aligned} xPPy &\iff x, y \in S_i \text{ and } x \lneq y, \\ xDDy &\iff y = m(1 - x), \\ xENy &\iff m(1 - x) \lneq y, \\ xDNy &\iff m(1 - x) \gneq y, \\ xEDy &\iff y \text{ is in a component different from that of } x \text{ or } DD(x). \end{aligned}$$

Figure 7: An ordering for $\mathcal{S}_2$



If $C = -(DN \cup DD)$, then S_2 is isomorphic to the algebra generated by C. We have not been able to find an intuitive spatial explanation of this situation. An indication of the atoms of $\mathcal{S}_2$ is given in Fig. 7.

We can also have $ED \circ ED = 1$; in this case, we need (at least) six components, and, otherwise, use the same definitions as for S_2.

5 Classical mereology

The classical mereology of Lesniewśki is based on a "part of relation" P which is a partial order. Two individuals x, y are called *discrete*, written as $xDRy$, if they have no common part; in relational terminology,

$$xDRy \iff \neg\, xP^{\smile} \circ Py.$$

x is the *fusion* of a collection X of individuals, if every element of X is a part of x, and if no part of x is discrete from all elements of X. To capture classical mereology, we need an additional axiom for our contact structures, namely

$$P^{\smile} \circ P = C. \tag{5.1}$$

Then, $O = C$, and the notion of fusion is equivalent to that of (4.1) on page 10. This has been mentioned in the literature; a proof of this follows from the following

Proposition 5.1. *Let* $\langle L, +, \cdot, - \rangle$ *be a model of mereology for which* $C = P^{\smile} \circ P$*; furthermore let* $\emptyset \neq X \subseteq L \setminus \{1\}$*, and* $x_1, x_2 \in L \setminus \{1\}$ *such that for all* $y, z \in L \setminus \{1\}$*,*

$$x_1 Cy \iff (\exists t)(t \in X \wedge tCy), \tag{5.2}$$

$$y \in X \Rightarrow yPx_2, \tag{5.3}$$

$$zPx_2 \Rightarrow (\exists s)(s \in X \wedge zCs). \tag{5.4}$$

Then, $x_1 = x_2$.

Proof. First, we show that $P^{\smile} \circ C \subseteq C$: Let $xP^{\smile}zCy$, and assume that $xDCy$. then, $x(DC \circ C)z$; since $P^{\smile} = -(DC \circ C)$, this contradicts $xP^{\smile}z$.

To show $x_1 = x_2$, it is sufficient to show that $x_1Cy \iff x_2Cy$ for all $y \in L \setminus \{1\}$. Let x_1Cy; then, by (5.2), there is some $z \in X$ with zCy. By (5.3) we have zPx_2, and therefore, $x_2(P^{\smile} \circ C)y$; it follows that x_2Cy. Conversely, suppose that x_2Cy; then, there is some t such that $x_2P^{\smile}tPy$. By (5.4), there is some $z \in X$ with tCz, and therefore, $yP^{\smile}tCz$. This implies zCy, and hence, x_1Cy by (5.2). □

$C = P^{\smile} \circ P$ implies that the relations EC and TPP are empty. If $\langle U, C, \sum \rangle$ satisfies (3.1), (3.2), (4.2), and (5.1), we call it a *model of classical mereology* (CM).

Models of classical mereology arise from complete Boolean algebras B with the 0 element removed (see Tarski, 1935, p. 190f, footnote 5), where P is the Boolean order; it is not hard to see that

(5.5) $$xCy \iff x \cdot y \neq 0.$$

Let B be a (complete) atomless Boolean algebra; completeness does not play a role in our considerations. We shall assume that our base set is $U \stackrel{\mathrm{def}}{=} B \setminus \{0, 1\}$, and the relations are restricted to U. Since P is the basic relation of classical mereology, there is only one RA associated to CM, when the Boolean algebra is atomless.

In addition to the relations defined in Section 3, we define the following relations:

$$\begin{aligned}
\# = -(P \cup P^{\smile}) \quad & = \{\langle x, z\rangle : x \text{ and } z \text{ are incomparable w.r.t. } \leq\} \\
T = -(P \circ P^{\smile}) \quad & = \{\langle x, z\rangle : x + z = 1\} \\
PON = O \cap \# \cap -T \quad & = \{\langle x, z\rangle : x\#z,\ x \cdot z \neq 0,\ x + z \neq 1\} \\
POD = O \cap \# \cap T \quad & = \{\langle x, z\rangle : x\#z,\ x \cdot z \neq 0,\ x + z = 1\}
\end{aligned}$$

where $x, z \in U$. Since $C = O$, there is no external connection. We now have

Proposition 5.2. *Let B be an atomless Boolean algebra. Then, the relations*

$$1',\ PP,\ PP^{\smile},\ PON,\ POD,\ DN,\ DD$$

as defined above are the atoms of the algebra $\mathcal{G}$ on $B \setminus \{0, 1\}$ generated by the Boolean order P whose composition is given in Table 10.

Proof. Clearly, these relations partition $U \times U$. The computations are straightforward, if somewhat tedious, and are left to the reader. □

Table 10: The algebra $\mathcal{G}$

$\circ$	O				D	
	PP	$PP^{\smile}$	PON	POD	DN	DD
PP	PP	$-(POD \cup DD)$	PP, PON, DN	PP, PO, D	DN	DN
$PP^{\smile}$	$1', O$	$PP^{\smile}$	$PP^{\smile}, PO$	POD	$PP^{\smile}, PO, D$	POD
PON	PP, PO	$PP^{\smile}, PON, DN$	1	PP, PO	$PP^{\smile}, PON, DN$	PON
POD	POD	$PP^{\smile}, PO, D$	$PP^{\smile}, PO$	$1', O$	$PP^{\smile}$	$PP^{\smile}$
DN	PP, PO, D	DN	PP, PON, DN	PP	$-(POD \cup DD)$	PP
DD	POD	DN	PON	PP	$PP^{\smile}$	$1'$

In the algebra $\mathcal{G}$, there are two possibilities to define a contact relation: We can take either $C = O$ or $C = O \cup DD$. In both cases, $P = C \setminus C$. In the first case, (5.1) is also fulfilled, so that we obtain a model of classical mereology. If $\mathcal{A}$ is a CRA and $C \neq O$, then, because $\mathcal{A}$ is integral and DD is a function, it is not hard to see that $DD \subseteq C$. It follows that in the realm of atomless Boolean algebras with $P = \leq$, the algebra $\mathcal{G}$ is the smallest CRA coming from a model of classical mereology if $C = O$. If $C = O \cup DD$, then we do not obtain a model of mereology, since in such models, a region is never in contact with its complement.

At any rate, whenever a CRA assumes an underlying atomless Boolean algebra with the Boolean ordering as the "part – of" relation (such as the RCC mentioned below), then the relations of $\mathcal{G}$ must be present. Indeed, every relation C on an atomless Boolean algebra which satisfies (3.1) and (3.2) with $\leq = P$, must satisfy $O \subseteq C$, since $P^{\smile} \circ P \subseteq C$ by Lemma 2.2(2).

Another calculus for spatial reasoning, the *Region Connection Calculus* (RCC) of Randell et al. (1992), also has as a foundation a quasi–Boolean structure, and hence, the algebra $\mathcal{G}$ is a subalgebra of RAs obtained from the RCC. We describe some of the relational properties of the RCC in Düntsch et al. (1999a).

The situation when (a representation of) $\mathcal{G}$ is Galois closed is understood. Recall that a Boolean algebra B is called *homogeneous* if every nontrivial relative algebra $B \upharpoonright x$ is isomorphic to B. In case B has more than four elements, it is known that this is equivalent to the fact that the stabilizer H of $\{0, 1\}$ in the automorphism group of B is transitive (see e.g. Koppelberg, 1989, p.135). Furthermore, H is just the group of base automorphisms of $\mathcal{G}$. Now,

Proposition 5.3. *Düntsch et al. (1999a)*
$\mathcal{G}$ is Galois closed if and only if B is homogeneous. In particular, $\mathcal{G}$ is Galois closed over the BA of regular open sets of a Euclidean space.

6 Conclusion and outlook

We have introduced contact relations and their algebras (CRAs) which are based on the relations arising in mereology as defined by Leśniewski and extended by Clarke; these relations play a prominent role in contemporary qualitative spatial reasoning.

We have given natural spatial models of CRAs using circles in the Euclidean plane; these have made clear the conceptual relationship of CRAs to the interval algebra. We have also given minimal CRAs as well as the (unique) CRA associated with models of classical mereology.

In this introductory article, many important problems have not been discussed, and there is much room for further research. The CRAs for standard ontologies of mereotopology and their expressiveness are currently being investigated by Düntsch et al. (1999b). These include the standard model of the RCC as the collection of all nonempty regular closed sets on a regular connected spaces, as well as the polygonal algebras of Pratt & Schoop (1998, 1999).

A logic for CRAs with a complete proof system has been presented by Düntsch & Orłowska (1999a), and modal logics for frames with a contact relation are being developed in Düntsch & Orłowska (1999b).

Finally, we should like to draw the reader's attention to the following open questions:

- For which partial orders P is there a contact relation C such that $P = C \setminus C$? When can C be chosen as $P^{\smile} \circ P$?
- Investigate the complexity of CRAs. This is an important question, relating to the feasibility of relational reasoning in QSR (Bennett et al., 1997). There have been investigations for the algebra of time intervals and its relatives (Nebel & Bürckert, 1993, Ladkin & Maddux, 1994, Hirsch, 1997), as well as for RA-like structures related to the RCC (Renz & Nebel, 1997, 1998, Jonsson & Drakengren, 1997). In connection with the different representations of subalgebras of the interval algebra, it is also of interest to investigate the network satisfaction problem for the given algebras and their representations (Hirsch, 1997).
- Look at vagueness of spatial regions. This seems especially important for applications such as geographical information systems (Worboys, 1998). The rough relations of Comer (1993) and Düntsch (1994), or the uncertainty approach of Düntsch & Gediga (1997) may come in useful. It should also be worthwhile to investigate the connections of rough mereology (Polkowski & Skowron, 1994) to this problem.

7 Acknowledgements

We should like to thank Brandon Bennett, Tony Cohn, and Dominik Schoop for fruitful discussions and pointers to useful references, and Peter Jipsen for re–programming his RA scratchpad to suit our needs. We are especially grateful to Michael Winter who checked the composition tables, and spotted (and corrected) several inaccuracies.

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