Equidivisible consecutive integers*

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1 Introduction

Let $d : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the *divisor function*, defined by

$$d(n) := |\{a \in \mathbb{Z}^+ : a|n\}|.$$

Here, $|U|$ denotes the cardinality of a set U, and $a|n$ means that a divides n.

We say that $n, m \in \mathbb{Z}^+$ are *equidivisible*, if they have the same number of divisors, i.e. if $d(m) = d(n)$. In this paper, we use elementary methods to study runs of equidivisible numbers, that is, sequences of consecutive positive integers which happen to be equidivisible. (Unless explicitly stated otherwise, *numbers* in this paper are assumed to be positive integers.) We call a run of equidivisible numbers *maximal*, if it is not properly contained in any such longer run.

Evaluation of the divisor function is, in principle, very simple. Let $\mathbb{P}$ be the set of primes. For any integer $n \geq 0$,

$$d(p^\alpha) = \alpha + 1 \text{ for any } p \in \mathbb{P}.$$

It is well known that d is a multiplicative function, so

$$d(n) \prod_{p^\alpha \| n} d(p^\alpha) = \prod_{p^\alpha \| n} (\alpha + 1),$$

where the products are taken over the maximal prime power divisors of n. In particular, it immediately follows that $d(n)$ is odd precisely when n is a square, and $d(n) \geq 2$ if $n \geq 2$.

For each $k \in \mathbb{Z}^+$, let $D(k)$ be the set of positive integers which begin maximal runs of equidivisible numbers with exactly k divisors. Then, $D(1) = \{1\}$, and

$$D(k) := \{a \in \mathbb{Z}^+ : d(a) = k, d(a-1) \neq k\},$$

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for each $k \geq 2$. For example, $d(n) = 2$ precisely when n is prime, so it easily follows that $D(2) = \mathbb{P} \setminus \{3\}$. Again, $d(n) = 3$ precisely when n is the square of a prime; no two squares are consecutive in $\mathbb{Z}^+$, so, $D(3) = \{p^2 : p \in \mathbb{P}\}$.

Theorem 1.1. *For each integer $k \geq 2$, the set $D(k)$ is infinite.*

Proof. Since $d(p^{k-1}) = k$ for any $p \in \mathbb{P}$ and $k \in \mathbb{Z}^+$, if $k \geq 2$, there are infinitely many positive integers with exactly k divisors. The Theorem easily follows. □

2 The sets $\mathbf{D}(k,m)$

We can partition $D(k)$ into disjoint subsets according to the length of the run of equidivisible numbers starting at each member of $D(k)$. More precisely, for each $k, m \in \mathbb{Z}^+$, let

$$D(k,m) := \{a \in D(k) : d(a+i) = k \text{ for } 0 \leq i < m,\ d(a+m) \neq k\}.$$

Then,

$$D(k) = \bigcup_{m \geq 1} D(m,k).$$

Lemma 2.1. *For each $k \in \mathbb{Z}^+$, every run of 2^k positive integers includes at least one member with more than k divisors.*

Proof. Every run of 2^k integers includes a multiple of 2^k. Every such number has moe than k divisors, since $d(2^k r \geq d(2^k) = k+1$ for every $r \in \mathbb{Z}^+$. □

As an immediate consequence we have

Theorem 2.2. $D(k,m) = \emptyset$ *for each* $m \geq 2^k, k \in \mathbb{Z}^+$.

Thus, $D(k)$ is the union of only finitely many nonempty sets of the form $D(k,m)$. In particular, $D(2) = D(2,1) \cup D(2,2)$, and $D(3) = D(3,1)$. For each $k \in \mathbb{Z}^+$, let

$$M(k) := \max\{m \in \mathbb{Z}^+ : D(k,m) \neq \emptyset\}.$$

Then, $M(k)$ is the greatest length achieved by runs of equidivisible numbers with exactly k divisors. By Theorem 2.2, $M(k) < 2^k$ for each $k \in \mathbb{Z}^+$.

Theorem 2.3. *If $k \in \mathbb{Z}^+$ is odd, then*

1. $D(k) = D(k,1)$,

2. $M(k) = 1$.

Proof. For any odd $k \in \mathbb{Z}^+$, recall that $d(n) = k$ is only possible if n is a square. Since no two squares are consecutive in $\mathbb{Z}^+$, the claims follow. □

It is easy to explicitly describe $D(k,1)$ for odd $k \geq 3$, since it compriese every number with exactly k divisors. Let $\overrightarrow{a}$ be any sequence of numbers $a_1,\ldots,a_s$ where $a_1\cdot\cdots\cdot a_s = k$, and $3 \leq a_1 \leq \cdots \leq a_s$; correspondingly, let $\overrightarrow{p}$ be any sequence of s distinct primes. Then,

$$D(k,1) = \bigcup_{\overrightarrow{a}}\bigcup_{\overrightarrow{p}}\{\prod_{{p_i}^{a_i-1}} : a_i \in \overrightarrow{a}, p_i \in \overrightarrow{p}\}.$$

Recall that each a_i is odd, so each number in $D(k,1)$ is a square, and $\bigcup\{D(k,1) : k \text{ odd}, k \in \mathbb{Z}^+\} = \{n^2 : n \in \mathbb{Z}^+\}$.

Theorem 1.1 asserts that $D(k)$ is infinite when $k \geq 2$, so $D(k,m)$ must be infinite for at keast one m. For each $k \geq 2$, let

$$M^*(k) := \max\{m \in \mathbb{Z}^+ : D(k,m) \text{ is infinite}\}.$$

Clearly, $M^*(k) \leq M(k)$, and $M^*(k) = M(k) = 1$ for any odd $k \in \mathbb{Z}^+$ by Theorem 2.3. However, $M^*(k)$ is not necessarily equal to k: For example, $D(2,1) = \mathbb{P}\setminus\{2,3\}$, and $D(2,2) = \{2\}$; so, $M^*(2) = 1$ and $M(2) = 2$.

We know that $D(k)$ is infinite when $k \geq 2$, but we do not know whether $D(k,m)$ is infinite for each $m \in [1, M^*(k)]$, though it seems likely that this is so. As a small step in this direction, we shall prove

Theorem 2.4. *For each integer $k \geq 2$, the set $D(k,1)$ is infinite.*

Proof. When k is odd, the claim is immediate from Theorem 1.1. Thus, suppose that k is even. Let b be the smallest positive integer satisfying

$$b \equiv \begin{cases} 1 & \mod 2^k \\ -1 & \mod 3^k \end{cases}$$

By Dirichlet's Theorem on primes in arithmetic progression, there are infinitely many primes p which satisfy

$$p \equiv b \mod 6^k.$$

We shall now show for each such prime that $p^{k-1} \in D(k,1$ from which the claim follows.

Clearly, $d(p^{k-1}) = k$. By our choice of p, the predecessor of p^{k-1} is a multiple of 2^k, since $2^k|(p-1)$, and $(p-1)|(p^{k-1}+1$, since k is even. Hence,

$$d(p^{k-1}+1) \geq d(p+1) \geq d(3^k) > k.$$

As the predecessor and the successor of p^{k-1} both have more than k divisors, it follows that $p^{k-1} \in D(k,1)$. □

Erdös & Mirsky (1952) studied the asymptotic number of distinct values assumed by the divisor function $d(n)$ in $[1,x]$. They remarked that their methods could not estimate the length of the longest run of equidivisible numbers in $[1,x]$, and, indeed, could not show that there are infinitely many pairs of consecutive numbers which are equidivisible. Subsequently, Spiro (1981) made significant progress on the latter problem, and it was eventually fully proved by Heath-Brown (1984). In view of Theorem 2.3, this implies that the set

$$\bigcup\{D(2k,m) : k \in \mathbb{Z}^+,\ m \in [2, M(2k)]\}$$

is infinite. But this is unfortunately not strong enough to imply the infinitude of a single set $D(2k,m$ with $m \in [2, M(2k)]$.

3 The function $\mathbf{M}(k)$

Let us now turn to the question of determining $M(k)$ when k is even. It is convenient here to use some terminology which, though natural, might confuse the reader if we did not explain: For $r \in \mathbb{Z}^+$, by an *odd multiple* of 2^r we shall mean a number of the form $2^r \cdot s$, where s is odd.

Lemma 3.1. *For any $k \in \mathbb{Z}^+$, let $r \in \mathbb{Z}^+$ be the smallest non–divisor of k. Then, no odd multiple of 2^{r-1} has exactly k divisors.*

Proof. For any odd $s \in \mathbb{Z}^+$, we have $d(2^{r-1} \cdot s) = r \cdot d(s)$, so r is a factor of the number of divisors of every odd multiple of 2^{r-1} in $\mathbb{Z}^+$. Since $r \nmid k$, it follows that no odd multiple of 2^{r-1} has exactly k divisors. □

Theorem 3.2. *For any $k \in \mathbb{Z}^+$, let $r \in \mathbb{Z}^+$ be the smallest non–divisor of k. Then, $M(k) \leq 2^{r-1}$.*

Proof. Every run of 2^r consecutive numbers includes an odd multiple of 2^{r-1}, and thus, it includes a numbers which does not have exactly k divisors by Lemma 3.1. Hence, $M(k) \leq 2^r - 1$. □

This result implies that long runs of equidivisible consecutive integers can only occur when the number of divisors is a multiple of all intgers in some initial interval of $\mathbb{Z}^+$. Put $L_n := \text{lcm}[1,n]$. Then,

Corollary. *$M(k) \geq 2^n$ is only possible if $L_n \mid k$ for any $n \in \mathbb{Z}^+$.*

Proof. If $r \nmid k$ for some positive integer $r \leq n$, then $M(k) \leq 2^n - 1$ whence the claim follows. □

Theorem 3.3. *For any $k \in \mathbb{Z}^+$, if $M(k) \geq 8$ then $12 \mid k$.*

Proof. If $M(k) \geq 8$, the Corollary to Theorem 3.2 shows that $6 \mid k$. To demonstrate the Theorem, it therefore suffices to show that $4 \mid k$ also holds. If $r \in \mathbb{Z}^+$ is odd, then $d(2r) = 2 \cdot d(r)$, so $4 \mid d(2r)$ holds unless r is a perfect square. No two squares in $\mathbb{Z}^+$ differ by 2, so any two consecutive odd multiples of 2 in $\mathbb{Z}^+$ include at least one with a multiple of 4 divisors. But in any run of 8 positive integers there are two consecutive odd multiples 2. Thus, if all are equidivisible with exactly k divisors, then $4 \mid k$. Hence, $M(k) \geq 8$ implies $4 \mid k$ as required. □

Theorem 3.4. *For any $k \in \mathbb{Z}^+$, if $M(k) \geq 32$ then $120 \mid k$.*

Proof. The Corollary to Theorem 3.2 shows that $M(k) \geq 32$ implies that $60 \mid k$, so for the present Theorem it suffices to show that $M(k) \geq 32$ implies that $8 \mid k$. If r is odd and not a perfect square, then $d(8r) = 4 \cdot d(r)$, and thus, $d(8r) \equiv 0 \mod 8$.

No two consecutive odd integers are both squares, so any two consective odd multiples of 8 include one with a multiple of 8 divisors. It follows that, if they are all equidivisible with exactly k divisors, then $8 \mid k$. ☐

Ideed, numerical evidence suggests that runs of much shorter length than 32 can only occur if $8 \mid k$, but we have no proof of this. Perhaps it is even true that $M(k) \geq 8$ is only possible if $8 \mid k$.

4 Numbers with exactly 4 divisors

Now, let us look at numbers with exactly 4 divisors, the first case not fully resolved so far. The following result is relevant:

Lemma 4.1. *Every multiple* 4 *greater than* 16 *has at least* 6 *divisors.*

Proof. For any $k \in \mathbb{Z}^+$,

$$d(32k) \geq d(32) = 6.$$

If $k \geq 3$ is odd, then

$$\begin{aligned} d(4k) &= 3 \cdot d(k) \geq 6, \\ d(8k) &= 4 \cdot d(k) \geq 8, \\ d(16k) &= 5 \cdot d(k) \geq 10. \end{aligned}$$

Every multiple of 4 greater than 16 is covered by one of these cases. ☐

Theorem 4.2. $M(4) = 3$.

Proof. By Lemma 4.1, then only multiple of 4 with exactly 4 divisors is 8. But $8 \in D(4,1)$, so no multiple of 4 can occur in a run of two or more equidivisible integers with exactly 4 divisors. Hence, $M(4) \leq 3$.

Finally, $33 \in D(4,3)$ shows that $M(4) = 3$. ☐

It is relevant to report that 213 and 217 both belong to $D(4,3)$: Six of the seven numbers in $[213,219]$ have exactly 4 divisors, and $d(216) = 16$.

The composition of the sets $D(4,m)$ is somewhat elucidated by the following result:

Theorem 4.3. $p^3 \in D(4,1)$ *for any prime* $\neq 3$.

Proof. By direct calculation, $8 \in D(4,1)$ and $27 \notin D(4,1)$, since $26 \in D(4,2)$.

Suppose that $p \geq 5$; clearly, $d(p^3) = 4$. Let $r := p^2 + p + 1$, so the predecessor of p^3 is $(p-1) \cdot r$. If $g := \gcd\{p-1, r\}$, then g is a factor of $(p-1) + r = p \cdot (p+2)$. But $\gcd\{p-1, p\} = 1$, which shows that $g \mid (p+2)$. Hence, g divides the difference

$$(p+2) - (p-1) = 3,$$

and thus, $g \in \{1, 3\}$. It follows that $g = 3$ precisely when $p \equiv 1 \mod 3$.

If $p \not\equiv 1 \mod 3$, then $p-1$ and r are coprime and $p-1$ is composite. Thus,

$$d(p^3 - 1) = d(p-1) \cdot d(r) \geq 3 \cdot 2 = 6.$$

If $p \equiv 1 \mod 3$, then $r \equiv 3 \mod 9$. Hence, $3p-3$ and $\frac{r}{3}$ are coprime, and $3p - 3 = 9s$ for some $s \geq 2$. It follows that

$$d(p^3 - 1) = d(3p-3) \cdot d(\frac{r}{3}) = d(9s) \cdot d(\frac{r}{3}) \geq 4 \cdot 2 = 8.$$

By an entirely analoguous argument, $p^3 + 1$ has at least 6 divisors when $p \geq 5$, and it follows that $p^3 \in D(4,1)$. □

Corollary. *Except for* 27*, any positive integer in a run of two or more equidivisible numbers with exactly* 4 *divisors is a product of two distinct primes.*

Proof. This is immediate from Theorem 4.3, since $d(n) = 4$ is only possible when $n = p^3$ or $n = p \cdot q$, where $p, q \in \mathbb{P},\ p \neq q$. □

5 Numbers with exactly 6 divisors

Let us now look at runs of equidivisible integers with exactly 6 divisors. By the same method used to prove Lemma 4.1 it can be shown that

Lemma 5.1. *Every multiple of* 8 *greater than* 32 *has at least* 7 *divisors.*

Another relevant result is

Lemma 5.2. *For any odd prime p, at most one of any two consecutive odd multiples of* 2 *has exactly* $2p$ *divisors.*

Proof. For any odd $k \in \mathbb{Z}^+$, $d(2k) = 2p$ can only hold when $d(k) = p$, that is, $k = q^{p-1}$ for some $q \geq 3$. However, two consecutive odd numbers cannot be of this form, since perfect $(p-1)^{\text{st}}$ powers in $\mathbb{Z}^+$ must differ by at least $2^{p-1} - 1 > 2$. □

Theorem 5.3. $M(6) = 5$.

Proof. By Lemma 5.1, it easily follows that 32 is the only positive multiple of 8 with exactly 6 divisors. But $32 \in D(6,1)$, so no multiple of 8 occurs in any run of two or more equidivisible consecutive numbers with exactly 6 divisors.

Between any two consecitive multiples of 8 there are two odd multiples of 2, at most one of which has exactly 6 divisors by Lemma 5.2. Hence, $M(6) \leq 5$.

The other direction follows from $10,093,613,546,512,121 \in D(6,5)$. □

The starting numbers of $D(6,1) - D(6,4)$ can be found in Table 2. It is of interest to note that the first member of $D(6,4)$ is smaller than the first member of $D(6,3)$.

A companion result to Theorem 4.3 is

Theorem 5.4. *For any prime $p \neq 3$, $p^5 \in D(6,1)$.*

Proof. By direct calculation we obtain $2^5 = 32 \in D(6,1), 5^5 = 3125 \in D(6,1)$, and $3^5 = 243 \notin D(6,1)$, since $242 \in D(6,4)$; thus, let $p \geq 7$.

Since $d(p^5) = 6$, it only remains to show that $d(p^5-1) \neq 6$ and $d(p^5+1) \neq 6$. Let

$$r := p^4 + p^3 + p^2 + p + 1,$$

so that $p^5 - 1 = (p-1) \cdot r$ and $\gcd\{p-1, r\} \in \{1, 5\}$. If $p \not\equiv 1 \mod 5$, then $p-1$ and r are coprime, also, $d(p-1) \geq 4$ since $p-1$ is even and greater than 4. Hence,

$$d(p^5-1) = d(p-1) \cdot d(r) \geq 4 \cdot 2 = 8.$$

If $p \equiv 1 \mod 5$, then $r \equiv 5 \mod 25$, and $5p - 5 = 25s$ for some $s \geq 2$. It follows that

$$d(p^5-1) = d(5p-5) \cdot d(\frac{r}{5}) = d(25s) \cdot d(\frac{r}{5}) \geq 4 \cdot 2 = 8.$$

Similarly it is shown that $d(p^5+1) \geq 8$. □

Corollary 5.5. *Except for* 243*, any positive integer in a run of two or more equidivisible numbers with exactly* 6 *divisors has the form $p^2 \cdot q$, where p and q are distinct primes.*

Proof. This follows because $d(n) = 6$ exactly when $n = p^5$ or $p^2 \cdot q$ with $p, q \in \mathbb{P}$. □

6 Numbers with exactly $2p$ divisors, p ≥ 5

It is easy to generalise ideas used in the proof of Theorem 5.3 to show that $M(2p) \leq 5$ for any $p \in \mathbb{P}$. However, we shall establish a stronger result:

Lemma 6.1. *Suppose that $p \in \mathbb{P}, p \geq 5$. If three consecutive numbers are equidivisible with exactly $2p$ divisors, then only one can be even.*

Proof. If $d(n) = 2p$, then $n = q^{2p-1}$ or $n = q^{p-1} \cdot r$, where $q, r \in \mathbb{P}$. Consequently, if two consecutive even numbers both have exactly $2p$ divisors, the one which is divisible by 4 must be 2^{2p-1} or $2^{p-1}r$. But the former is not possible, since

$$d(2^{2p-1} \pm 2) = 2d \cdot (2^{2p-2} \pm 1) = 2p$$

implies

$$d(4^{p-1} \pm 1) = 1.$$

It follows that $4^{p-1} \pm 1 = s^{p-1}$ for some $s \in \mathbb{P}$. However, this cannot occur since any two $(p-1)^{\text{st}}$ powers in $\mathbb{Z}^+$ differ by at least $2^{p-1} - 1 \geq 1$.

Hence, if $d(a-1) = d(a) = d(a+1) = 2p$ and $2 \nmid a$, then we must have $\{a-1, a+1\} = \{2q^{p-1}, 2^{p-1} \cdot r\}$, where $q, r \in \mathbb{P} \setminus \{2\}$.

Let $2t := p - 1$; then, $t \geq 2$ by our assumption on p. Since $2q^{2t} \equiv 2 \mod 8$ and $2^{2t} \cdot r \equiv 0 \mod 8$, it follows that

$$a - 1 = 2^{2t} \cdot r, \; a + 1 = 2q^{2t}.$$

Let us identify which of $a-1, a, a+1$ is a multiple of 3.

1. If $r = 3$, then $a + 1 = 3 \cdot 2^{2t} + 2 = 2q^{2t}$, and therefore $2^{2t} + 2^{2t-1} + 1 = q^{2t}$. But the left side of this equation is less than 3^{2t}, so it cannot equal q^{2t}.

2. If $q = 3$, then $a - 1 = 2 \cdot 3^{2t} - 2 = 2^{2t} \cdot r$, and therefore

   $$2^{2t-1} \cdot r = 3^{2t} - 1 = (3^t - 1) \cdot (3^t + 1).$$

   This requires one of $3t \pm 1$ to be a power of 2. As $t \geq 2$, the only possibility is $t = 2$, since $8, 9$ are the largest consecutive integers with no prime factor greater than 3. (This is a consequence of Størmer's Theorem, see for example Ecklund & Eggleton (1972).) Then, $d(a+1) = d(162) = 10$, but $d(a-1) = d(160) = 12$, a "near miss".

It now follows that $3 \mid a$. If $9 \mid a$, then $a + 1 = 2q^{2t}$ would imply $2q^{2t} \equiv 1 \mod 9$. Consideration of quadratic residues $\mod 9$ shows that this is impossible. Hence, $a = 3s^{p-1}$, and $q, r, s \in \mathbb{P} \setminus \{2, 3\}$.

Now, $a - 1 = 2^{2t} \cdot r$ implies $3s^{2t} \equiv 1 \mod 8$; again considering quadratic residues, we see that is not possible. It now follows by contradiction that $2 \mid a$ if $d(a-1) = d(a) = d(a+1) = 2p$. ☐

The restriction to primes $\neq 3$ is necessary, since $7442 \in D(6, 3)$.

Theorem 6.2. *$M(2p) \leq 3$ for any prime $p \neq 3$.*

Proof. When $p = 2$, the claim follows from Theorem 4.2; when $p \geq 5$ it follows from Lemma 6.1. ☐

Corollary 6.3. *For any prime $p \neq 3$, if $D(2p, 3)$ is not empty it contains only odd numbers.*

Proof. Lemma 4.1 implies that no multiple of 4 can occur in a run of two or more equidivisible integers with exactly 4 divisors; thus, only one even number can occur in any run of three such numbers. This settles the case $p = 2$. For $p \geq 5$, the claim is explicit in Lemma 6.1. $\square$

Theorem 6.4. $M(10) = M(14) = 3$.

Proof. We have checked that $7,939,375 = \min D(10,3)$, and $76,571,890,623 \in D(14,3)$. $\square$

We have not been able to show $D(2p,3) \neq \emptyset$ for any $p \geq 11$.

7 Numbers with exactly 2^n divisors, n ≥ 3

Next, we consider numbers with exactly 8 divisors. With the publication of tables of $d(n)$ for $n \leq 10^4$ by Glashier (1940), it was known that $M(8) \geq 4$ and $|D(8,4) \cap [1,10^4]| = 7$. Later, Mycielski showed that $M(8) \geq 5$ and $|D(8,5) \cap [1,10^5]| = 2$, see Sierpinski (1988), p. 169. It appears from the accounts given in Guy (1981), **B18**, and Sierpinski (1988) that the corresponding runs of length 5 may be the longest known runs of equidivisible numbers known to date.

We can now show

Theorem 7.1. *1.* $M(8) = 7$,

2. $M(16) = 7$,

3. $5 \leq M(32) \leq 7$.

Proof. By Theorem 3.2, $M(k) \leq 7$ for $k \in \{8,16,32\}$. The rest follows from

$$171,893 \in D(8,7),$$
$$17,476,613 \in D(16,7),$$
$$57,645,182 \in D(32,5).$$

The values are the smallest possible. $\square$

8 Values of $\mathbf{M}(k), k \leq 32$ in other cases

Let us briefly look at those even $k \leq 32$ which have not yet been covered. Theorems 3.3 and 3.4 immediately give us

Theorem 8.1. *1.* $M(20), M(28), M(30) \leq 7$.

2. $M(24) \leq 31$.

For $k = 12$ and $k = 18$ we can somewhat improve the bounds.

Lemma 8.2. *If* $d(8n) = 12$, *then* $d(8(n+1)) \neq 12$ *or* $d(8(n+2)) \neq 12$.

Proof. If $d(8m) = 12$, then m has one of the following forms:

$$m = 2^8,$$
$$m = p^2,$$
$$m = 4p$$

for some prime p.

Assume that for some $n \in \mathbb{Z}^+$, $d(8n) = d(8(n+1)) = d(8(n+2)) = 12$. Inspection rules out that 2^8 is one of these numbers, and at most one of them can have the form $8p^2$. Therefore, $n, n+1, n+2$ are three consecutive numbers, at least two of which are divisible by 4, a contradiction. ☐

Corollary 8.3. $M(12) \leq 23$.

Proof. 24 consecutive numbers contain 3 consecutive multiples of 8. ☐

Now, let us look at $k = 18$.

Theorem 8.4. $M(18) \leq 5$

Proof. If $d(n) = 18$, then n has one of the following forms:

$$p^{17},\ p \cdot q^8,\ p \cdot q^2 \cdot r^2,\ p^2 \cdot q^5,$$

for some primes p, q, r. Since we need only consider even n, inspection rules out the case 2^{17}.

Suppose that for $0 \leq i \leq 5$, $d(n+i) = 18$, and let $m \in N := \{n, n+1, \ldots, n+5\}$. First, assume that $m \equiv 6 \mod 8$. Then, $m = 2 \cdot p^8$ or $m = 2 \cdot p^2 \cdot q^2$ for some odd primes p, q. However, if r is odd, then $r^2 \equiv 1 \mod 8$, which shows that $2 \cdot r^2 \not\equiv 6 \mod 8$.

Next, assume that $m \equiv 0 \mod 8$, and that $m + 2 \in N$. There are two cases:

1. $m = 32 \cdot p^2$ for some odd prime p. Then, $\frac{m}{2}$ is a perfect square, and $m + 2 = 2 \cdot q^8$ or $m = 2 \cdot q^2 \cdot r^2$ for odd primes q, r. In either case, $\frac{m+2}{2} = \frac{m}{2} + 1$ is a perfect square as well, a contradiction.

2. $m = 2^8 \cdot p$ for some odd prime p. As above, $m + 2 = 2 \cdot t^2$ for some $t > 2$, so that

   $$2^7 \cdot p = t^2 - 1 = (t+1) \cdot (t-1).$$

   This is only possible for $p = 31$, but $2^8 \cdot 31 + 1$ is prime.

Thus, N does not contain a number $m \equiv 6 \mod 8$, and cannot contain two numbers $\equiv 0 \mod 8$ and $\equiv 2 \mod 8$. It follows that N has at most 5 elements. ☐

The lower bounds we have been able to obtain are shown in Table 2.

9 Long runs of equidivisible numbers

We have already seen runs of 7 equidivisible numbers with exactly 8 or 16 divisors, and noted that apparently the longest previously known runs of equidivisible numbers are of length 5. Now, let us investigate how long such runs can be.

For any prime p and any $n \in \mathbb{Z}^+$ we call ε the *exponent of p in n*, if $p^\varepsilon \mid n$ and $p^{\varepsilon+1} \nmid n$. Let p_n be the n–th prime, and let $\{a_n : n \in \mathbb{Z}^+\}$ be a sequence of positive integers defined by

$$a_n := \prod_{i<n} p_i^{\varepsilon(n,i},$$

where $\varepsilon(n,i)$ is th exponent of p_i in $n-i$. Thus, for example,

$$a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 1, a_5 = 12, a_6 = 1, a_7 = 2, \ldots$$

For $m \in \mathbb{Z}^+$ we define

$$\begin{aligned} L(m) &:= \operatorname{lcm} d(a_n) : n \in [1,m], \\ \lambda(m,n) &:= m \cdot \frac{L(m)}{d(a_n)}, \\ M(m) &:= \prod_{n \le m} p_n^{\lambda(m,n)}, \\ M(m,n) &:= \frac{M(m)}{a_n \cdot p_n^{\lambda(m,n)-1}}. \end{aligned}$$

Observe that $\lambda(m,n)$ is a positive multiple of m, and therefore, $\lambda(m,n) \ge m$, and $p_n^{\lambda(m,n)-1} \le m$. If $1 \le i < n \le m$, then

$$p_i^{\lambda(m,i)-1} \ge m > n-i \ge p_i^{\varepsilon(n,i)},$$

and thus, $\lambda(m,i) - 1 \ge \varepsilon(n,i)$. It follows that $M(m,n)$ is an integer, and it is divisible by each prime p_i with $i \in [1,m]$, and no others.

By the Chinese Remainder Theorem, there is a smallest positive integer x_m such that $x \equiv x_m \mod M(m)$ is the general solution to the system of simultaneous congruences

$$x + n \equiv p_n^{\lambda(m,n)-1} \mod p_n^{\lambda(m,n)}, \qquad n \in [1,m].$$

Then, $x_m + n = b_{m,n} \cdot p_n^{\lambda(m,n)-1}$ for some $b_{m,n} \in \mathbb{Z}^+$, and $p_n \nmid b_{m,n}$.

We shall now show that $b_{m,n} = a_n \cdot q(m,n)$, where $q(m,n) \in \mathbb{Z}^+$, and prime factors of $q(m,n)$ are greater than p_m. If $1 \le i < n \le m$, then $\varepsilon(n,i)$ is th eexponent of p_i in $x_m + n = (x_m + i) + (n - i)$, since $\lambda(m,i) - 1 > \varepsilon(n,i)$; hence, the factor of $b_{m,n}$ composed of primes less than p_n is precisely a_n.

If $1 \le n < i \le m$, then $x_m + n = (x_m + i) - (i - n)$, and

$$p_i \mid (x_m + i), \text{ but } p_i \nmid (i - n),$$

since $p_i > i-n > 0$ and therefore $p_i \nmid b_{m,n}$. It follows that $b_{m,n} = a_n \cdot q(m,n)$, where $p_i \nmid q(m,n)$ for all $i \in [1,m]$.

Since $M(m,n)$ is only divisible by primes p_i with $i \leq m$, while none of thes eis a factor of $q(m,n)$, it follows that every prime factor of $r \cdot M(mm,n) + q(m,n)$ is greater than p_m. Hence,

$$\begin{aligned} d(r \cdot M(m) + x_m + n) &= d(a_n) \cdot \lambda(m,n) \cdot d(r \cdot M(m,n) + q(m,n)), \\ &= m \cdot L(m) \cdot d(r \cdot M(m,n) + q(m,n)). \end{aligned}$$

Let

$$\begin{aligned} Q(m,r) &:= \{r \cdot M(m,n) + q(m,n) : n \in [1,m]\}, \\ R(m,r) &:= \{r \cdot M(m) + x_m + n : n \in [1,m]\}. \end{aligned}$$

If there is an r such that all m numbers in $Q(m,r)$ are equidivisible, then $R(m,r)$ is a run of equidivisible numbers. We have proved

Lemma 9.1. *Let $m \in \mathbb{Z}^+$. If there is an integer $r \geq 0$ such that all m numbers in $Q(m,n)$ are equidivisible, then there is a run of m equidivisible numebrs.*

Because $M(m,n)$ and $q(m,n)$ are coprime, Dirichlet's Theorem guarantees that $r \cdot M(m,n) + q(m,n)$ is prime for infinitely many values of r, but it is not strong enough to ensure that there is a value of r for which even two members of $Q(m,r)$ are prime.

We shall prove that Schinzel's Conjecture H (see Sierpinski (1988), p. 133) implies the existence of infinitely many values of r for which all members of $Q(m,r)$ are prime. To shop this, it suffices to prove that there is no prime p which devides the product of numbers in $Q(m,r)$ for every $r \geq 0$.

First, note that all members of $Q(m,0)$ are coprime, because any factor common to $x_m + i$ and $x_m + n$ with $i,n \in [1,m]$ must be less than m, and hence, less than p_m. Now, assume there is a prime p which divides some number in each $Q(m,r), r \in [0,m]$. Then, by the Pigeonhole Principle, there is some $n \in [1,m]$ such that p divides $r_1 \cdot M(m,n) + q(m,n)$ and $r_2 \cdot M(m,n) + q(m,n)$, where $0 \leq r_1 < r_2 \leq m$. Then, p divides $(r_2 - r_1) \cdot M(m,n)$, and therefore, $p \leq p_m$. This contradicts the fact that no member of any $Q(m,r)$ is divisible by such a prime, hence, Conjecture H applies.

When all members of $Q(m,r)$ are prime, they are certainly equidivisible, and each member of $R(m,r)$ has $2m \cdot L(m)$ divisors. It follows that $M^*(2m \cdot L(m)) \geq m$. We have shown

Theorem 9.2. *Conjecture H implies that $\{M^*(k) : k \in \mathbb{Z}^+\}$ is unbounded.*

The construction used for Lemma 9.1 can be made the basis for a computer search for long runs of equidivisible numbers. In practice, it is convenient to modify the construction to reduce the size of the search modulus $M(m)$ to more manageable proportions; however, the details would only have added extra complications to the proof of Lemma 9.1. As a result of such calculations, we have found that

$$\begin{aligned} 4,751,909,738,598,652,780,445 &= 5 \cdot 17^2 \cdot 3319 \cdot 5,299,069 \cdot 186,979,291 \in D(48,7), \\ 6,213,958,594,795,772,370,845 &= 5 \cdot 17^2 \cdot 1733 \cdot 5737 \cdot 432,530,856,701 \in D(48,8), \end{aligned}$$

and, as our most exciting numerical discovery,

Theorem 9.3. $M(48) \geq 9$.

Proof. $N = 17,796,126,877,482,329,126,044 \in D(48,9)$ as witnessed by

$$
\begin{aligned}
N &= 2^2 \cdot 7 \cdot 4327 \cdot 456,293 \cdot 321911699243, \\
N+1 &= 5 \cdot 17^2 \cdot 47 \cdot 53 \cdot 4,944,062,119,125,691, \\
N+2 &= 2 \cdot 3^2 \cdot 179 \cdot 5171 \cdot 1,068,133,213,285,138, \\
N+3 &= 11^5 \cdot 23 \cdot 107 \cdot 44,900,425,217,777, \\
N+4 &= 2^5 \cdot 19 \cdot 4,590,338,339 \cdot 6,376,424,429, \\
N+5 &= 3 \cdot 13^2 \cdot 241 \cdot 557 \cdot 261,484,106,225,711, \\
N+6 &= 2 \cdot 5^2 \cdot 11831 \cdot 189043 \cdot 159,137,830,837, \\
N+7 &= 7^5 \cdot 29 \cdot 351,121 \cdot 103,987,345,177, \\
N+8 &= 2^2 \cdot 3 \cdot 149 \cdot 991,723 \cdot 10,036,160,394,373.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
N-1 &= 3 \cdot 73 \cdot 2381 \cdot 63,678,479 \cdot 535,956,203, \\
N+9 &= 449 \cdot 11,618,801 \cdot 3,411,283,698,997,
\end{aligned}
$$

which proves our claim. □

10 Tables

Here, we give lower and upper bounds for $M(k)$, $2 \leq k \leq 32$, k even, which we have been able to obtain.

Table 1: Values of $M(k)$

k	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
$M(k)$	2	3	5	7	3	≤ 23	3	7	≤ 5	≤ 7	≤ 3	≤ 31	≤ 3	≤ 7	≤ 7	≤ 31

Lower bounds can be obtained from Table 2

References

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Table 2: Values of $D(k,m)$

k	$D(k,1)$	$D(k,2)$	$D(k,3)$	$D(k,4)$	$D(k,5)$	$D(k,6)$	$D(k,7)$
2	5	2	•				
4	6	14	33	•			
6	12	44	603	242	10,093,613,546,512,121	•	
8	24	104	230	3655	11605	28374	171893
10	48	2511	7939375	•			
12	60	735	1274	19940	204323	?	?
14	192	29888	76,571,890,623	•			
16	120	2295	8294	153,543	178,086	5,852,870	17,476,613
18	180	6075	959,075	?	?	•	
20	240	5264	249,750	?	?	?	?
22	3072	?	?	•			
24	360	5984	72224	2,919,123	15537948	?	?
26	12288	?	?	•			
28	960	156,735	?	?	?	?	?
30	720	180,224	?	?	?	?	?
32	840	21735	318,680	6,800,934	57,645,182	?	?

The smallest member of each class is given except, possibly, for$D(6,5)$ and $D(14,3)$

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