On the maximal number of non C - endorigid equivalence relations

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Abstract

In [1] it was shown that a maximal non C - endorigid family of equivalence relations on a set U with $|U| = n \geq 3$ has at least $2 \cdot eq(n-1)$ elements, where $eq(m)$ is the number of equivalence relations on a set with m elements, and it was asked whether any larger family of such relations exists. In this note we show that the given bound is the best possible.

Key words: endorigid relation, equivalence relation

1 Preliminaries

Throughout, U is a finite set with $n \geq 3$ elements. ω is the set of natural numbers, and we write ω^+ for $\omega \setminus \{0\}$. If $\varphi : U \to U$ is a mapping, $ran\ \varphi = \{x\varphi : x \in U\}$ is the *range* of φ, and if $K \subseteq U$, then K^φ is the set $\{a\varphi : a \in K\}$. $Sym(U)$ denotes the symmetric group on U, and $1'$ the identity function. We say that $\varphi \in Sym(U)$ is *reduced* if the cardinality of each non trivial cycle of φ is prime. It is well known that any $1' \neq \varphi \in Sym(U)$ has a reduced power which is not the identity.

$Rel(U)$ is the set of all binary relations on U. For a mapping $\varphi : U \to U$ and $R \in Rel(U)$ we denote by R^φ the set $\{(a\varphi, b\varphi) : (a,b) \in R\}$. If $R^\varphi \subseteq R$, then φ is called an *endomorphism* of R. For $Q \subseteq Rel(U)$, we define $End\ Q = \{\varphi : U \to U : R^\varphi \subseteq R \text{ for all } R \in Q\}$, and Q is called *endorigid* if $End\ Q = \{1'\}$.

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Let C be the set of all constant functions on U; we call Q C - *endorigid* if $End\ Q \subseteq \{1'\} \cup C$.

$Eq(U)$ is the set of all equivalence relations on U. If $k \in \omega^+$, we set $eq(k) = |Eq(V)|$, where V is a k - element set; following convention, we set $eq(0) = 1$. For $n \geq 1$, $eq(n)$ is given by

$$eq(n) = \frac{1}{e} \cdot \sum_{r=1}^{\infty} \frac{r^n}{r!},$$

see [2], p. 555.

If $R \in Eq(U)$, then the number of equivalence relations contained in R is given by $\prod_{K \in Part(R)} |Eq(K)|$, where $Part(R)$ is the set of blocks of R.

To illustrate the behaviour of $eq(n)$, we list $eq(n)$ for a few small numbers:

n	1	2	3	4	5
$eq(n)$	1	2	5	15	52

n	6	7	8	9	10
$eq(n)$	203	877	4140	21147	115975

In the sequel we shall need two combinatorial results. The proof of the first one, which may be known, is a straightforward, if rather lengthy and tedious induction and will be omitted.

Lemma 1.1 *If* $0 < n < m$, *then* $eq(n) \cdot eq(m) \geq eq(n+1) \cdot eq(m-1)$. □

Lemma 1.2 *If* $k \geq 1$ *and* $a_0 + \ldots + a_k = n, a_i \geq 1$, *then* $eq(a_0) \cdot \ldots \cdot eq(a_k) \leq eq(n-k)$.

Proof. Let $k = 1$, and w.l.o.g. $a_0 \leq a_1$. Then, $a_1 = n - a_0$, and

$$eq(n-1) = eq(1) \cdot eq(n-1) \geq \ldots \geq eq(a_0) \cdot eq(n-a_0),$$

by 1.1.

Now, suppose that the claim is true for $k \geq 1$, and let $a_0 + \ldots + a_k = m, a_i \geq 1$, and $eq(a_0) \cdot \ldots \cdot eq(a_k) \leq eq(m-k)$. Suppose that $a_{k+1} \geq 1$ and $m + a_{k+1} = n$. Then,

$$\begin{aligned} eq(a_0) \cdot \ldots \cdot eq(a_k) \cdot eq(a_{k+1}) &\leq eq(m-k) \cdot eq(a_{k+1}) \\ &\leq eq(m-k+a_{k+1}-1) \\ &= eq(n-(k+1)). \end{aligned}$$

□

2 Maximal non C - endorigid families of equivalence relations

Let $|U| = n$ and $\mu(n) = max\ \{|Q| : Q \subseteq Eq(U)$ and Q is not C - endorigid$\}$. This differs slightly from the notation in [1], but it is clear how to move from one concept to the other. In [1] it was shown that $\mu(n) \geq 2 \cdot eq(n-1)$ for $n \geq 3$, and the problem was posed to exactly determine $\mu(n)$. The aim of this note is to show that the given bound is the best possible.

For $\varphi : U \to U$, let $Eq(\varphi) = \{R \in Eq(U) : R^\varphi \subseteq R\}$. The strategy we shall pursue to find $\mu(n)$ is the following: Instead of directly considering families of equivalence relations on U, we look at certain canonical types of transformations φ on U and determine $Eq(\varphi)$. This is justified by the following observation:

Lemma 2.1 *Let Q be a maximal non C - endorigid family of equivalence relations on U. Then, there is some $\varphi : U \to U$ such that $Q = Eq(\varphi)$, $|ran\ \varphi| \geq 2$, $\varphi \neq 1'$, and φ has one of the following forms:*

1. *φ is a reduced permutation of U,*
2. *$ran\ \varphi \neq U$ and $\varphi|(ran\ \varphi)$ is the identity on $ran\ \varphi$,*
3. *$ran\ \varphi \neq U$ and $\varphi|(ran\ \varphi)$ is a constant function.*

Proof. Since Q is not C - endorigid, there is some $\varphi : U \to U$ such that $|ran\ \varphi| \geq 2$, $\varphi \neq 1'$, and $R^\varphi \subseteq R$ for all $R \in Q$. Then, $Q \subseteq Eq(\varphi)$, and hence $Q = Eq(\varphi)$ by the maximality of Q. The rest now follows from the fact that $End(Q)$ is a semigroup with respect to composition, and that some power of φ has one of the prescribed forms. $\square$

We first look at the case when φ is a permutation of U. In what follows, let $1' \neq \varphi \in Sym(U)$ be reduced, R be an equivalence relation on U, and $Part(R)$ its set of blocks. If $\psi \in Sym(U)$, we denote by $\psi_R : Part(R) \to \mathcal{P}(U)$ the mapping $K \mapsto K^\psi$.

Lemma 2.2 1. *$R^\varphi \subseteq R$ implies $R^\varphi = R$.*

2. *$R^\varphi = R$ if and only if φ_R is a permutation of $Part(R)$.*
3. *If M is a cycle of φ, K a block of R, $K \cap M \neq \emptyset$ and $R^\varphi = R$, then $M \subseteq K$ or $|M \cap K| = 1$.*

Proof. 1. Since φ is a permutation of U, $|R^{\varphi}| = |R|$, and the conclusion follows from the fact that U is finite.

2. "$\Rightarrow$": Let K be a block of R. If $a, b \in K^{\varphi}$, there are $c, d \in K$ such that $c\varphi = a$, $d\varphi = b$. Since cRd and φ preserves R, we have aRb. If aRe, then $(a, e)\varphi^{-1} = (c, e\varphi^{-1}) \in R$, which implies $e \in K^{\varphi}$. Thus, K^{φ} is a block of R.

"$\Leftarrow$": Let aRb, $a, b \in K$, a block of R. Since K^{φ} is a block of R, we have $a\varphi R b\varphi$.

3. Suppose that $a, b \in M \cap K$, $a \neq b$, and $a^{\phi^k} = b$. Then, $K^{\phi^k} = K$, and hence, $K = K^{\phi^{n \cdot k}}$ for any $n \in \omega$. Since φ is reduced, $|M|$ is a prime, and therefore $M \subseteq K$. $\Box$

Proposition 2.3 [1] *If $\varphi \neq 1'$ is a reduced permutation of U, then $|Eq(\varphi)| \leq eq(n-1) + eq(n-2)$.*

Proof. The strategy we shall pursue is to replace φ with a permutation which contains only one non trivial cycle, and then show that for such permutations the conclusion holds.

Task 1: Replace φ be a permutation with only one non trivial cycle:

Let $(a_0 \dots a_{m-1})$ be a non trivial cycle of φ, and set $\psi = (a_0 \dots a_{m-1})$, $M = \{a_0, \dots, a_{m-1}\}$. We shall exhibit an injective mapping $\nu : Eq(\varphi) \to Eq(\psi)$.

Let $R \in Eq(\varphi)$. We consider two cases:

1. If $M \subseteq K$ for some block K of R or if $\{a_i\}$ is a block of R for some (and hence for all) $i < m$, we set $\nu(R) = R$. Note that in the first case $K^{\varphi} = K$.

2. Otherwise, there is some $K_0 \in Part(R)$ such that $K_0 \cap M = \{a_0\}$, and $K_0 \setminus \{a_0\} \neq \emptyset$. Consider the cycle $(K_0 \dots K_t)$ of φ_R: We have $|M \cap K_i| = 1$ by 2.2, and $K_i \setminus M \neq \emptyset$ for all $i \leq t$; furthermore, for any block K of R, $K \cap M \neq \emptyset$ if and only if $K = K_i$ for some $i \leq t$. The classes of $\nu(R)$ now are defined as follows:

   (a) $K_0 \cup M$

   (b) $K_i \setminus M$ for $0 < 1 \leq t$

   (c) K, if $K \in Part(R)$, $K \neq K_0, \dots, K_t$

   Observe that in this case, $\nu(R) \notin Eq(\varphi)$, since $(K_0 \cup M)^{\varphi} \cap (K_0 \cup M) \neq \emptyset$, and $(K_0 \cup M)^{\varphi} \cap -(K_0 \cup M) \neq \emptyset$. Furthermore, $K \in Part(R) \cap Part(\nu(R))$ if and only if $K \cap M = \emptyset$

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[1] I should like to thank the referee for pointing out a gap in the original proof.

Since M is contained in a block of $\nu(R)$ or M is discrete in $\nu(R)$, we have $\nu(R)^{\psi} = \nu(R)$ in both cases.

Suppose that $R_0, R_1 \in Eq(\varphi)$, $R_0 \neq R_1$. If, say, $\nu(R_0) \in Eq(\varphi)$, then $\nu(R_0) = R_0$, and either $R_1 = \nu(R_1)$ or $\nu(R_1) \notin Eq(\varphi)$. In both cases, $\nu(R_0) \neq \nu(R_1)$. Thus, let $\nu(R_0)$, $\nu(R_1) \notin Eq(\varphi)$. Then, both $\nu(R_0)$ and $\nu(R_1)$ are obtained by construction 2. above. There are two cases:

1. There is some $K \in Part(R_0) \cup Part(R_1)$ such that
   (a) $K \cap M \neq \emptyset$,
   (b) $K \notin Part(R_0) \cap Part(R_1)$.

   Suppose w.l.o.g. that K is a block of R_0. Then $a_0 \in K^{\rho}$ for some power ρ of φ, and $K \notin Part(R_1)$ implies that $K^{\rho} \notin Part(R_1)$. Let$K'$ be the block of R_1 containing a_0. Since $K^{\rho} \neq K'$ and $K^{\rho} \cap M = K' \cap M = \{a_0\}$, we have $K^{\rho} \cup M \neq K' \cup M$, and hence, $\nu(R_0) \neq \nu(R_1)$.

2. Otherwise, suppose that for each $K \subseteq U$ with $K \cap M \neq \emptyset$ we have

   $$K \in Part(R_0) \iff K \in Part(R_1).$$

   Let w.l.o.g. $K \in Part(R_0) \setminus Part(R_1)$; then, $K \cap M = \emptyset$, and hence $K \in \nu(R_0)$.

   Assume that $K \in \nu(R_1)$. Since $K \notin Part(R_1)$ and $K \cap M = \emptyset$, by the construction of $\nu(R_1)$ there is some a_i such that $K' = K \cup \{a_i\} \in Part(R_1)$. By our hypothesis, we have $K \cup \{a_i\} \in Part(R_0)$, a contradiction. It follows that $\nu(R_0) \neq \nu(R_1)$.

Thus, we can suppose that φ has exactly one non trivial cycle M of prime cardinality m.

Task 2: Show that $|Eq(\varphi)| \leq eq(n-1) + eq(n-2)$:

Let $R^{\varphi} = R$. Then, either M is a contained in a block of R or $|Ra| = 1$ for any $a \in M$. The first case gives us at most $eq(n-m+1)$ relations, and from the second case we obtain at most $eq(n-m)$ possibilities. Since $m \geq 2$, we see that $|Eq(\varphi)| \leq eq(n-1) + eq(n-2)$. $\square$

This takes care of the case when φ is a permutation, since $eq(n-1) + eq(n-2) < 2 \cdot eq(n-1)$. The other cases are covered by

Proposition 2.4 *Let* $\varphi : U \to U$, *such that ran* $\varphi = T$, $T \neq U$, $|T| \geq 2$, *and*

1. $\varphi|T$ *is the identity on* T, *or*

2. $\varphi|T$ is a constant function.

Then $|Eq(\varphi)| \leq 2 \cdot eq(n-1)$.

Proof. For each $S \in Eq(T)$, let θ_S be the equivalence on U defined by

$$(x,y) \in \theta_S \overset{\text{def}}{\Longleftrightarrow} (x\varphi, y\varphi) \in S.$$

The classes of θ_S have the form $\varphi^{-1}(K)$, where K is a class of S. If $(x,y) \in S$, then $x, y \in T$ and either $x\varphi = x$, $y\varphi = y$, or $x\varphi = y\varphi \in T$; at any rate, $(x,y) \in \theta_S$, so that $S \subseteq \theta_S$. For each $S \in Eq(T)$, set

$$\nu(S) = \{R \in Eq(U) : S \subseteq R \subseteq \theta_S\}.$$

Claim 1: $Eq(\varphi) = \bigcup_{S \in Eq(T)} \nu(S)$:

Proof: "$\subseteq$": Let $R \in Eq(\varphi)$, and set $S = R \cap T^2$. Then, $S \subseteq R$, and $S \in Eq(T)$. We need to show that $R \subseteq \theta_S$: Let $(x,y) \in R$; since *ran* $\varphi = T$, we have $(x\varphi, y\varphi) \in T^2$, and it follows from $R^\varphi \subseteq R$ that $(x\varphi, y\varphi) \in R$. Hence, $(x\varphi, y\varphi) \in R \cap T^2 = S$, and thus $(x,y) \in \theta_S$.

"$\supseteq$": Suppose that $S \in Eq(T)$, $R \in Eq(U)$, and $S \subseteq R \subseteq \theta_S$. We need to show that $R^\varphi \subseteq R$: If $(x,y) \in R$, then $R \subseteq \theta_S$ implies that $x\varphi S y\varphi$, and $S \subseteq R$ implies $x\varphi R y\varphi$. • *(Claim 1)*

Let $|T| = k$, $S \in Eq(T)$, $\{K_i : i \leq m\}$ be the set of blocks of S, and $A_i = \varphi^{-1}(K_i)$, $i \leq m$. Then, $\{A_i : i \leq m\}$ is the set of blocks of θ_S. We are going to find an upper bound for $|\nu(S)|$:

Claim 2: $|\nu(S)| \leq eq(n-k+1)$:

Proof: If $S \subseteq R \subseteq \theta_S$, then each block of S is contained in a block of R, and each block of R is contained in some A_i. Collapse each K_i to a point, and let U', $\theta_S^{'}$, $A_i^{'}$ be the resulting sets. Furthermore, set $|A_i^{'}| = a_i$, and $r = a_0 + \cdots + a_m$. Then, $r = n - k + m + 1$, and

$$\begin{aligned}
|\nu(S)| &= |\{R \in Eq(U) : S \subseteq R \subseteq \theta_S\}| \\
&= |\{R' \in Eq(U') : R' \subseteq \theta_S^{'}\}| \\
&= \prod_{i \leq m} eq(a_i) \\
&\leq eq(r-m) \qquad \text{(by 1.2)} \\
&= eq(n-k+m+1-m) \\
&= eq(n-k+1). \qquad \bullet \quad (\textit{Claim 2})
\end{aligned}$$

Now we are ready to prove the result:

$$\begin{aligned}
|Eq(\varphi)| &= |\bigcup_{S \in Eq(T)} \nu(S)| \qquad \text{(by Claim 1)} \\
&\leq \sum_{S \in Eq(T)} |\nu(S)| \\
&\leq eq(k) \cdot eq(n-k+1) \qquad \text{(by Claim 2)} \\
&\leq eq(k-(k-2)) \cdot eq(n-k+1+(k-2)) \qquad \text{(by 1.1)} \\
&= eq(2) \cdot eq(n-1) \\
&= 2 \cdot eq(n-1).
\end{aligned}$$

This completes the proof. □

Collecting all results, we arrive at

Proposition 2.5 *Let $n \geq 3$. Then, $\mu(n) = 2 \cdot eq(n-1)$.*

Proof. 2.3 and 2.4 show that $\mu(n) \leq 2 \cdot eq(n-1)$. The bound is attained by the following example from [1]:

Let $T = \{a, b\} \subseteq U$, and define $\varphi : U \to U$ by

$$x\varphi = \begin{cases} a, & \text{for } x = a, \\ b, & \text{otherwise.} \end{cases}$$

Then, $\varphi|T$ is the identity on T, and $|Eq(\varphi)| = eq(n-1) + eq(n-1)$. □

References

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[2] Ledermann, W. *Handbook of applicable Mathematics, vol V(B).* Wiley, 1985.