Moving Spaces

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Abstract

Boolean contact algebras constitute a convenient approach to a region based theory of space. In this paper we want to extend this approach to regions moving in time - called timed contact structures. We study their canonical models using topological spaces. As the main contribution we prove a general representation theorem for this kind of algebras.

1. Introduction

The origins of mereotopology go back to the works of [11] on mereology and, on the other hand, the works of [9], [12], and [16] who used regions instead of points as the basic entity of geometry. In this "pointless geometry", points are now second order definable as sets of regions, similar to the representation of Boolean algebras, where elements can be recovered as ultrafilters. Whitehead's addition to the mereological structures of Leśniewski (which were, basically, complete Boolean algebras B without a smallest element) was a "connection" (or "contact") relation C among nonempty regions, which, in its simplest form is a reflexive and symmetric relation satisfying an additional extensionality axiom. Historically, standard (models for) mereotopological structures were collections of regular closed (or regular open) sets of topological spaces $\langle X, \tau\rangle$ with the *standard (Whiteheadian) contact* among regions defined by

$$uCv \iff u \cap v \neq \emptyset.$$

The primary example is the collection of all nonempty regular closed sets of the Euclidean plane.

The most simple algebraic counterpart of mereotopology are contact algebras, appearing in different papers under various names (see for instance [1, 2, 3, 5, 6, 7, 13, 14, 15]) which are Boolean algebras extended with the contact relation C satisfying some axioms. The elements of the Boolean algebra symbolize regions while the contact relation C corresponds to the additional topological aspect.

In this paper we want to extend this approach to regions moving in time. We model those regions by a collection of snapshots. First, we fix a structure of static regions, i.e. the static *world*. Given a set of points in time T a moving region is just a function from T to the static world. In other words, a region is described by a collection of static regions at any point in time t - its (spatial) extent at time t. We want to illustrate this idea by two examples. Notice that not all regions may be 'visible' at any given time; this situation is modelled by the fact that such a region is not in contact with anything, including itself, at that point in time.

Example 1. We choose the Euclidean plane as static world and the set $\mathbb{R}^{\geq 0}$ of the non-negative real numbers as the model of time. Now, a moving region is a function from $\mathbb{R}^{\geq 0}$ to the regular closed sets of the plane. For example, r is the region so that $r(x)$ is the closed disc with radius 1 at the point (x, x). This region starts at the origin and moves along the diagonal without changing its shape. Another example is the region s where $s(x)$ is the closed disc at the origin with radius x. This region starts as the empty region and grows constantly bigger. It does not move at all.

Example 2. The second example is finite. Consider the set $X = \{1, 2, 3, 4, 5, 6\}$ with a contact structure based an the following picture

a	b	c
d	e	f

i.e. a is in contact with b and d, b is in contact with a, c and e etc. We say that two subsets A, B of X are in contact (in symbols: ACB) iff there is an $a \in A$ and an $b \in B$ which are in contact. This structure - the powerset $\mathcal{P}(X)$ of X together with the relation C - establishes a (finite) Boolean contact algebra (cf. next section). $\mathcal{P}(X)$ serves as the static

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world in our example. Time will also be discrete, and we choose $T = \{0, 1, 2, 3\}$. Similar to the previous example a moving region is a function from T to $\mathcal{P}(X)$. For example, r defined by

$$r(0) = \{b\},\ \ r(1) = \{a\},\ \ r(2) = \{d\},\ \ r(3) = \{e\}$$

is the region that starts at b and moves to e via a and d. Also in this example it is possible for regions to grow or shrink. The region s defined by

$$s(0) = \{b\},\ \ s(1) = \{b, c\},\ \ s(2) = \{c\},\ \ s(3) = \emptyset$$

starts at b grows to $\{b, c\}$ and finally disappears.

2. Boolean Contact Algebras

For any set X and $Y \subseteq X$ we denote by $X - Y$ the complement of Y in X. If X is clear from the context we simply write $-Y$. For a binary relation R on D, and $x \in D$, we let $R(x) = \{y : xRy\}$, the *range of x with respect to R.*

If $\sim$ is an equivalence relation on a set A, we will denote the equivalence class of an element $a \in A$ by $[a]$, i.e. $[a] = \{x \in A \mid a \sim x\}$.

2.1. Contact Relations

Throughout this paper, $(B, 0, 1, +, \cdot, *)$ be a Boolean algebra; if no confusion can arise we sometimes denote algebras by their base set.

In the remainder we will also use the symmetric difference $a \triangle b := a \cdot b^* + a^* \cdot b$ of two elements a and b.

A binary relation C on a Boolean algebra B is called a *contact relation* (CR) if it satisfies:

C0. $(\forall a)0(-C)a$;

C1. $(\forall a)[a \neq 0 \Rightarrow aCa]$;

C2. $(\forall a)(\forall b)[aCb \Rightarrow bCa]$;

C3. $(\forall a)(\forall b)(\forall c)[(aCb \text{ and } b \leq c) \Rightarrow aCc]$;

C4. $(\forall a)(\forall b)(\forall c)[aC(b + c) \Rightarrow (aCb \text{ or } aCc)]$.

The pair $\langle B, C\rangle$ is called a *Boolean Contact Algebra* (BCA).

Axioms C0 and C1 imply that 0 is the only element with $C(0) = \emptyset$, i.e. that is not connected to any element. We are interested in a weakening of this property. Therefore, we call a relation C on a Boolean algebra B a pre-contact relation if it satisfies C0,C2-4 and

P1. $(\forall a)[(\exists b)aCb \Rightarrow aCa]$;

Any contact relation is a pre-contact relation.

Notice that the range of C distributes over $+$ for any (pre-)contact relation, i.e.

$$\begin{aligned} x \in C(a+b) &\iff (a+b)Cx \\ &\iff aCx \text{ or } bCx && \text{by C2-4} \\ &\iff x \in C(a) \cup C(b). \end{aligned}$$

Lemma 1 *Let C be a pre-contact relation on a Boolean algebra B. Then, the relation $\sim$ on B defined by*

$$a \sim b \ :\iff\ C(a \triangle b) = \emptyset,$$

is a congruence relation, and the relation $C/_\sim$ on $B/_\sim$ defined by

$$[a](C/_\sim)[b] \ :\iff\ aCb,$$

is a contact relation.

Proof. Clearly, $\sim$ is reflexive and symmetric. Let $a \sim b$ and $b \sim c$. Then, by definition we have

$$C(a \cdot b^*) = C(a^* \cdot b) = C(b \cdot c^*) = C(b^* \cdot c) = \emptyset.$$

We have to show $C(a \triangle c) = C(a \cdot c^*) \cup C(a^* \cdot c) = \emptyset$. Now,

$$\begin{aligned} C(a \cdot c^*) &= C(a \cdot c^* \cdot b + a \cdot c^* \cdot b^*) \\ &= C(a \cdot c^* \cdot b) \cup C(a \cdot c^* \cdot b^*) && \text{see above} \\ &\subseteq C(c^* \cdot b) \cup C(a \cdot b^*) && \text{C3} \\ &= \emptyset. \end{aligned}$$

The second case is analogous.

Since $a \triangle b = a^* \triangle b^*$, $\sim$ preserves complementation. If $a \sim b$ and $a' \sim b'$, then

$$\begin{aligned} &C(a \cdot a' \cdot (b \cdot b')^*) \\ &= C(a \cdot a' \cdot (b^* + b'^*)) \\ &= C(a \cdot a' \cdot b^* + a \cdot a' \cdot b'^*)) \\ &= C(a \cdot a' \cdot b^*) \cup C(a \cdot a' \cdot b'^*)) && \text{see above} \\ &\subseteq C(a \cdot b^*) \cup C(a' \cdot b'^*)) && \text{C3} \\ &= \emptyset. \end{aligned}$$

The other follows analogously so that $\sim$ preserves meet. Finally, $C(0) = \emptyset$ by C0, and, hence, $\sim$ is a Boolean congruence.

To show that $C/_\sim$ is well-defined we assume that $a \sim b$ and aCc. We conclude $C(a \triangle b) = \emptyset$, which is equivalent to $C(a \cdot b^*) = \emptyset$ and $C(a^* \cdot b) = \emptyset$. Now,

$$\begin{aligned} C(a) &= C(a \cdot b + a \cdot b^*) \\ &= C(a \cdot b) \cup C(a \cdot b^*) && \text{see above} \\ &= C(a \cdot b) && C(a \cdot b^*) = \emptyset \\ &\subseteq C(b) && \text{C3} \end{aligned}$$

shows that $c \in C(b)$, and, hence, bCc.

C0,2-4: These properties follow immediately from the definition of $C/_\sim$ and the corresponding property of C.

C1: Let $[a] \neq [0]$, i.e. $C(a) \neq \emptyset$. By P1 we conclude aCa, and, hence $[a](C/_\sim)[a]$. □

Notice that the congruence relation $\sim$ is not the same as the relation given by $C(x) = C(y)$. Even though the ideal of all elements equivalent to 0 is the same for both relations, the latter is not necessarily a congruence. In Example 2 $\sim$ is the identity, whereas $C(\{b, e\}) = \{a, b, c, d, e, f\} = C(\{a, b, c, d, e, f\})$. In addition, $C(\{a\}) \cap C(\{c\}) = \{b\}$ but $C(\{a\} \cap \{c\}) = \emptyset$.

Every BCA $\langle B, C\rangle$ corresponds exactly to a relational structure on the dual space $\langle \text{Ult}(B), \tau, R_C\rangle$ of B, where R_C is a closed, reflexive and symmetric relation [8]. We call R_C the dual relation of C, and they are related by the following property:

$$aCb \iff \exists F, G \in \text{Ult}(B) : a \in F \wedge b \in G \wedge F R_C G.$$

2.2. Topological models

First we recall some notions from topology. By a topological space (X, τ) we mean a set X provided with a family τ of subsets, called open sets, which contains the empty set and the whole set X, and is closed with respect to arbitrary unions and finite intersections. A subset $a \subseteq X$ is called *closed* if it is the complement of an open set.

In every topological space one can define the following operations on subsets $a \subseteq X$:

1. $Int(a) = \bigcup\{o \in \tau \mid o \subseteq a\}$ (the interior of a), i.e., the union of all open sets contained in a.

2. $Cl(a) = \bigcap\{c \text{ is closed} \mid a \subseteq c\}$ (the closure of a), i.e., the intersection of all closed sets containing a.

Cl and Int are interdefinable, i.e. $Cl(a) = -Int(-a)$ and $Int(a) = -Cl(-a)$.

A subset a of X is called *regular closed* if $Cl(Int(a)) = a$. We denote by $RC(X)$ the family of regular closed sets of X. It is a well known fact that $RC(X)$ is a Boolean algebra with respect to the following operations and constants:

$$0 = \emptyset, 1 = X, a + b = a \cup b \text{ and } a \cdot b = Cl(Int(a \cap b)).$$

$RC(X)$ naturally provides a contact relation C defined by aCb if and only if $a \cap b \neq \emptyset$. C is called the standard (or Whiteheadean) contact relation on $RC(X)$.

A topological space is called semi-regular if it has a basis of regular closed sets. It is called compact if for every non-empty set $\{A_i \mid i \in I\}$ of closed sets the following property holds: If $\bigcap_{j \in J} A_j \neq \emptyset$ for every finite subset J of I, then $\bigcap_{i \in I} A_i \neq \emptyset$.

Notice that all BCA's are representable in certain topological spaces. The following theorem can be found in [4, 6].

Theorem 1 *Let $\langle B, C\rangle$ be a BCA. Then there is a compact and semi-regular T_0 space $\langle X, \tau\rangle$ and a (Boolean) embedding $h : B \to RC(X)$ with aCb if and only if $h(a) \cap h(b) \neq \emptyset$.*

3. Timed Contact Structures

A timed contact structure is given by a set modelling time, a Boolean algebra of regions and a ternary contact relation.

Definition 1 *We call $\langle T, B, (C_t)_{t \in T}\rangle$ a timed contact structure if T is an arbitrary set, B Boolean algebra, and C_t is a pre-contact relation on B for every $t \in T$.*

The property aC_tb indicates that the regions a and b are in contact at time t.

Notice that the family of pre-contact relations could be replaced by a ternary relation $C \subseteq T \times B \times B$ defined by $C = \bigcup_{t \in T} \{t\} \times C_t$.

A suitable definition of a family of pre-contact relations is obvious in both example. The regions r and s from Example 1 are always in contact, i.e. rC_ts for all $t \in \mathbb{R}^{\geq 0}$. For the regions r and s from Example 2 we just have rC_0s and rC_1s.

Lemma 1 shows that for all $t \in T$ there is a quotient BCA $\langle B_t, C_t/_{\sim_t}\rangle$ of B. Notice that the quotients need not be isomorphic. Each algebra describes the static world at time t. Given an element $a \in B$ we will denote the derived element in the quotient B_t by a_t.

The equivalence relation $\sim_t$ is, in general, not the identity. For example, the region s from Example 2 is at $t = 3$ equivalent to the empty region, i.e. $s \sim_3 \emptyset$.

3.1. Topological models

The spatial component is given by a topological space $\langle X, \tau \rangle$ with the Whitheadean contact C^w. A region in time now is a function $f : T \to RC(X)$, i.e. a function providing the spatial extent of the region for every point in time.

We denote the set of all such function by X_T and define $C_t \subseteq X_T \times X_T$ by

$$fC_tg : \iff f(t) \cap g(t) \neq \emptyset.$$

Notice that for each $t \in T$ the relation C_t is defined by the standard contact relation on B, i.e. fC_tg if and only if $f(t)C^wg(t)$.

Lemma 2 *X_T is a Boolean algebra.*

Proof. This follows immediately from the fact that $RC(X)$ is a Boolean algebra and the component-wise definition of the Boolean operators in X_T. □

Theorem 2 *$\langle T, X_T, (C_t)_{t\in T}\rangle$ is a timed contact structure.*

Proof. C0: We have $f(t)\cap 0(t) = f(t)\cap\emptyset = \emptyset$, and, hence, $f(-C_t)0$.

P1: Let be fC_tg, i.e. $f(t) \cap g(t) \neq \emptyset$. In particular, $f(t) \neq \emptyset$, and, hence, fC_tf.

C2: trivial.

C3: Assume fC_tg and $g \leq h$. Then we have $f(t)\cap g(t) \neq \emptyset$ and $g(t) \subseteq h(t)$. This implies $f(t) \cap h(t) \neq \emptyset$, and, hence, fC_th.

C4: Assume $fC_t(g + h)$. Then we have

$$\begin{aligned} \emptyset &\neq f(t) \cap (g + h)(t) \\ &= f(t) \cap (g(t) \cup h(t)) \\ &= (f(t) \cap g(t)) \cup (f(t) \cap h(t)) \end{aligned}$$

so that either $f(t) \cap g(t) \neq \emptyset$ or $f(t) \cap h(t) \neq \emptyset$. □

4. Representability of Timed Contact Structures

By fixing a $t \in T$ we obtain a BCA $\langle B_t, C_t\rangle$ on equivalence classes of B. As already mentioned in Section 3 different B_t's need not to be isomorphic. We will use the free product of Boolean algebras in order to embed all B_t's in a common structure.

Let $(B_i)_{i\in I}$ be a family of Boolean algebras. The dual space of the free product B is given by the product of the dual space of the B_i's [10], i.e. it is the set $\prod_{i\in I}$ Ult(B_i) with the product topology.

Theorem 3 *Let $\langle B_i, C_i\rangle_{i\in I}$ be a family of BCAs, B be the free product of the B_i's and $e_i : B_i \to B$ the canonical embedding of B_i in B. Then there is a contact relation C on B so that aC_ib if and only if $e_i(a)Ce_i(b)$ for all $i \in I$.*

Proof. In this proof we will identify the two sets $(\prod_{i\in I}$ Ult$(B_i)) \times (\prod_{i\in I}$ Ult$(B_i))$ and $\prod_{i\in I}($Ult$(B_i) \times$ Ult$(B_i))$. Therefore, we can define $R_C := \prod_{i\in I} R_i$ where R_i is the dual of the contact relation on B_i. R_C is obviously reflexive and symmetric. Furthermore, R_C is closed since it is a product of closed sets. This shows that R_C is the dual of a contact relation C on the free product B.

Recall that the embedding e_i is defined as the dual of the projection p_i from the product $\prod_{i\in I}$ Ult(B_i) to Ult(B_i). It satisfies the property:

$$(*) \quad h(e_i(a)) = p_i^{-1}[h_i(a)],$$

where $h_i(a) = \{F \in \text{Ult}(B_i) \mid a \in F\}$ and $h(b) = \{F \in \prod \text{Ult}(B_i) \mid b \in F\}$ are the Stone embeddings and $p_i^{-1}[h(a)]$ denotes the inverse image of p_i applied to the set $h(a)$, i.e.

$$p_i^{-1}[h(a)] = \{F \in \prod_{i\in I} \text{Ult}(B_i) \mid p_i(F) \in h(a)\}.$$

Assume $e_i(a)Ce_i(b)$. By the definition of the dual R_C of C there are ultrafilters $e_i(a) \in F$, $e_i(b) \in G$ with FR_CG. ¿From $e_i(a) \in F$ we conclude $F \in h(e_i(a))$, and, hence, $F \in p_i^{-1}[h(a)]$ by $(*)$. This implies that $p_i(F) \in h(a)$, and hence $a \in p_i(F)$. Analogously, we get $b \in p_i(G)$. From the definition of R_C and FR_CG we conclude $p_i(F)R_ip_i(G)$. Together we obtain aC_ib.

Conversely, assume aC_ib. Then there are ultrafilters $a \in F$, $b \in G$ with FR_iG. For each $j \in I$ fix one ultrafilter H_j and define

$$\bar{F} = (\prod_{i\neq j\in I} H_j) \times F, \quad \bar{G} = (\prod_{i\neq j\in I} H_j) \times G.$$

Then $p_i(\bar{F}) = F$, $p_i(\bar{G}) = G$ and $p_j(\bar{F}) = p_j(\bar{G})$ for all $i \neq j \in I$. We conclude $\bar{F}R_C\bar{G}$, $\bar{F} \in p_i^{-1}[h(a)]$ and $\bar{G} \in p_i^{-1}[h(b)]$. From $\bar{F} \in p_i^{-1}[h(a)]$ we conclude $\bar{F} \in h(e_i(a))$, and, hence, $e_i(a) \in \bar{F}$. $e_i(b) \in \bar{G}$ follows analogously. This shows $e_i(a)Ce_i(b)$. □

In the following we will apply the previous theorem to the quotients $\langle B_t, C_t\rangle$ of a timed contact structure $\langle T, B, C\rangle$.

Theorem 4 *Let $\langle T, B, (C_t)_{t\in T}\rangle$ be a timed contact structure. Then there is a compact and semi-regular T_0 space $\langle X, \tau\rangle$ and an embedding $k : B \to X_T$ with aC_tb if and only if $k(a)(t) \cap k(b)(t) \neq \emptyset$.*

Proof. By Lemma 1 the quotients $\langle B_t, C_t \rangle$ are BCA's. From Theorem 3 we obtain a contact relation on the free product B of the B_t's. B can be represented by a compact and semi-regular T_0 space $\langle X, \tau \rangle$ and a (Boolean) embedding $h : B \to RC(X)$ using Theorem 1. Now, define $k : B \to X_T$ by

$$k(a)(t) := h(e_t(a_t)).$$

k is a (Boolean) embedding since e_t and h are. Therefore, it remains to show that

$$aC_t b \iff k(a)(t) \cap k(b)(t) \neq \emptyset.$$

This follows from

$k(a)(t) \cap k(b)(t) \neq \emptyset$	
$\Leftrightarrow h(e_t(a_t)) \cap h(e_t(b_t)) \neq \emptyset$	
$\Leftrightarrow e_t(a_t) C e_t(b_t)$	property of h
$\Leftrightarrow a_t C_t b_t$	Theorem 3
$\Leftrightarrow aC_t b$	Lemma 1.

This completes the proof. □

5. Conclusion and Outlook

In this paper we have introduced timed contact structures as an extension of Boolean contact algebras. We studied their canonical models using topological spaces, and we proved a representation theorem.

In concrete timed contact structures regions are arbitrary functions from the time domain into the regions of the static world. Therefore, such a region need not move 'smoothly' through the world. In future work we will investigate additional axioms related to certain continuity properties with respect to movement.

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