LOGICS OF COMPLEMENTARITY IN INFORMATION SYSTEMS[1]

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Abstract: Each information system (or data table) leads to a hierarchy of binary relations on the object set in a natural way; these relational systems can serve as frames for the semantics of modal logics. While relations of indiscernibility and their logics have been frequently studied, the situation in the case of relations which distinguish objects is much less clear. In this paper, we present complete logical systems for relations of complementarity derived from information systems.

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1 Introduction

Widely used methods of knowledge representation are

$$\text{Object} \rightarrow \text{Attribute}$$

systems, in which each object x is described by an attribute (feature) vector

$$a(x) = \langle a_1(x), \ldots, a_n(x) \rangle;$$

each $a_i(x)$ is a set of values that x may take under the attribute a_i. These lead to various relations on the object set in a natural way; for example, two objects x, y are *indiscernible* under the attributes in A, if

$$a(x) = a(y) \text{ for all } a \in A.$$

Similarly, we can say that x, y are *A – distinguishable*, if

$$a(x) \neq a(y) \text{ for some } a \in A.$$

Relations arising from these considerations are called *information relations*; an introduction to information relations can be found in Orlowska (1997b, 1998). If R is an information relation on an object set OB, then we can regard <OB,R> as a frame,

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which can serve as a semantic structure for a modal logic. If, for example, R is an indiscernibility relation as defined above, then R is easily seen to be an equivalence, and the corresponding modal logics are just the S5 logics. Modal logics for various types of information relations have been widely studied, and the interested reader is invited to consult Orlowska (1997b).

In this paper, we describe the logic of frames arising from information relations of complementarity; these were introduced by Demri and Orlowska (1998). The paper is organized as follows: Section 2 gives a brief introduction to information systems, and their associated families of parametrized relations. Section 3 contains a short overview of standard modal logics with a necessity operator, and their frame semantics. Modal logics with sufficiency operators are introduced in Section 4. Section 5 introduces complementarity frames and a modal logic L1 for such frames. Sections 6 and 7 present a multi-modal logic for complementarity and incomplementarity, and a proof system, based on the technique of relational methods (Orlowska 1988), which is both sound and complete for the intended models. Section 8 is concerned with logics of relative complementarity and incomplementarity.

2 Information systems and relations of complementarity

Information systems are collections of information items, which describe objects in terms of their properties. More formally, by an *information system* we mean a structure

$$S=(OB, AT, \{VAL_a: a\in AT\})$$

such that OB is a nonempty finite set of objects, and AT is a finite nonempty set of functions $a: OB\rightarrow Sb(VAL_a)$, where each VAL_a is a nonempty set of values of attribute a, and $Sb(VAL_a)$ its powerset. If each a(x) is a singleton set, then system S is said to be *deterministic,* otherwise S is called *nondeterministic*. We shall usually identify singleton sets with the element that they contain. In particular, if a(x) is a singleton set, say {v}, we omit the parentheses and write a(x)=v.

Any set a(x) can be viewed as a set of properties of an object x corresponding to attribute a. For example, if attribute a is 'color' and a(x)={green}, then x possesses the property of 'being green'; if a is 'languages spoken' and if a person x speaks, Polish (Pl), German (D), and French (F), then a(x)={Pl, D, F}.

In this setting, any set a(x) is referred to as the set of a–properties of object x and its complement VAL_a–a(x) is said to be the set of negative a–properties of x.

Note that in the example above we interpret the nondeterminism given by a set of a-values as a multi-valued situation, i.e. by a conjunctive interpretation. It is also possible to interpret a(x) as an indeterministic set in the sense that x speaks Polish or German or French. These ontological differences, though of course decisive for applications, do not play a role in our logical considerations, which can be applied to both.

Since in an information system both the set of objects and the set of attributes are assumed to be finite, we regard such a system as a data table with rows labeled by objects, and columns labeled by attributes; the cell entry <x,a> contains the value set a(x) of attribute a for object x. As an example, consider a file containing information about the academic degrees of persons P1,...,P6, and the languages that these persons speak:

	Lan	Deg
P1	F, D	BS, MS, Ph.D.
P2	H, R	BS
P3	F, D, S	BS, MS
P4	F	BS, MS
P5	F, D	BS
P6	R	BS

The set of objects is OB={P1,...,P6}, and the given properties of these objects are of the form *'speaking a language'* and *'having a degree'*. Thus, we have the attribute set AT={Language (Lan), Degree (Deg)}, and the sets VAL_{Deg}={BS, MS, Ph.D.}, VAL_{Lan}={D, F, H, R, S} of values of these attributes. According to the information given in our file, P2 possesses the property Lan(P2) = {H, R} of speaking Hungarian and Romanian, while P3 does not possess those properties. Indeed, Lan(P2) is also the set of negative Lan–properties of P3.

Nondeterministic information systems where descriptions of objects are tuples consisting of subsets of values of attributes were introduced by Lipski (1976, 1979) under the name *systems with incomplete information*. They are also used in symbolic data analysis, e.g. Diday (1987), Diday and Roy (1988), Prediger (1997) and in rough set–based data analysis, e.g. Orlowska (Ed) (1997b), Wang , Düntsch, Bell (1998a), Wang, Düntsch, Gediga (1998b).

Apart from the explicit information given in an information system, any such system contains implicit information. This information has the form of relationships among the objects of set OB which are determined by the properties of the objects. Typically, the relationships have the form of binary relations, and they are referred to *as information relations derived from an information system*. There are two major groups of information relations:

- Relations that reflect various forms of indistinguishability of objects in terms of their properties.
- Relations that indicate distinguishability of the objects.

The primary relation of indistinguishability is one of indiscernibility, in which objects cannot be distinguished by the given attributes:

$$(x,y) \in ind(a) \text{ iff } a(x) = a(y) \text{ for } a \in OB.$$

Whereas relations of indistinguishability have been frequently studied and are well understood, the situation of relations of indistinguishability is much less clear. Complementarity and incomplementarity relations are typical examples of indistinguishability relations; they are defined as follows: Let an information system S=(OB, AT, $\{VAL_a: a \in AT\}$) be given, and suppose that $A \subseteq AT$. We define

Strong (weak) complementarity:

$$(x,y) \in com(A)\ (wcom(A)) \text{ iff } a(x) = -a(y) \text{ for all (some) } a \in A$$

Strong (weak) incomplementarity:

$$(x,y) \in icom(A)\ (wicom(A)) \text{ iff } a(x) \neq -a(y) \text{ for all (some) } a \in A.$$

We clearly have

$$com(A) = -wicom(A) \text{ and } wcom(A) = -icom(A).$$

Many other families of information relations can be found in Orlowska (1998).

In our exemplary file we have, among others, (P2, P5) $\in$ icom(Lan) which means that, up to our present knowledge, P2 and P5 are not 'completely' distinct with respect to the attribute Lan, because the set of Lan–properties of P2 is not equal to the set of negative Lan–properties of P5. We also have Lan(P3)=–Lan(P2), and hence, (P3, P2)$\in$com(Lan).

A *decision table* is an information system (OB, AT, d, $\{VAL_a: a \in AT\}$) with a special attribute d adjoined, referred to as *decision attribute*. Sometimes, a finite family of decision attributes is also considered. Usually, it is assumed that d(x) is a singleton set for every object x. The values d(x) are referred to as *decisions*. The attributes from AT are referred to as *condition* or *independent* attributes. As an example, we extend our previous example by a column Emp, which we interpret as *has found employment*.

	Lan	Deg	Emp
P1	F, D	BS, MS, Ph.D.	Yes
P2	H, R	BS	No
P3	F, D, S	BS, MS	Yes
P4	F	BS, MS	No
P5	F, D	BS	Yes
P6	R	BS	No

Every row of a decision table determines a decision rule in the following way: Suppose that AT={a_1, a_2,...,a_n}. By *the rule determined by an object x∈OB* we mean the following statement:

r_x If the value of a_1 for x is $a_1(x)$, and ..., and the value of a_n for x is $a_n(x)$ then decide d(x).

We abbreviate such a rule as

$$(a_1,a_1(x)) \&\ldots\& (a_n, a_n(x))\to d(x).$$

Hence, the rules are statements determining a decision that depends on the properties of objects. At our level of analysis, we do not assume a specific interpretation of the nature of nondeterminism, and thus, rules could be interpreted either way.

The fact that the relations derived from information systems are indexed with subsets of the attribute set suggests that, in a general setting, a hierarchy of relative relations may be constructed determined by the powerset hierarchy of a set PAR of parameters. Such a hierarchy was presented in Orlowska (1988a), and it is further investigated in Balbiani (1997), Demri (1998), Demri and Gore (1998), Demri and Konikowska (1998), Balbiani and Orlowska (1999), Konikowska (1997). Its basic idea is as follows: Let L_1(PAR)=PAR and for n≥2 let L_n(PAR) be the family of finite subsets of L_{n-1}(PAR). Then we can consider families of relations indexed with the elements of any level L_n(PAR). Relations indexed with the elements of L_1(PAR) form a family of relations indexed by the elements of PAR. Information relations derived from an information system as defined above are examples of relations indexed with the elements of L_2(PAR), where PAR is the set of attributes of the information system. Relative relations of any level greater than 2 appear naturally in representation of hierarchical information, see Green et al. (1996).

In this paper we present logics of complementarity and incomplementarity relations of level 1 and 2.

3 Standard modal logics

To make the paper more self-contained, we present in this section a brief introduction to standard modal logics and the technique of copying. The alphabet of the language $\mathrm{LAN}_{<>}$ of any standard propositional modal logic consists of an infinite, denumerable set VP of propositional variables and propositional connectives of disjunction ($\vee$), conjunction ($\wedge$), implication ($\rightarrow$), negation ($\neg$), and possibility ($<>$). The *set of formulas of* $\mathit{LAN}_{<>}$ is the smallest set that includes VP and is closed with respect to the connectives. As usual, we define the connectives of *equivalence* ($\leftrightarrow$) and *necessity* ($[]$) as follows:

$$F\leftrightarrow G := (F\rightarrow G)\wedge(G\rightarrow F)$$

$$[]F:= \neg<>\neg F.$$

A $\mathit{LAN}_{<>}$ *–frame* is a structure K=(W, R) such that W is a nonempty set (of states) and R is a binary relation on W; we write R(w) to denote the set $\{w'\in W: (w,w')\in R\}$. A $\mathit{LAN}_{<>}$ *–model based on K* is a triple M=(W, R, m) such that m: VP$\rightarrow$Sb(W) is a meaning function which assigns sets of states to propositional variables. Intuitively, m(p) is the set of states at which p is true. We extend the meaning function to all the formulas of $\mathrm{LAN}_{<>}$; for the sake of simplicity we denote this extended mapping by m as well:

$$m(\neg F)=W-m(F)$$

$$m(F\vee G)=m(F)\cup m(G)$$

$$m(F\wedge G)=m(F)\cap m(G)$$

$$m(F\rightarrow G)=(W-m(F))\cup m(G)$$

$$m(<>F)=\{w\in W: R(w)\cap m(F)\neq\emptyset\}$$

The definition of [] now implies that

$$m([]F)=(w\in W: R(w)\subseteq m(F)\}.$$

Satisfiability of formulas in a model by a state is defined inductively as follows:

M, w sat p iff $w\in m(p)$ for $p\in VP$

M, w sat $F \vee G$ iff M, w sat F or M, w sat G

M, w sat $F \wedge G$ iff M, w sat F and M, w sat G

M, w sat $F \rightarrow G$ iff M, w sat F implies M, w sat G

M, w sat $\neg F$ iff not M, w sat F

M, w sat $<>F$ iff there is w' such that $(w,w') \in R$ and M, w' sat F.

We clearly have

$$w \in m(F) \text{ iff M, w sat F.}$$

A formula F is said to *be true in model M* if m(F) =W. A formula F is *true in frame K* if it is true in every model based on K.

Let C be a class of $LAN_{<>}$ –frames. A formula F is *C–valid* if it is true in every frame $K \in C$. The *logic L(C)* of the class C of frames is the set of formulas that are true in every member of C.

A *standard normal modal logic L* is a set of formulas of $LAN_{<>}$ that includes the following formulas:

(1n) All tautologies of the classical propositional calculus,

(2n) $[]F \wedge [](F \rightarrow G) \rightarrow []G$,

and which is closed with respect to:

(3n) *modus ponens* rule (if F and $F \rightarrow G \in L$ then $G \in L$),
necessitation rule (if $F \in L$ then $[]F \in L$),
substitution (if $F(p) \in L$ where p is a propositional variable occurring in F, then $F(G) \in L$, where F(G) is obtained from F(p) by substituting a formula G for every occurrence of p).

The *minimal normal modal logic LK* is the smallest set of formulas of $LAN_{<>}$ that satisfies the conditions (1n), (2n), (3n).

A logic L is said to be *sound* with respect to a class C of $LAN_{<>}$ –frames if for every formula F of $LAN_{<>}$, $F \in L$ implies that F is C–valid. The logic L is *complete* with respect to class C if for every formula F of $LAN_{<>}$, if F is C–valid then $F \in L$. It is well known that LK is sound and complete with respect to the set of all $LAN_{<>}$ –frames.

Let (prop) be a property of binary relations and let C(prop) be the class of $LAN_{<>}$ –frames (W, R) such that relation R satisfies the property (prop). The class C(prop) is said to be *definable* in $LAN_{<>}$ iff there is a formula F of $LAN_{<>}$ such that for every frame $K \in C$,

$$K\in C(prop) \text{ iff } F \text{ is true in } K.$$

Recall that the composition R;S of two relations R,S on W is defined as

$$R;S = \{(x,z)\text{: There is some } y \in W \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\}.$$

For later reference, we mention the well known definability of frames (W,R), where R is symmetric or 3 – transitive, i.e. where $R;R;R\subseteq R$:

3.1 Proposition

(a) $K\in C(sym)$ iff $F\to[]<>F$ is true in K

(b) $K\in C(3\text{–}tran)$ iff $<><><>F\to<>F$ is true in K. (QED)

In what follows we recall the notion of copying introduced in Vakarelov (1998) and the related theorems that enable us to prove completeness of the logic L1 of complementarity in Section 5.

Let K=(W, R) and K'=(W', R') be two $\mathrm{LAN}_{<>}$ –frames and let I be a nonempty set of functions f: $W\to W'$. I is *a copying from K to K'* if

(1c) $W'=\bigcup\{f(W)\text{: } f\in I\}$.

(2c) If f(x)=g(y) then x=y.

(3c) If $(x,y)\in R$ then for all $f\in I$ there is $g\in I$ such that $(f(x), g(y))\in R'$.

(4c) If $(f(x), y')\in R'$ then there is $g\in I$ and there is $y\in W$ such that y'=g(y) and $(x,y)\in R$.

Let M and M' be models based on K and K', respectively. I is a *copying from M to M'* if for all $p\in VP$, for all $w\in W$ and for all $f\in I$,

$$M, w \text{ sat } p \text{ iff } M', f(w) \text{ sat } p.$$

3.2 Proposition (Vakarelov, 1998)

If I is a copying from M=(W, R, m) to M'=(W', R', m') then for every formula F of $\mathrm{LAN}_{<>}$, for every $x\in W$ and for every $f\in I$, M,

$$w \text{ sat } F \text{ iff } M', f(x) \text{ sat } F.$$

Proof: The proof is by induction with respect to the complexity of a formula. (QED)

We invite the reader to consult Chellas (1980) as a standard reference for modal logic.

4 Modal logics with sufficiency operators

Gargov et al. (1987) noted that

"Kripke's mathematical interpretation of 'p is necessary (true) in x' ... only sharpens but does not satisfy one's desire to formally handle the 'sufficiency' phenomena as well. The first and trivial attempt is to grammatically reduce the 'sufficiency' to 'necessity' saying that 'x is sufficient for p' iff 'p is necessary for x', and this surely will not enrich our knowledge."

To remedy this deficiency they introduce sufficiency operators. The language $LAN_{<<>>}$ of modal logics with sufficiency operators differs from $LAN_{<>}$ in that instead of operations <> and [], we use *sufficiency operators* <<>> and [[]]. Semantic notions for $LAN_{<<>>}$ are similar to those defined in Section 3 for $LAN_{<>}$. A *$LAN_{<<>>}$ –frame* is a structure of the form K = (W, R), and a *$LAN_{<<>>}$ –model* based on K has the form (W, R, m), where the meaning function m extends to the formulas with sufficiency operators as follows: Let −R(w) be an abbreviation for $(W^2-R)(w)$; then,

$$m(<<>>F)= \{w\in W: -R(w)\cap(W-m(F))\neq\emptyset\}$$
$$m([[]]F)=\{w\in W: -R(w)\subseteq m(F)\}.$$

Satisfiability of formulas with sufficiency operators is defined as follows:

M, w sat <<>>F iff there is w'∈W such that (w,w')∉R and not M, w' sat F

M, w sat [[]]R iff for all w'∈W, if M, w' sat F then (w,w')∈R.

Any modal logic L with sufficiency operators is a set of formulas of $LAN_{<<>>}$ that includes the following formulas:

(1s) All the tautologies of the classical propositional calculus

(2s) $[[]]F\wedge[[]](\neg F\wedge G)\rightarrow [[]]G$,

and which is closed with respect to

(3s) the modus ponens rule, sufficiency rule (if F∈L then [[]]¬F∈L) and substitution.

The *minimal modal logic LK** is the smallest set of formulas of $LAN_{<<>>}$ that satisfies the conditions (1s), (2s), and (3s).

Definability of classes of frames in $LAN_{<<>>}$ is defined in the way analogous to definability in $LAN_{<>}$: Let (prop) be a property of binary relations and let C(prop) be the class of $LAN_{<<>>}$ –frames (W, R) such that the relation R satisfies (prop). Class C(prop) is said to be *definable* in $LAN_{<<>>}$ iff there is a formula F of $LAN_{<<>>}$ such that for every frame K∈C, K∈C(prop) iff F is true in K. By way of example we show definability of irreflexive frames:

4.1 Proposition

$K \in C(irref)$ iff $F \to <<>> \neg F$ is true in K.

Proof: ($\to$) Let K=(W, R) be a frame such that R is an irreflexive relation, and suppose that the formula F is not true in some model M=(W, R, m) based on K. Hence, there is some $x \in W$ such that (i) M, x sat F and (ii) not M, x sat $<<>> \neg F$.

It follows from (ii) that for all $y \in W$, either $(x,y) \in R$ or M, y sat $\neg F$. In particular, if y = x we get $(x,x) \in R$ which contradicts the assumption, or M, x sat $\neg F$ which contradicts (i).

($\leftarrow$) Let the formula F be true in K, and suppose that R is not irreflexive, that is $(a,a) \in R$ for some $a \in W$. Consider the model M=(W, R, m) based on K such that for some propositional variable p, $m(p)=\{y \in W: (a,y) \in R\}$. We have M, a sat p, so we also have M, a sat $<<>> \neg p$. But this means that there is some $x \in W$ such that $(a,x) \notin R$ and $(a,x) \in R$, a contradiction. (QED)

The following relationships between the logics LK and LK* are presented in Gargov et al. (1987). Let t be a translation mapping from the set of formulas of $LAN_{<>}$ onto the set of formulas of $LAN_{<<>>}$ defined as follows:

$t(p)=p$ for $p \in VP$,

$t(\neg F)=\neg t(F)$,

$t(F \bullet G)=t(F) \bullet t(G)$ for $\bullet \in \{\vee, \wedge, \to, \leftrightarrow\}$,

$t(<>F)=<<>> \neg t(F)$,

$t([]F)=[[]] \neg t(F)$.

4.2 Proposition (Gargov et al, 1987)

For every formula F of $LAN_{<>}$, $F \in LK$ iff $t(F) \in LK^*$. (QED)

Let M=(W, R, m) be a $LAN_{<>}$ –model, and consider the $LAN_{<<>>}$ –model $M'=(W, W^2-R, m)$. Then the following holds:

4.3 Proposition

M, w sat F iff M', w sat t(F), i.e. a formula F of $LAN_{<>}$ is true in M iff t(F) is true in M'. (QED)

Propositions 4.2 and 4.3 and the properties of the logic LK immediately lead to the following result:

4.4 Proposition

The logic LK* is sound and complete with respect to the class of all $LAN_{<<>>}$ –frames. (QED)

The relationship between normal modal logics and logics with sufficiency operators is the following: Let C be a class of $LAN_{<>}$ –frames, and consider the class C* of $LAN_{<<>>}$ –frames defined by $C^*=\{(W, W^2-R): (W, R)\in C\}$.

4.5 Proposition

Let (prop) be a property of binary relations. A class C(prop) of $LAN_{<>}$ –frames is definable in $LAN_{<>}$ with a formula F iff C* (prop) is definable in $LAN_{<<>>}$ with the formula t(F). (QED)

For example, it is known that reflexivity of relations is definable in $LAN_{<>}$ with the formula $F\rightarrow <>F$. In view of the above proposition, reflexivity of the complement of a relation (and hence irreflexivity of the relation) is definable in $LAN_{<<>>}$ with the formula $F\rightarrow <<>>\neg F$.

Similarly, in view of Proposition 3.1, 3–transitivity of the complement of a relation is definable in $LAN_{<<>>}$ with the formula $<<>>[[]]<<>>F\rightarrow <<>>F$.

5 A Logic for complementarity

Let R be a binary relation on a set W. By a *standard complementarity frame* we mean any frame (W, R) such that R is an irreflexive, symmetric and 3–transitive relation on W. Let COM_s be the family of all standard complementarity frames. *By a general complementarity frame* we mean a frame whose relation is symmetric and 3–transitive. Let COM_g be the family of all general complementarity frames.

Let L1 be the smallest set of $LAN_{<>}$ formulas which includes

(1com) All tautologies of the classical propositional calculus,

(2com) $[]F\wedge[](F\rightarrow G)\rightarrow []G$,

(3com) $F\rightarrow[]<>F$,

(4com) $[]F\rightarrow[][][]F$,

and which is closed with respect to modus ponens, necessitation and substitution.

Dimiter Vakarelov (1998) has communicated the following result to us:

5.1 Proposition

For every $K \in COM_g$ there are some $K' \in COM_s$ and a copying I from K to K'.

Proof: Let K=(W, R); we will construct K'=(W', R'). Let $W'=W\times\{1,-1\}$. For $a \in \{1, -1\}$ let f_a: $W \rightarrow W'$ be defined by $f_a(x) = (x,a)$; observe that each element of W' has the form $f_a(x)$ for some $x \in W$, $a \in \{1, -1\}$.
For a, $b\in\{1,-1\}$; we define the relation R' on W' by

$$((x,a), (y,b)) \in R' \text{ iff } (x,y) \in R \text{ and } a = -b.$$

It is easy to see that R' is irreflexive, symmetric and 3–transitive, and hence K' = $(W',R') \in COM_s$. Furthermore, it is easy to verify that $I=\{f_1, f_{-1}\}$ is a copying from K to K'. (QED)

5.2 Proposition

The logic L1 is sound with respect to the classes COM_s and COM_g. (QED)

5.3 Proposition

(a) The logic L1 is complete with respect to COM_g

(b) The logic L1 is complete with respect to COM_s.

Proof: Since all the properties of a relation in general frames are definable in $LAN_{<>}$, the proof of (a) can be easily obtained using standard techniques of modal logic.

The proof of (b) is due to D. Vakarelov (1998) and uses the copying method. Assume that a formula F is true in all frames of COM_s, and suppose that $F\notin L1$. By (a) there is a general frame K=(W, R) such that F is not true in K. Hence, there is a model M=(W, R, m) based on K such that F is not true in M, that is, for some $x\in W$, we have not M, x sat F. By Proposition 5.1, there are a standard frame K' = (W', R'), and a copying I from K to K'. We can now define the model M' based on K' such that $m'(p)=\{f(x)\in W': x\in m(p), f\in I\}$. It follows that I is a copying from M to M'. By Proposition 3.2 we have not M', f(x) sat F, which contradicts the assumption. (QED)

Proposition 5.3 shows that the logic L1 is too weak to distinguish between standard and general complementarity frames.

6 A logic of complementarity and incomplementarity

Let $LAN_{<>,<<>>}$ be the join of the languages $LAN_{<>}$ and $LAN_{<<>>}$. Let CI (Complementarity + Incomplementarity) be the class of frames (W, R, S) such that

(i) W is a nonempty set,

(ii) R is a symmetric and 3–transitive relation on W,

(iii) S is a reflexive relation on W, and

(iv) $R \cup S = W^2$, $R \cap S = \emptyset$.

Observe that these frames have the status of standard frames. Although neither irreflexivity of R nor symmetry of S are assumed explicitly, irreflexivity of R is guaranteed by reflexivity of S, and symmetry of S is guaranteed by symmetry of R, since R = -S. The standard notion of modal definability of frames assumes implicitly that the formula that defines a property of a relation contains a modal operator that is determined by that relation. The considerations of this paper suggest that in fact this notion should be broader. In the context of multi–modal logics, one might admit a relative definability, that is, a definability of a property of a relation, say A, via a property of some other relation(s) together with a relationship between A and those relations.

Let L2=L(CI) be the logic of the class CI of frames. In the sequel, we present a relational proof system for this logic, and we show its completeness with respect to the class of relational models determined by CI. Relational formalization of nonclassical logics and relational proof systems were suggested in Orlowska (1988, 1996, 1997a).

Suppose that VR is an infinite, denumerable set of relational variables, and R, S are relational constants (representing the accessibility relations of CI frames). Relational terms are generated by $VR \cup \{R, S\}$ with the relational operations of union, intersection, complement, and relative product (;). A relational logic ReL2 for L2 is the logic whose formulas are of the form xAy, where x, y are individuum variables taken from an infinite set VI, and A is a relational term.

Models for the relational logic ReL2 are structures of the form M=(W, m), where W is a nonempty set, and $m: VR \cup \{R, S\} \rightarrow Sb(W \times W)$ is a meaning function that assigns binary relations on W to relational variables and constants, and satisfies the following conditions:

(i) $m(A);(W\times W)=m(A)$ for $A\in VR$ (that is the relation variables are mapped into right ideal relations),

(ii) m(R) is a symmetric and 3−transitive relation,

(iii) m(S) is a reflexive relation,

The function m extends to all the relational terms in a homomorphic way, that is

$$m(-A)=-m(A),\ m(A\cup B)=m(A)\cup m(B),\quad m(A\cap B)=m(A)\cap m(B),\quad m(A;B)=m(A);m(B),$$

$$\text{and, moreover, } m(R)\cup m(S)=W^2 \text{ and } m(R)\cap m(S)=\emptyset.$$

By a *valuation* in M we understand a function $v:VI\to W$ assigning elements of W to the individuum variables. A relational formula xAy is *satisfied by v in M*, written as M,v sat xAy, whenever $(v(x),v(y))\in m(A)$, i.e.

$$M,v \text{ sat } xAy \text{ iff } (v(x),v(y))\in m(A).$$

A formula xAy is *true in M* iff M,v sat xAy for all valuations v in M, and xAy is *valid in ReL2* iff it is true in all models for ReL2. In other words, the formula xAy is true in a model M whenever $m(A)=W^2$.

Next, we define a relational translation RT of formulas of L2 into formulas of ReL2. Let $t': VP\to VR$ be a bijection from propositional variables to relational variables. Then we define

$RT(p)=t'(p)$

$RT(\neg F)=-RT(F)$

$RT(F\vee G)=RT(F)\cup RT(G)$

$RT(F\wedge G)=RT(F)\cap RT(G)$

$RT(F\to G)=-RT(F)\cup RT(G)$

$RT(F\leftrightarrow G)=RT(F\to G)\cap RT(G\to F)$

$RT(<>F)=R;RT(F)$

$RT([]F)=-(R;-RT(F))$

$RT(<<>>F)=-S;-RT(F)$

$RT([[]]F)=-(-S;RT(F))$.

The semantic relationship between the logic L2 and the relational logic ReL2 is established in the following results.

6.1 Proposition

For every model M=(W, R, S, m) of L2 there is a model $M^{`}$=(W, $m^{'}$) of the relational logic ReL2 such that for any formula F of L2 and for any $w \in W$ we have:

(i) M,w sat F iff $(w,z) \in m^{'}(t(F))$ for all $z \in W$.

Proof: We define the desired model $M^{'}$ as follows: Its universe coincides with the universe W of M. If $P \in VR$ and $P=t^{'}(p)$ for a propositional variable p, then we set $m^{'}(P)=m(p)\times W$. For the constants R and S we set $m^{'}(R)=R$, $m'(S)=S$, that is, the meaning of the constants in the relational model are the relations from the model M, denoted by the respective constants; we use the same symbols for both of them. The proof of the required condition is by induction with respect to the complexity of F. We only show the induction step for a formula of the form $<<>>F$, since the rest is straightforward. We have M, x sat $<<>>F$ iff there is $y \in W$ such that $(x,y) \notin S$ and not M, y sat F. By the induction hypothesis $(x,z) \notin m^{'}(t(F))$ for all $z \in W$ which yields $(x,z) \in -m^{'}(S);-m^{'}(t(F))=m^{'}(-S;-t(F))=m^{'}(t(<<>>F))$. (QED)

6.2 Proposition

For every model $M^{'}$=(W, $m^{'}$) of the relational logic ReL2, there is a model M of L2 such that condition (i) of Proposition 6.1 is satisfied.

Proof: We define the model M as follows: Its universe coincides with the universe W of $M^{'}$. Accessibility relations in M are the relations $m^{'}(R)$ and $m'(S)$. For any propositional variable p we put m(p) = domain of $m^{'}(P)$ where $P = t^{'}(p)$. By induction on the complexity of a formula F one can easily show that condition (i) is satisfied. (QED)

6.3 Proposition

A formula F of the logic L2 is CI–valid iff the relational formula xRT(F)y is valid in ReL2.

Proof: ($\rightarrow$) Let F be CI–valid and suppose that there is a model M=(W, m) of ReL2 and a valuation v in M such that not M, v sat xRT(F)y. It follows that there are a, $b \in W$ such that $(a,b) \notin m'(RT(F))$. By Proposition 6.2 there is a model M' of L2 with the universe W such that for any $w \in W$, M', w sat F iff $(w,z) \in m'(RT(F))$ for all $z \in W$. Consequently, not M', a sat F, a contradiction.

($\leftarrow$) Assume that for every model (W, m) of ReL2 we have $m'(RT(F))=W^{2}$, and suppose that there are a frame K=(W, R, S) $\in CI$, a model M based on K, and some $w \in W$ such that not M, w sat F. By Proposition 6.1 there is a model M'=(W, m') of ReL2 such that

M, w sat F iff $(w,z) \in m'(RT(F))$ for all $z \in W$. It follows that $(w,z) \notin m'(RT(F))$ for some z, a contradiction. (QED)

7 A relational proof system for the logic L2

The proof system for the relational logic ReL2 is a Rasiowa–Sikorski style system (Rasiowa and Sikorski 1963); it consists of the rules that apply to finite sequences of relational formulas. There are two groups of rules, namely, *decomposition rules* and *specific rules*. Decomposition rules enable us to decompose formulas into a sequence of simpler formulas; we shall see that decomposition depends on relational operations occurring in a formula. As a result of decomposition, we obtain finitely many new sequences of formulas.

The specific rules enable us to modify a sequence to which they are applied; they have a status of structural rules. The role of axioms is played by what is called *fundamental sequences*. In what follows, K and H denote finite, possibly empty, sequences of formulas of the relational logic. A variable is said to be *restricted in a rule* whenever it does not appear in any formula of the upper sequence in that rule.

(DEC) Decomposition rules:

(∪) K, xA∪By, H
K, xAy, xBy, H

(−∪) K, x−(A∪B)y, H
K, x−Ay, H   K, x−By, H

(∩) K, xA∩By, H
K, xAy, H   K, xBy, H

(−∩) K, x−(A∩B)y, H
K, x−Ay, x−By, H

(−−) K, x−−Ay, H
K, xAy, H

(;) K, xA;By, H
K, xAz, H, xA;By   K, zBy, H, xA;By,   where z is a variable

(−;) K, x−(A;B)y, H
K, x−Az, z−By, H,   where z is a restricted variable

(SPE) Specific rules:

(ideal) K, xAy, H

K, xAz, H, xAy, where z is a variable, A∈VR

(symR) H, xRy, H

H, yRx, H

(3−tranR) K, xRy, H

K, xRz, H, xRy K, zRt, H, xRy K, tRy, H, xRy

where z, t are variables

(–R) K, x–Ry, H

K, xSy, H, x–Ry

(–S) K, x–Sy, H

K, xRy, H, x–Sy

(cut) K

K, xAy K, x–Ay for A ∈ {R, S}

(FND) Fundamental sequences:

A sequence of formulas is said to be *fundamental* whenever it contains a subsequence of one of the following forms:

(f1) xAy, x–Ay, where A is a relational term

(f2) xSx

(f3) xRy, xSy

A sequence *K* of relational formulas is *valid in ReL2* iff for every model (W,v) of the relational logic ReL2, there is a formula in *K* which is satisfied in (W,v); it follows that sequences of formulas are interpreted as (metalevel) disjunctions of their elements. A relational rule of the form $K/\{H_t: t\in T\}$ is *admissible* in ReL2 whenever

The sequence *K* is valid in ReL2 iff for all t∈T the sequence H_t is valid in ReL2.

7.1 Proposition

(a) The rules given above are admissible in ReL2.

(b) The fundamental sequences are valid in ReL2.

Proof: (a) Admissibility of decomposition rules follows from the definitions of the relational operations. The rules (–R) and (–S) are admissible iff $S \subseteq -R$ and $R \subseteq -S$, respectively, and these conditions hold due to the fact that $R \cap S = \emptyset$ is satisfied in CI frames. The remaining specific rules are admissible due to the respective properties of relations reflected in the names of the rules.

The sequence (f2) is valid due to reflexivity of S. The sequence (f3) is valid due to the condition $S \cup R = W^2$ which holds in CI frames. (QED)

Relational proofs have the form of trees. Given a relational formula xAy, where A might be a compound relational expression, we successively apply decomposition or specific rules. In this way, we form a tree whose root consists of xAy and whose nodes consist of finite sequences of relational formulas. We stop applying rules to the formulas in a node after obtaining a fundamental sequence, or when none of the rules is applicable to the formulas in this node. A branch of a proof tree is said to be *closed* whenever it contains a node with a fundamental sequence of formulas. A tree is closed iff all of its branches are closed.

7.2 Proposition (Completeness theorem)

A relational formula xAy is valid in ReL2 iff there is a closed proof tree with root xAy.

Proof: (→) Suppose that there is no closed proof tree for xAy, and consider a tree satisfying the following conditions for every non–closed branch b. We write $G \in b$ whenever a formula G is a member of a sequence of formulas in a certain node of b.

(b1) $xAy \in b$

(b2) If $x(B \cup C)y$ ($x-(B \cap C)y) \in b$, then both xBy $(x-By) \in b$ and xCy $(x-Cy) \in b$ are obtained by application of rule (∪) (resp. (–∩)).

(b3) If $x-(B \cup C)y$ $(x(B \cap C)y) \in b$, then, either $x-By$ $(xBy) \in b$ or $x-Cy$ $(xCy) \in b$ is obtained by application of rule (–∪) (resp. (∩)).

(b4) If $x(B;C)y \in b$, then, for every $z \in VI$, either $xBz \in b$ or $zCy \in b$ is obtained by application of rule (;).

(b5) If $x-(B;C)y\in b$, then, for some $z\in VI$, $x-Bz\in b$ and $z-Cy\in b$ are obtained by application of rule
(–;).

(b6) If $x--By\in b$, then $xBy\in b$ is obtained by application of rule (––)

(b7) If $xBy\in b$ with $B\in VR$, then, for every $z\in VI$, we have $xBz\in b$ obtained by application of rule (ideal).

(b8) If $xRy\in b$, then $yRx\in B$ is obtained by application of rule (symR).

(b9) If $xRy\in b$, then, for every $z, t\in VI$, either of xRz, zRt, tRy belongs to b obtained by application of rule (3–tranR)

(b10) If $x-Ry\in b$, then $xSy\in b$ is obtained by application of rule (–R)

(b11) If $x-Sy\in b$, then $xRy\in b$ is obtained by application of rule (–S)

(b12) For every $x, y\in VI$, either $xAy\in b$ or $x-Ay\in b$, for A=R, S, obtained by application of rule (cut).

Any tree satisfying conditions (b1),...,(b12) is referred to as a *complete proof tree*. The standard proof–theoretic construction shows that for every formula there is a complete proof tree with this formula in a root.

Let b be a non–closed branch of a complete proof tree. We define the system $M^b=(W^b, m^b)$ such that

1. $W^b = VI$
2. $m^b(P) = \{(x,y)\in W^b\times W^b: xPy \notin b\}$ for $P \in VR\cup\{R, S\}$.

We extend m^b in a homomorphic way to all relational expressions. Observe that

(i) *$m^b(R)$ is a symmetric relation on set W^b.*

Let $(x,y)\in m^b(R)$, hence $xRy\notin b$. If $(y,x)\notin m^b(R)$, then $yRx\in b$, and by (b8) we have $xRy\in b$, a contradiction.

(ii) *$m^b(R)$ is a 3–transitive relation on W^b:*

The proof is by an easy verification using condition (b9).

(iii) *$m^b(S)$ is a reflexive relation on W^b:*

Otherwise, assume that for some x we have $(x,x)\notin m^b(S)$. It follows that $xSx\in b$, and then branch b would be closed, a contradiction.

(iv) *$m^b(R)\cup m^b(S)=W^b\times W^b$:*

Suppose that there are x, y∈VI such that $(x,y)\notin m^b(R)$ and $(x,y)\notin m^b(S)$. It follows that xRy∈b and xSy∈b, and then branch b would be closed, a contradiction.

(v) $m^b(R)\cap m^b(S)=\emptyset$:

Suppose that there are x, y such that $(x,y)\in m^b(R)$ and $(x,y)\in m^b(S)$. It follows that xRy∉b and xSy∉b. By (b12) x–Ry∈b and by (b10) xSy∈b, a contradiction.

(vi) $m^b(P)$ *is an ideal relation for any P∈VR:*

This condition follows from (b7).

Let v^b be a valuation in M^b such that $v^b(x)=x$ for every individual variable x. We say that formula xBy is *indecomposable* whenever B∈VR∪{R, S}. Let IND^b be the set of all indecomposable formulas occurring in the nodes of branch b. From the definition of m^b we obtain

(vii) *For every zBt∈*IND^b *we have not* M^b,v^b *sat zBt.*

Next, we define an ordering of relational terms as follows:

1. If P is a relational variable then ord(P)=ord(R)=ord(S)=1.
2. If ord(B)=n then for any unary relational operation * we define ord(*B)=n+1.
3. If ord(B)≤n and ord(C)≤n and at least one of the inequalities is =, then for every binary relational operation # we define ord(B#C)=n+1.

We will show that:

(viii) *not* M^b,v^b *sat xAy:*

Otherwise, assume the negation, and let X^b be the set of formulas zBt on b such that M^b,v^b sat zBt. X^b is nonempty since, by our assumption, it contains xAy. Let C be a term of minimal order such that uCw is in X^b for some variables u, w. We show that C must be either a relational variable or a relational constant: C cannot be of the form u–Pw for a relational variable P or P=R, S: Otherwise we would have u–Pw ∈ b and M^b,v^b sat u–Pw; by definition of m^b, the latter is equivalent to uPw ∈ b, and then branch b would be closed.

Suppose that C is of the form C1;C2. Hence, the conditions

(a1) uC1;C2w ∈ b and

(a2) M^b,v^b sat uC1;C2w

hold. From (a1) and (b4) we conclude that for all z either $uC1z \in b$ or $zC2w \in b$. From (a2) we have that there is some t such that M^b,v^b sat uC1t and M^b,v^b sat tC2w. Hence, either $uC1t \in X^b$ or $tC2w \in X^b$, and C1,C2 have a smaller value of ord than C, a contradiction.

In a similar way we show that C is neither an expression built with any other relational operators, nor a complemented compound expression.

In view of the above, $uCw \in IND^b$, and hence, by (vii), we have not M^b,v^b sat uCw, a contradiction. This completes the proof of (viii).

We conclude that M^b is a model of the relational logic such that xAy is not true in M^b, a contradiction.

The proof of part (←)follows from Proposition 7.1. (QED)

Next, we present two examples of relational proofs of formulas of the logic L2. Let G be the formula

$$[[]]\neg F \to []F.$$

This formula is true in a frame (W, R, S) iff $R \subseteq -S$. For the sake of simplicity, let us denote TR(F) by F. Then the relational translation of G is $RT(G)=(-S;-F) \cup -(R;-F)$.

xRT(G)y

(∪)

x(−S;−F)y, x−(R;−F)y

(−;) z is a restricted variable

x(−S;−F)y, x−Rz, zFy,...

(;) new variable:= z

x−Sz, x−Rz, zFy,...　　　　z−Fy, x−Rz, zFy,...

(−S)　　　　　　　　　　fundamental (f1)

xRz, x−Rz, zFy,...

fundamental (f1)

As another example consider the formula $G = F \to [[]][[]]F$. It is true in the frame (W, R, S) iff the relation S is symmetric. Its relational translation is $RT(G)=-F \cup -(-S;-(-S;F))$.

xRT(G)y

(∪)

x–Fy, x–(–S;–(–S;F))y

(–;) z is a restricted variable

x–Fy, xSz, z–S;Fy

(;) new variable:=x

x–Fy, xSz, z–Sx,... x–Fy, xSz, xFy,...

(–S) fundamental (f1)

x–Fy, xSz, zRx,...

(symR)

x–Fy, xSz, xRz,...

fundamental (f3)

8 Logics of relative complementarity and incomplementarity

In this section we discuss logics for the relations of complementarity and incomplementarity of level 2. Let K=(W,{R(P):P⊆PAR}) be a relational system such that U is a nonempty set, PAR is a nonempty finite set, and each R(P) is a binary relation on W.

We say that K is an *information frame with strong relations*, if for all P,Q ⊆ PAR the relations of K satisfy the following conditions:

1. R(P∪Q)=R(P)∩R(Q)
2. R(∅)=U×U.

K is called an *information frame with weak relations* if for all P,Q ⊆ PAR the relations of K satisfy the following conditions:

1. R(P∪Q)=R(P)∪R(Q)
2. R(∅)=∅

We now define complementarity and incomplementarity frames with relative relations:

Strong complementarity frame (SCOM):

Strong, symmetric, 3–transitive and irreflexive relations.

Weak complementarity frame (WCOM):

Weak, symmetric and irreflexive relations, and for all a∈A, R({a}) is 3–transitive

Strong incomplementarity frame (SICOM):

Strong, symmetric and reflexive relations such that –R({a}) is 3–transitive for all a∈A.

Weak incomplementarity frame (WICOM):

Weak, symmetric and reflexive relations whose complements are 3–transitive.

Let T(PAR) be the set of terms over the Boolean algebra of subsets of PAR. The adequate language for expressing properties of the above relations of level 2 is a multimodal language $LAN(PAR)_{<>,<<>>}$ with multiple modal operators of possibility $<R(P)>$ and sufficiency $<<R(P)>>$, where $P \in T(PAR)$, and each operator is determined by a relative complementarity or incomplementarity relation.

8.1 Proposition

Let $K=(W, \{R(P): P \subseteq PAR\})$ be a frame with relative relations. Then

(a) $<<R(P\cup Q)>>F \to <<R(P)>>F \vee <<R(Q)>>F$ is true in K iff $R(P) \cap R(Q) \subseteq R(P\cup Q)$

(b) $<<R(P)>>F \vee <<R(Q)>> \to <<R(P\cup Q)>>F$ is true in K iff $R(P\cup Q) \subseteq R(P) \cap R(Q)$

(c) $<R(P\cup Q)>F \to <R(P)>F \vee <R(Q)>F$ is true in K iff $R(P\cup Q) \subseteq R(P) \cup R(Q)$

(d) $<R(P)>F \vee <R(Q)>F \to <R(P\cup Q)>F$ is true in K iff $R(P) \cup R(Q) \subseteq R(P\cup Q)$

(e) $\neg <<R(\emptyset)>>false$ is true in K iff $R(\emptyset)=W\times W$ where 'false' denotes a formula of the form $F \wedge \neg F$

(d) $\neg <R(\emptyset)>true$ is true in K iff $R(\emptyset)=\emptyset$ where 'true' denotes a formula of the form $F \vee \neg F$.

Proof: By way of example we prove (a) and (e).

(a)($\to$) Suppose that there are $a,b \in W$ such that $(a,b) \in R(P) \cap R(Q)$ and $(a,b) \notin R(P\cup Q)$. Consider a model based on K, such that

$$m(p)=\{y \in W: (a,y) \notin R(P) \cap R(Q)\}$$

for some propositional variable p. It is easy to show that M, a sat $<<R(P\cup Q)>>p$ and not M, a sat $<<R(P)>>p \vee <<R(Q)>>p$, a contradiction.

(a)($\leftarrow$) This part of the proof is by an easy verification.

(e) Truth in K of this formula means that for every model M based on K and for every $x \in W$, not M, x sat $<<R(\emptyset)>>false$. Equivalently, for every $x \in W$ it is not the case that there is $y \in W$ such that not M, y sat false and $(x,y) \notin R(\emptyset)$. We conclude that for every $x \in W$, for every $y \in W$, not $(x,y) \notin R(\emptyset)$ which completes the proof. (QED)

A relational formalization of logics of strong (weak) complementarity and incomplementarity frames contains the rules and fundamental sequences that reflect strongness or weakness of the respective relations:

(s) K, $xR(P\cup Q)y$, H
K, $xR(P)y$, H K, $xR(Q)y$, H

(−s) K, $x{-}R(P\cup Q)y$, H
K, $x{-}R(P)y$, $x{-}R(Q)y$, H

(w) K, $xR(P\cup Q)y$, H
K, $xR(P)y$, $xR(Q)y$, H

(−w) K, $x{-}R(P\cup Q)y$, H
K, $x{-}R(P)y$, HK, $x{-}R(Q)y$, H

(fs) $xR(\emptyset)y$
(fw) $x{-}R(\emptyset)y$

Let L(SCOM) be the logic of strong complementarity frames. By ReL(SCOM) we mean the relational logic for L(SCOM) defined in a simalar way as the logic ReL2 presented in section 6. Then the theorems analogous to Propositions 6.1, 6.2, 6.3 hold. The translation from formulas of L(SCOM) into relational formulas is analogous to the translation RT and it will be denoted by RT as well.

The relational proof system D(SCOM) for the logic L(SCOM) consists of the following rules: all the decomposition rules from (DEC), (s), (–s), (ideal), and, for every R(P) with $P \in T(PAR)$, the rules of the form (symR), (3–tranR) and (cut). The fundamental sequences are: (f1), (fs), and

(f4) any sequence containing $x{-}R(P)y$ for $P\in T(PAR)$.

8.2 Proposition

A relational formula xAy is valid in ReL(SCOM) iff there is a closed proof tree with the root xAy built according to the rules of the proof system D(SCOM).

Proof: This is similar to the proof of Proposition 7.2: To the definition of the complete proof tree we add the conditions:

(b13) If $xR(P\cup Q)y\in b$ then either $xR(P)y\in b$ or $xR(Q)y\in b$ obtained by application of rule (s)
(b14) If $x{-}R(P\cup Q)y\in b$ then both $x{-}R(P)y\in b$ and $x{-}R(Q)y\in b$ obtained by application of rule (–s).

Then we show that in a model M^b the following conditions are satisfied:

1) $m^b(R(P\cup Q))= m^b(R(P))\cap m^b(R(Q))$:

($\subseteq$) Let (i) $(x,y)\in m^b(R(P\cup Q))$, and suppose that (ii) $(x,y)\notin m^b(R(P))\cap m^b(R(Q))$. It follows from (i) that $xR(P\cup Q)y\notin b$. By (b12), $x{-}R(P\cup Q)y\in b$. By (b14), we have $x{-}R(P)y\in b$ and $x{-}R(Q)y\in b$. It follows from (ii) in the proof of Proposition 7.2, that $(x,y)\notin m^b(R(P))$ or $(x,y)\notin m^b(R(Q))$, which yields $xR(P)y\in b$ or $xR(Q)y\in b$; however, in this case, branch b would contain a fundamental sequence, a contradiction.

($\supseteq$) Let $(x,y)\in m^b(R(P))\cap m^b(R(Q))$. It follows that $xR(P)y\notin b$ and $xR(Q)y\notin b$. Suppose that $(x,y)\notin m^b(R(P\cup Q))$. We obtain $xR(P\cup Q)y\in b$. By (b13) $xR(P)y\in b$ or $xR(Q)y\in b$, a contradiction.

2) $m^b(R(\emptyset))=W^b\times W^b$:

Suppose that there are x, y $\in$ VI such that $(x,y)\notin m^b(R(\emptyset))$. This means that $xR(\emptyset)y\in b$, but then b would contain a fundamental sequence of the form (fs), a contradiction.

3) $m^b(R(P))$ *is irreflexive*:

Suppose that there is some $x\in$VI such that $(x,x)\in R(P)$. It follows that $xR(P)x\notin b$. By (b12) we have $x{-}R(P)x\in b$; but then, branch b would contain a fundamental sequence of the form (f4), a contradiction.

The proof of symmetry and 3–transitivity of relations $m^b(R(P))$ is the same as in the proof of Proposition 7.2. (QED)

The relational proof system for the logic L(WCOM) of weak complementarity frames consists of rules (DEC), (w), (–w), (ideal), for every R(P) with $P\in$T(PAR), the rules of the form (symR) and (cut), and for every R({a}) with $a\in$PAR, of the rule of the form (3–tranR); the fundamental sequences are (f1), (fw) and (f4). The proof of the completeness theorem is similar to the proof of 8.2. In the definition of the complete proof tree we add the conditions:

(b13') If $xR(P\cup Q)y\in b$, then both $xR(P)y\in b$ and $xR(Q)y\in b$ are obtained by application of rule (w).
(b14') If $x{-}R(P\cup Q)y\in b$, then either $x{-}R(P)y\in b$ or $x{-}R(Q)y\in b$ are obtained by application of rule (–w).

Then we show that in model M^b the following conditions are satisfied:

(a) $m^b(R(P\cup Q))= m^b(R(P))\cup m^b(R(Q))$
(b) $m^b(R(\emptyset))=\emptyset$
(c) $m^b(R(\emptyset))$ *is irreflexive.*

By way of example we show condition (b): Suppose that there are x, y $\in$ VI such that $(x,y)\in m^b(R(\emptyset))$. It follows that $xR(\emptyset)\notin b$. By (b12), we have $x{-}R(\emptyset)y\in b$, a contradiction because a fundamental sequence of the form (f4) cannot belong to b.
The rest of the proof follows the lines of the proof of Proposition 8.2.

The rule reflecting 3–transitivity of the complement of a relation is as follows:

(3–tran–R) K, x–Ry, H

K, x–Rz, H, x–Ry K, z–Rt, H, x–Ry K, t–Ry, H, x–Ry

z, t are variables

Clearly, the rule is admissible in a relational logic iff the relation −R is 3−transitive in all models of that logic.

The relational proof system for logic L(SICOM) of strong incomplementarity frames consists of the rules (DEC), (s), (−s), (ideal), for every R(P) with P∈T(PAR) of the rules of the form (symR) and (cut), and for every R({a}) with a∈PAR of the rule of the form (3−tran−R). The fundamental sequences are: (f1), (fs) and

(f5) any sequence containing formula xR(P)x for P∈T(PAR).

The relational proof system for the logic L(WICOM) of weak incomplementarity frames consists of the rules (DEC), (w), (−w), (ideal), and for every R(P) with P∈T(PAR) of the rules of the form (symR), (cut) and (3−tran−R). The fundamental sequences are: (f1), (fw) and (f5).

The respective completeness proofs are analogous to the proof of Proposition 7.2 extended in a similar way to that presented for the logics of complementarity relations in Proposition 8.2.

In what follows we present two examples of relational proofs in the logics presented above: Consider the formula

$$G = <<R(\{a\})>>[[R(\{a\})]]<<R(\{a\})>>F \rightarrow <<R(\{a\})>>F.$$

We show that G∈L(SICOM). For the sake of simplicity in the formulas below we shall write R and F instead of R({a}) and RT(F), respectively. The relational translation of G is

$$RT(G) = -(-R;(-R;(-R;-F))) \cup (-R;-F).$$

Now,

xRT(G)y

(∪)

x−(−R;(−R;(−R;−F)))y, x−R;−Fy

(−;) z is a restricted variable

xRz, z−(−R;(−R;−F))y, x−R;−Fy

(−;) t is a restricted variable

xRz, zRt, t−(−R;−F)y, x−R;−Fy

(−;) u is a restricted variable

xRz, zRt, tRu, uFy, x−R;−Fy

(;) new variable :=u

xRz, zRt, tRu, uFy, x−Ru,... | xRz, zRt, tRu, uFy, u−Fy,...

(3−tran−R) new variables:=z, t | fundamental (f1)

.....x−Rz,... | ....z−Rt,... | ....t−Ru,...

fund. (f1) | fund. (f1) | fund. (f1)

Next, we show that formula

$$G=\langle R(\emptyset)\rangle\langle R(\emptyset)\rangle F\rightarrow\langle R(\emptyset)\rangle F$$

belongs to the logics of strong frames of complementarity or incomplementarity. We have

$$RT(G)=-(R;(R;F))\cup(R;F).$$

Now,

$$xRT(G)y$$
$$(\cup)$$
$$x-(R;(R;F))y,\ xR;Fy$$
$$(-;)\ z \text{ is a restricted variable}$$
$$x-Rz,\ z-(R;F)y,\ xR;Fy$$
$$(-;)\ t \text{ is a restricted variable}$$
$$x-Rz,\ z-Rt,\ t-Fy,\ xR;Fy$$
$$(;)\ \text{new variable} := t$$

$x-Rz,\ z-Rt,\ t-Fy,\ xRt,\ldots$	$x-Rz,\ z-Rt,\ t-Fy,\ tFy,\ldots$
fundamental (fs)	fundamental (f1)

9 Conclusion and outlook

In this paper we proposed a relational methodology for analyzing and reasoning about complementarity of objects in information systems. We presented relation-based concepts of complementarity/incomplementarity, and we exhibited their role in knowledge representation. We showed how various forms of complementarity/incomplementarity appear in information systems depending on the properties of their objects. We developed modal formalisms for reasoning about families of complemetarity/incomplementarity relations. Their semantics is based on complementarity/incomplementarity frames. For these logics we constructed relational proof systems and we proved their completeness.

In the forthcoming paper (Düntsch and Orlowska, 1999) we continue investigating complementarity, and we present classes of information algebras with the information operators determined by complementarity relations. The algebras are referred to as the complementarity algebras. We develop the Jónsson-Tarski style duality theory for the complementarity frames and the complementarity algebras.

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