Chapter 1

AXIOMS, ALGEBRAS, AND TOPOLOGY

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1. Introduction

This work explores the interconnections between a number of different perspectives on the formalisation of space. We begin with an informal discussion of the intuitions that motivate these formal representations.

1.1 Axioms *vs* Algebras

Axiomatic theories provide a very general means for specifying the logical properties of formal concepts. From the axiomatic point of view, it is symbolic formulae and the logical relations between them — especially the entailment relation — that form the primary subject of interest. The vocabulary of concepts of any theory can be interpreted in terms of a domain of entities, which exemplify properties, relations and functional mappings corresponding to the formal symbols of the theory. Moreover, by interpreting logical operations as functions of these seman-

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tic denotations, such an interpretation enables us to evaluate the truth of any logical formula built from these symbols. An interpretation is said to satisfy, or be a *model* of a theory, if all the axioms of the theory are true according to this evaluation.

In general an axiomatic theory can have many different models exhibiting diverse structural properties. However, in formulating a logical theory, we will normally be interested in characterising a particular domain and a number of particular properties, relations and/or functions that describe the structure of that domain; or, more generally, we may wish to characterise a family of domains that exhibit common structural features, and which can be described by the same conceptual vocabulary.

From the *algebraic* perspective, it is the domain of objects and its structure that form the primary subject of investigation. Here again, we may be interested in a specific set of objects and its structure, or a family of object sets exemplifying shared structural features. And the nature of the structure will be described in terms of properties, relations and functions of the objects. To specify a particular structure or family of structures, one will normally give an axiomatic theory formulated in terms of this vocabulary, such that the algebraic structures under investigation may be identified with the models of the theory.

Hence, axiom systems and algebras are intimately related and complementary views of a conceptual system. The axiomatic viewpoint characterises the meanings of concepts in terms of true propositions involving those concepts, whereas the algebraic viewpoint exemplifies these meanings in terms of a set of objects and mappings among them. Moreover, the models of axiomatic theories can be regarded as algebras, and conversely algebras may be characterised by axiomatic theories.

Having said this, the two perspectives lead to different emphasis in the way a conceptual system is articulated. If one starts from axiomatic propositions, one tends to focus on relational concepts (formalised as predicates), whereas, if one starts from objects and structures, the focus tends to be on functional concepts corresponding to mappings between the objects. Indeed, the term 'algebra' is sometimes reserved for structures that may be characterised without employing any relational concept apart from the logical equality relation. And the most typical algebras are those specified purely by means of universally quantified equations holding between functional terms.

1.2 Representing Space

1.2.1 Classical Approaches. Our modern appreciation of space is very much conditioned by mathematical representations. In particular, the insights into spatial structure given to us by Euclid and Descartes are deeply ingrained in our understanding.

Euclid described space in terms several distinct categories of geometrical object. These include *points*, *lines* and *surfaces* as well as *angles* and *plane figures*. These entities may be said to satisfy a number of basic properties (e.g. lines may be *straight* and surfaces may be *planar*) and relationships (e.g. a point may be *incident* in a line or surface, two lines may meet at a point or be inclined at an angle). The nature of space was then characterised by postulates involving these basic concepts, which were originally stated in ordinary language. Euclid proceeded to define many further concepts (such as different types of geometrical figure) in terms of the basic vocabulary.

From Descartes came a numerical interpretation of space, with points in an n-dimensional space being associated with n-tuples of numerical values. According to this Cartesian model, the basic elements of space are points. Their structure and properties can be axiomatised in terms of the metrical relation of equidistance (see e.g. [101, 104]), and interpreted in terms of numerical coordinates.

If points are taken as the primary constituents of the universe, lines and regions have a derivative status. Two distinct points determine a line, and polygonal figures can be represented by sequence of their vertex points. To get a more general notion of 'region' we need to refer to more or less arbitray collections of points. This is the representational perspective of classical point-set topology.

1.2.2 Region-Based Approaches. Although the point-based analysis has become the dominant approach to spatial representation, there are a number of motivations for taking an alternative view, in which *extended regions* are considered as the primary spatial entities.

An early exponent of this approach was Alfred North Whitehead, who shared with Bertrand Russell the view that an adequate theory of nature should be founded on an analysis of *sense data*, and that elements of perception can be the only referents of truly primitive terms. On this basis, Whitehead in his book *Concept of Nature* [111] argued that extended regions are more fundamental that points: whereas regions may be perceived as the spatial correlates colour patches in the visual field, points cannot be perceived directly but are only constructed by cognitive abstraction.

This motivation shares some common ground with that of Stanislaw Leśniewski, who also wanted to bring the theoretical analysis of the world more closely in line with phenomenological conceptions of reality, and believed that perceiving the integrity of extended objects is basic to our interpretation of the world. *Mereology*, a formal theory of the part-whole relation was originally presented by Leśniewski [66] in his own logical calculus, which he called *Ontology*.

Whitehead identified the relation of *connection* between two spatial or spatio-temporal regions as of particular importance to the phenomenological description of reality. In further work he attempted to use this concept as the fundamental primitive in a logical theory of space and time. A formal theory of this relation was presented in [112].[3]

The earliest completely rigorous and fully formalised theory of space where regions are the basic entity is the axiomatisation of a *Geometry of Solids* that was given by Tarski in 1929 [98]. Subsequently, a number of other formalisations have been developed. These include the *Calculus of Individuals* proposed by Leonard and Goodman [45, 64] (which are close to Leśniewski's mereology) and the spatial theories of Clarke [16, 17] (which are based on Whitehead's connection relation.

More recently, region-based theories have attracted attention from researchers working on Knowledge Representation for Artificial Intelligence systems (AI). The so-called *Region Connection Calculus* [88] is a 1st-order formalism based on the connection relation and is a modification of Clarke's theory. AI researchers are motivated to study such representations by a belief that they may be useful as a vehicle for automating certain human-like spatial reasoning capabilities. From this point of view it has been argued that the region-based approach is closer to the natural human conceptualisation of space. In the context of describing and reasoning about spatial situations in natural language, it is common to refer directly to regions and the relations between them, rather than referring to points and sets of points. Therefore, treating regions as basic entities in a formal language can in many cases allow simpler representation of high-level human-like spatial descriptions.

1.2.3 Interdefinability of Regions and Points. Despite the difference in perspective, several formal results show that region and point-based conceptualisations are in fact interdefinable, given a sufficiently rich formal apparatus. Whitehead himself had noted that by considering classes of regions, points can be defined as infinite sets of

[3]This was subsequently found to be inconsistent and was posthumously corrected in a second edition [113].

nested regions which converge to a point. Pratt in [83] and [86] showed that any sufficiently strong axiomatisation of the polygonal regions of a plane can be interpreted in terms of the classical point-based model of the Euclidean plane. So in some sense the region-based theory is not 'ontologically simpler' in its existential commitments.

Later in this paper we shall adopt a similar approach, and by identifying a point with the sets of all regions region to which it belongs, we shall show that even much weaker and more general region-based theories can be interpreted in terms of point sets. Hence, the point and region based approaches should not be regarded as mutually exclusive, but rather as complementary perspectives.

Nevertheless, we believe that region-based theories deserve more attention than has traditionally been paid them, and that for certain purposes they have clear advantages.

There is an argument that regions are actually more powerful and flexible than points as a starting point for spatial theories. In Pratt [86] it is shown that if we have a domain of regions plus a spatial language with sufficient (in fact rather low) expressive power, we implicitly determine a corresponding domain of points. Roughly speaking, this is done as follows: within the region-based theory we can specify pairs of regions that have a unique 'point' of contact. Thus, relations among points can be recast in the guise of formulae which refer to these region pairs. In this way points are implicitly definable from regions by first-order means; whereas, if we start with points as the basic entities, the definition of regions requires set theory (unless we arbitrarily restrict the geometrical complexity of regions).

1.3 Alternative Logical Formalisms

We have seen that the representation of space allows alternative views that invert the perspective of the orthodox picture. The same is also true for the mode of application of formal representations themselves. First-order logic provides a standard alignment of syntactic and conceptual categories. Specifically, the basic nominal symbols of the formal language refer to what are considered to be the primitive entities of the 'domain' of a theory, while formal predicates correspond to properties and relations that hold among those entities. This alignment is widely held to be 'natural', in that it seems to accord in some respects with the syntactic expression of semantics found in natural languages. However, this intuition is difficult to establish conclusively. Moreover, by altering the correspondence between syntactic and conceptual categories, one may obtain alternative calculi that also have a meaningful interpretation and proof theory. For instance, one could interpret syntactically basic symbols as denoting 'properties', and represent 'individuals' formally as predicates (the extension of the predicate being the set of properties satisfied by the corresponding individual).

However, since relations between objects are not in general reducible to properties of individual objects, a much more powerful abstraction is obtained by taking *relations* as the basic entities of a formal system. This approach was first formalised in an algebraic framework by Tarski [99] and relation algebras are now a well-established alternative to standard 1st-order formalisms [5, 102].

From the point of view of logic and computation, there are significant advantages in treating relations as basic entities. In particular this mode of representation allows quantifier-free formalisation of many properties and inference patters, which would otherwise require quantification. This is one of the main themes of Algebraic Logic as it is elaborated in [1, 5, 78].

As we will be concerned with spatial representation based on the 'contact' relation, relation algebras are a natural system within which to formulate theories of this kind.

1.4 Structure of the Chapter

The organisation of this chapter is as follows. In Section 2 we shall present some basic formal structures and notations that will be used to develop the theory. These include, Boolean algebras, relation algebras, topological spaces and proximity spaces. Section 3 introduces the spatial *contact* relation, which is the primary focus of our investigation. We consider the fundamental axioms satisfied by a contact relation and give standard interpretations of contact in terms of point-set topology. We then see how the basic properties of this relation can be described in both first order and relation algebraic calculi.

In Section 4 we introduce Boolean Contact Algebras, which are Boolean algebras supplemented with a contact relation satisfying appropriate general axioms. Additional axioms are also considered, which characterise further properties of contact that are exhibited under typical spatial interpretations. We give representation theorems for the general class of Boolean Algebras in terms of both topological spaces and proximity spaces, and give more specific representation systems for algebras satisfying additional axioms. These theorems make concrete the correspondence between the relational approaches which focus on axiomatic properties of the contact relation, and the more well-known models of space in terms of point-set topology.

Section 5 looks at some other well known approaches to formalising topological relationships, in particular the Region Connection Calculus [88] and the 9-intersection model. Section 6 considers the problem of reasoning with topological relations. The methods presented are: compositional reasoning, equational reasoning, encoding into modal logic and a relation algebraic proof theory. Section 6 concludes the chapter with a consideration of the correspondences that have been established between different modes of formalising topological information, and of ongoing and future developments in this area.

2. Preliminary Definitions and Notation

In this section we give definitions and key properties of the basic formal structures that will underpin our analysis. We start with *Boolean algebras with operators*, which provide an extremely general framework for studying structured domains of objects. Two important special cases of these algebras are considered: modal algebras, and relation algebras.

2.1 Boolean Algebras

We assume that the reader has some familiarity with Boolean algebras (BAs), and here only revise the basic details and notation. Our standard reference for BAs is [61], and we will just review some basic concepts. Our signature for a BA will be $\langle B, \cdot, +, -, \mathbf{0}, \mathbf{1}\rangle$. We will usually refer to an algebra by its base set (in this case B).

Definition 2.1. *Boolean Algebra concepts and notations:*

i) *For all $a, b \in B$, $a \leq b$ holds iff $a + b = b$.*

ii) *If $A \subset B$, then $\sum_B A$ denotes the least upper bound of A relative to the $\leq$ ordering of B. If A is infinite this does not necessarily exist. Where the relevant algebra is clear, we may write simply $\sum A$.*

iii) *If A is a subalgebra of B, we denote this by $A \leq B$.*

iv) *The set of non-zero elements of B is denoted by B^+.*

v) *If M is a subset of B^+, then M* is dense in B, *iff*

$$(\forall b \in B^+)(\exists a \in M)\ a \leq b\ .$$

vi) *An* atom of B *is an element $a \in B^+$ such that*

$$(\forall c)[c \leq a \Rightarrow (c = \mathbf{0} \vee c = a)]\ .$$

vii) *The set of atoms of B will be written as* $\mathrm{At}(B)$.

viii) *B is* atomic, *iff* $\mathrm{At}(B)$ *is dense in* B^+.

ix) *If $f : B \to B$ is a mapping, then its* dual *is the mapping* $f^\partial : B \to B$ *defined by* $f^\partial(x) = -f(-x)$.

In general, a BA will contain elements corresponding to the meet and join of any *finite* subset of its domain. A BA is called *complete*, if arbitrary joins and meets exist. The *completion of B* is the smallest complete BA A which contains B as a dense subalgebra. It is well known that each B has a completion which is unique up to isomorphisms.

Definition 2.2. *An* ultrafilter *is a subset F of B such that:*

i) *If $x \in F$, $y \in B$ and $x \leq y$, then $y \in F$.*

ii) *If $x, y \in F$, then $x \cdot y \in F$.*

iii) *$x \in F$ if and only if $-x \notin F$.*

The set of all ultrafilters of B will be denoted by $Ult(B)$.

Ultrafilters are often employed as a means to represent 'point-like' entities that are implicit in the structure of a BA. From a purely algebraic point of view, the elements of a BA are abstract entities with no substructure. However, the elements may be and often are intended to correspond to composite objects (e.g. sets or spatial regions). Thus, as we shall see later, the elements of a BA are often *interpreted* as point sets in some space (e.g. topological space). In such a context, an ultrafilter can usually be thought of a set of all those elements of a BA that contain some particular point in the space over which the algebra is interpreted.

Perhaps the simplest example is the BA X^* whose elements are (interpreted as) all subsets of the set X (with the Boolean operations having their standard set-theoretic interpretation). In this case, for each $x \in X$ the set $\{Y \mid Y \subseteq X^* \wedge x \in Y\}$ is an ultrafilter of X^*.

Definition 2.3. *A* canonical extension *of B is an algebra B^σ, which is a complete and atomic BA containing an isomorphic copy of B as a subalgebra, and which satisfies the following the properties:*

i) *Every atom of B^σ is the meet of elements of B.*

ii) *If $A \subseteq B$ such that $\sum_{B^\sigma} A = 1$,*
then there is a finite *set of $A' \subseteq A$ such that $\sum_{B^\sigma} A' = 1$.*

It is well known, that each BA has a canonical extension which is unique up to isomorphism. One such construction is given by Stone's representation theorem for Boolean algebras: Let B^σ be the powerset algebra of the set of ultrafilters X of B, and embed B into B^σ by $b \mapsto \{U \in X : b \in U\}$. If $A \leq B$, then A is called a *regular subalgebra of* B, if B is a canonical extension of A.[4]

For more details and discussions we refer the reader to [56–58, 60].

2.2 Boolean Algebras with Operators

The structure of a Boolean algebra may be further elaborated by the introduction of additional operators. These *Boolean algebras with operators* arose from the investigation of relation algebras, and were first studied in detail by [60]; a survey can be found in [56]. Many useful structures have the form of such algebras.

Definition 2.4. *Some useful concepts for describing Boolean algebras with operators are defined as follows:*

i) *A function* $f : B^n \to B$ *on a BA is called* additive *in its i-th argument if* $f(x_0, \ldots, x_i, \ldots x_n) + f(x_0, \ldots, x'_i, \ldots x_n) = f(x_0, \ldots, (x_i + x'_i), \ldots x_n)$*, for all* $x_i \in B$*.*

ii) *A function* $f : B^n \to B$ *on a BA is called an* operator*, if it is additive in each of its arguments.*

iii) $f : B^n \to B$ *is called* normal*, if it is normal and its value is* 0 *if any of its arguments is* 0*.*

iv) *A structure* $\langle B, (f_i)_{i \in I}\rangle$ *is called a* Boolean Algebra with Operators *(BAO), if* B *is a BA, and all* f_i *are operators.*

v) *If all* f_i *are furthermore normal, then we speak of a* normal BAO*.*

vi) *A collection of algebras defined by a given signature and a set of universally quantified equations is called an* equational class *(or* variety*).*

Examples of normal BAOs are modal algebras, relation algebras (both of which will be discussed below), and cylindric algebras, which provided an algebraicisation of first order logic. We invite the reader to consult the classic monographs by Henkin et al. [50, 51] or the recent exposition by

[4]The notion of canonical extension is equivalent to that of 'perfect' extension introduced in [60]. Our notion of regular sub-algebra is also equivalent to that used in [60], which is different to that given in [61].

Andréka et al. [5], which provides a comprehensive (and comprehensible) introduction to Tarski's algebraic logic.

The concept of canonical extensions of Definition 2.3 can be extended to BAOs:

Definition 2.5. *Suppose that B is a BA, and f an n-ary normal operator on B. The* canonical extension f^σ *of f is defined by*

$$f^\sigma(x) = \sum \left\{ \prod \{f(y) : p \leq y \in B^n\} : p \in \mathrm{At}(B^\sigma)^n \text{ and } p \leq x \right\} \tag{2.1}$$

for all $x \in (B^\sigma)^n$. If $\langle B, (f_i)_{i\in I}\rangle$ is a normal BAO, we call $\langle B^\sigma, (f_i^\sigma)_{i\in I}\rangle$ the canonical extension of $\langle B, (f_i)_{i\in I}\rangle$.

Proposition 2.1. *[60] The canonical extension of a normal BAO $\langle B, (f_i)_{i\in I}\rangle$ is a complete and atomic normal BAO containing $\langle B, (f_i)_{i\in I}\rangle$ as a subalgebra.*

This is not the place to dwell on the preservation properties of canonical extensions of normal BAOs, and we refer the reader to [56] and [24] for details.

As the connection of unary normal operators to operators of *modal logics* (which will be examined in detail later) is somewhat special, we make the following convention:

Definition 2.6. *If f is a unary normal operator on the BA B, we call it a* modal operator *or* possibility operator*, and the structure $\langle B, f\rangle$ a* modal algebra.

Hence, modal algebras form an equational class (or variety) of algebras. That is the class of BAOs with one operator f that satisfy, in addition to the identities of Boolean algebra, the equations:

$$f(x + y) = f(x) + f(y) \tag{2.2}$$

$$f(0) = 0 \tag{2.3}$$

A special case of modal algebras (hence, of BAOs) are *closure algebras*:

Definition 2.7. *A possibility operator f on B which also satisfies, for all $a \in B$,*

$$a + f(a) = f(a), \tag{2.4}$$

$$f(f(a)) = f(a) \tag{2.5}$$

is called a closure operator*, and, in this case, $\langle B, f\rangle$ is a* closure algebra.

Incidentally, one dimensional cylindric algebras are a special case of closure algebras [50].

Definition 2.8. *Functions whose duals are possibility operators are called* necessity *operators. Thus a necessity operator on B is a function $g : B \to B$, for which*

(2.6)	$g(1) = 1,$		*Dually normal*
(2.7)	$g(a \cdot b) = g(a) \cdot g(b)$	*for all* $a, b \in B$	*Multiplicative*

Definition 2.9. *A necessity operator g is called an* interior operator *if for all $a \in B$ it satisfies*

$$g(a) + a = a, \tag{2.8}$$

$$g(a) = g(g(a)) \tag{2.9}$$

If g is an interior operator on B, then the structure $\langle B, g\rangle$ is called an interior algebra*.*

Modal algebras can be viewed as an algebraic counterpart to the relational structures known as *(Kripke) frames*:

Definition 2.10. *A* frame *is a pair $F = \langle U, R\rangle$, where R is a binary relation on U, called an* accessibility relation*.*

Every frame F has a corresponding algebra, called the *complex* algebra of F.

Definition 2.11. *If $F = \langle U, R\rangle$ is a frame, then the* complex algebra *of F is the structure $F^* = \langle 2^U, \Diamond_R\rangle$, where $\Diamond_R : 2^U \to 2^U$ is defined by*

$$\Diamond_R(X) = \{y \in U : (\exists x \in X)xRy\}, \tag{2.10}$$

It is not hard to see that F^* is a complete and atomic modal algebra. Conversely, we can construct a frame from a modal algebra:

Definition 2.12. *If $\langle B, f\rangle$ is a modal algebra, let $R_f \in \mathrm{Rel}(\mathrm{At}(B))$ be defined by*

$$aR_f b \iff a \leq f(b). \tag{2.11}$$

The structure $\langle \mathrm{At}(B), R_f\rangle$ is called the canonical frame *of $\langle B, f\rangle$, and R_f its* canonical relation*.*

We now have the following Representation Theorem:

Proposition 2.2. *[56, 57, 60] Let $\langle B, f\rangle$ be a complete and atomic modal algebra. Then, $\langle B, f\rangle$ is a regular subalgebra of the complex algebra of its canonical frame. Furthermore, if $\langle B, f\rangle$ is isomorphic to a regular* *subalgebra of a complex algebra of some frame* $\langle U, R\rangle$*, then* $\langle U, R\rangle \cong \langle \mathrm{At}(B), R_f\rangle$.

It may be worthy of mention that that all normal BAOs, not only the ones with unary operators, are representable as regular subalgebras of complex algebras of frames. Normality is essential, since non–normal BAOs do not admit such a representation [72].

Correspondence theory investigates, which relational properties of R can be expressed by its canonical modal operator and its dual (see e.g. [107]). We have, for example:

$$\text{(2.12)} \qquad R \text{ is reflexive} \iff (\forall X)[X \subseteq \Diamond_R(X)],$$

$$\text{(2.13)} \qquad R \text{ is symmetric} \iff (\forall X)[\Diamond_R(-\Diamond_R(-X)) \subseteq X],$$

$$\text{(2.14)} \qquad R \text{ is transitive} \iff (\forall X)[\Diamond_R\Diamond_R(X) \subseteq \Diamond_R(X)]$$

These correspondences, as well as the following result, have appeared already in [60]:

Proposition 2.3. *A modal algebra is a closure algebra if and only if its canonical relation is reflexive and transitive.*

2.3 Binary Relations and Relation Algebras

A *binary relation* R on a set U is a subset of $U \times U$, i.e. a set of ordered pairs $\langle x, y\rangle$ where $x, y \in U$. Instead of $\langle x, y\rangle \in R$, we shall often write xRy. The smallest binary relation on U is the empty relation $\emptyset$, and the largest relation is the universal relation $U \times U$, which we will normally abbreviate as U^2. The *identity relation* $\langle x, x\rangle : x \in U$ will be denoted by $1'$, and its complement, the *diversity relation*, by $0'$. *Domain* and *range* of R are defined by

$$\text{(2.15)} \qquad \mathrm{dom}(R) = \{x \in U : (\exists y \in U)xRy\},$$

$$\text{(2.16)} \qquad \mathrm{ran}(R) = \{x \in U : (\exists y \in U)yRx\}.$$

Furthermore, we let $R(x) = \{y \in U : xRy\}$.

The set of all binary relations on U will be denoted by $\mathrm{Rel}(U)$. Clearly, $\mathrm{Rel}(U)$ is a Boolean algebra under the usual set operations:

$$\text{(2.17)} \qquad -R = \{\langle x, y\rangle : \neg(xRz)\}$$

$$\text{(2.18)} \qquad R \cup S = \{\langle x, y\rangle : xRy \text{ or } xSy\}$$

$$\text{(2.19)} \qquad R \cap S = \{\langle x, y\rangle : xRy \text{ and } xSy\}$$

If $R, S \in \mathrm{Rel}(U)$, the *composition of* R *and* S is defined as

$$\text{(2.20)} \qquad R \, ; \, S = \{\langle x, y\rangle : (\exists z)[xRz \text{ and } zSy]\}$$

The *converse of* R, written as $R^{\smile}$, is the set

$$R^{\smile} = \{\langle y, x\rangle : xRy\}. \tag{2.21}$$

A detailed analysis of relation algebras can be found in [50], and an overview in [55]. The following lemma sets out some decisive properties of composition and converse.

Lemma 2.1.

i) $;$ *is associative and distributes over arbitrary joins.*

ii) $1' \;;\; R = R \;;\; 1' = R$.

iii) $^{\smile}$ *is bijective, of order two, i.e.* $R^{\smile\,\smile} = R$, *and distributes over arbitrary joins.*

iv) $(R \;;\; S)^{\smile} = S^{\smile} \;;\; R^{\smile}$.

v) $(R \;;\; S) \cap T = \emptyset \iff (R^{\smile} \;;\; T) \cap S = \emptyset \iff (T \;;\; S^{\smile}) \cap R = \emptyset$.

Note that any equation and any inequality between relations can be written as $T = U^2$ for some T. To do this, it is convenient to first to define the operation $R \otimes S$, which gives the *symmetric difference* of R and S:

$$R \otimes S = (R \cap -S) \cup (S \cap -R). \tag{2.22}$$

We then have the following equivalences:

$$R = S \iff -(R \otimes S) = U^2, \tag{2.23}$$

$$R \neq S \iff (U^2; ((R \otimes S); U^2)) = U^2. \tag{2.24}$$

Implicitly, we use here the concept of *discriminator algebras* which are a powerful instrument of algebraic logic, see[110] and also [59].

The *full algebra of binary relations on* U is the structure

$$\langle \mathrm{Rel}(U), \cap, \cup, -, \emptyset, U^2, \;;\; , {}^{\smile}, 1'\rangle\ .$$

A Boolean subalgebra of $\mathrm{Rel}(U)$ which is closed under $;$ and $^{\smile}$ and contains $1'$ will be called an *algebra of binary relations* (BRA).

Many properties of relations can be expressed by equations (or inclusions) among relations, for example,

$$\text{(2.25)}\quad R \text{ is reflexive} \iff (\forall x) xRx, \iff 1' \subseteq R.$$

$$\text{(2.26)}\quad R \text{ is symmetric} \iff (\forall x, y)[xRy \iff yRx], \iff R = R^{\smile}.$$

$$\text{(2.27)}\quad R \text{ is transitive} \iff (\forall x, y, z)[xRy \wedge yRz \Rightarrow xRz], \iff R \; ; \; R \subseteq R.$$

$$\text{(2.28)}\quad R \text{ is dense} \iff (\forall x) x(-R)x \wedge (\forall x, y)[xRy \Rightarrow (\exists z) xRzRy], \iff R \cap 1' = \emptyset \wedge R \subseteq R \; ; \; R, \iff R \cap (1' \cup -(R \; ; \; R)) = \emptyset.$$

$$\text{(2.29)}\quad R \text{ is extensional} \iff (\forall x, y)[R(x) = R(y) \Rightarrow x = y], \iff [-(R \; ; \; -R^{\smile}) \cap -(R^{\smile} \; ; \; -R)] \subseteq 1'.$$

One observes that all formulae above contain at most three variables. This is no accident, as the following result shows:

Proposition 2.4. *[44, 103]*

1 *The first order properties of binary relations on a set U that can be expressed by equations using the operators $\langle \cap, \cup, -, \emptyset, U^2, \; ; \;, {}^{\smile}, 1' \rangle$ are exactly those which can be expressed with at most three distinct variables.*

2 *If If $\mathcal{R}$ is a collection of binary relations on U, then, the closure of $\mathcal{R}$ under the operations $\langle \cap, \cup, -, \emptyset, U^2, \; ; \;, {}^{\smile}, 1' \rangle$ is the set of all binary relations on U which are definable in the (language of the) relational structure $\langle U, \mathcal{R} \rangle$ by first order formulae using at most three variables, two of which are free.*

If A is a complete and atomic BRA, in particular if A is finite, then the actions of the Boolean operators are uniquely determined by the atoms. To determine the structure of A it is therefore enough to specify the composition and the converse operation.

When dealing with an atomic BRA, it is often convenient to specify the composition operation by means of *composition table* (CT), which, for any two atomic relations R_i, R_j, specifies the relation $R_i ; R_j$ in terms of its constituent atomic relations. Formally, a composition table is a mapping $\text{CT} : \text{At}(A) \times \text{At}(A) \to 2^{\text{At}(A)}$ such that

$$\text{(2.30)}\qquad T \in \text{CT}(R, S) \iff T \subseteq R \; ; \; S.$$

Since A is atomic, we have

$$R \ ; \ S = \bigcup \mathrm{CT}(R,S). \tag{2.31}$$

CT can be conveniently written as a quadratic array (the *composition table of* A), where rows and columns are labelled with the atoms of A, and the cells contain $\mathrm{CT}(R,S)$.

BRAs are one instance of the class of *relation algebras*, which may be seen of an abstraction of algebras of binary relations [99]:

Definition 2.13. *A relation algebra (RA)*

$$\langle A, +, \cdot, -, 0, 1, \ ; \ , \ ^{\smile}, 1' \rangle$$

is a structure of type $\langle 2, 2, 1, 0, 0, 2, 1, 0 \rangle$ *which satisfies*

(R0) $\langle A, +, \cdot, -, 0, 1 \rangle$ *is a Boolean algebra.*

(R1) $x \ ; \ (y \ ; \ z) = (x \ ; \ y) \ ; \ z.$

(R2) $(x + y) \ ; \ z = (x \ ; \ z) + (y \ ; \ z).$

(R3) $x \ ; \ 1' = x.$

(R4) $x^{\smile\smile} = x.$

(R5) $(x + y)^{\smile} = x^{\smile} + y^{\smile}.$

(R6) $(x \ ; \ y)^{\smile} = y^{\smile} \ ; \ x^{\smile}.$

(R7) $(x^{\smile} \ ; \ -(x \ ; \ y)) \leq -y.$

Observe that BRAs and RAs are BAOs: An RA is a BAO where the additional operators $\langle \ ; \ , \ ^{\smile}, 1' \rangle$ form an involuted monoid, and the connection between this monoid and the Boolean operations is given by (R7). The somewhat cryptic character of (R7), can be made clearer by observing that, in the presence of the other axioms, it is equivalent to the cycle law

$$(x \ ; \ y) \cdot z = 0 \iff (x^{\smile} \ ; \ z) \cdot y = 0 \iff (z \ ; \ y^{\smile}) \cdot x = 0. \tag{2.32}$$

Tarski announced in the late 1940s that set theory and number theory could be formulated in the calculus of relation algebras:

> "It has even been shown that every statement from a given set of axioms can be reduced to the problem of whether an equation is identically satisfied in every relation algebra. One could thus say that, in principle, the whole of mathematical research can be carried out by studying identities in the arithmetic of relation algebras". [15]

We invite the reader to consult [103], and, for an overview [1] or [44]; another excellent reference for the theory of RAs is the book by Hirsch and Hodkinson [52].

2.4 Topological spaces

We will denote topological spaces by $\langle X, \tau \rangle$, where X is the base set, and τ the collection of open sets. If τ is understood, we will usually call X a topological space. The elements of X will be denoted by lower case Greek letters, where τ is reserved to denote a topology on X, and its subsets by lower case Roman letters.

If $x \subseteq X$, its interior is denoted by $\text{int}(x)$, and its closure by $\text{cl}(x)$. Observe that int and cl are an interior operator in the sense of (2.8) – (2.9), respectively, a closure operator in the sense of (2.4) – (2.5). A subspace y of X is *dense in* X, if $\text{cl}(y) = X$.

The *boundary* $\partial(x)$ of $x \subseteq X$ is the set $\text{cl}(x) \setminus \text{int}(x)$. If $\alpha \in X$, and $\alpha \in x \in \tau$, then x is called an *open neighborhood of* α. X is called *connected* if it is not the union of two disjoint nonempty open sets.

2.4.1 Separation Conditions. The general framework of topological spaces includes structures of many different kinds. In particular the open sets may be more or less densely distributed within the space. Significant, fundamental properties of this distribution can often be described in terms of the existence of disjoint separating arbitrary points and/or subsets of the space. Such properties are known as *separation* conditions.

Later in Section 4.2 we will show how axiomatic properties of spaces described in terms of the contact relation correspond to separation conditions of their topological interpretations. To this end, the following conditions are especially relevant:

T_1. A topological space X is a called T_1 *space*, if for any two distinct points α, β, there are $x, y \in \tau$ such that $\alpha \in x$, $\beta \notin x$ and $\beta \in y$, $\alpha \notin y$. This is equivalent to the fact that each singleton set is closed.

T_2 (Hausdorff). X is called a T_2 or Hausdorff space, if any two distinct points have disjoint open neighborhoods. It is well known that each T_2 space is a T_1 space, and that each regular T_1 space is a T_2 space.

Regular. A space X is *regular* if every point α and every closed set not containing α are respectively included in disjoint open sets.

It is well known [see e.g. 38] that X is regular, if and only if for each non–empty $u \in \tau$ and each $\alpha \in u$ there is some $v \in \tau$ such that $\alpha \in v \subseteq \text{cl}(v) \subseteq u$.

Semi-Regular. A space is *semi-regular* if it has a basis of regular open sets — i.e. every open set is a union of regular open sets.

Regularity implies semiregularity, but not vice versa.

Weakly Regular. We call X *weakly regular* if it is semiregular and for each non–empty $u \in \tau$ there is some non–empty $v \in \tau$ such that $\mathrm{cl}(v) \subseteq u$. Weak regularity may be called a "pointless version" of regularity, and each regular space is weakly regular.

Completely Regular. X is called *completely regular*, if for every closed x and every point $\alpha \notin x$ there is a continuous function $f : X \to [0, 1]$ such that $f(\beta) = 0$ for all $\beta \in x$, and $f(\alpha) = 1$.

Normal. X is called *normal*, if any two disjoint closed sets can be separated by disjoint open sets.

Weakly Normal. X is called *weakly normal*, if any two disjoint regular closed sets can be separated by disjoint open sets.[5]

Entailments among these properties are as follows:
X is *normal*
$\Longrightarrow$ X is *weakly normal*
$\Longrightarrow$ X is *completely regular*
$\Longrightarrow$ X is *regular*
$\Longrightarrow$ X is *weakly regular*
$\Longrightarrow$ X is *semiregular.*

None of these implications can be reversed, see [32] for examples.

A space which is T_1 and regular is called a T_3 space and a space which is T_1 and normal is a T_4 space. The various conditions T_i are successively stricter as i increases. Thus, $T_4 \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1$.

2.4.2 Regular Sets and their Algebras. A set $x \subseteq X$ is called *regular open*, if $x = \mathrm{int}(\mathrm{cl}(x))$, and *regular closed*, if $x = \mathrm{cl}(\mathrm{int}(x))$. Clearly, the set complement of a regular open set is regular closed and vice versa. The collection of regular open sets (regular closed sets) will be denoted by $\mathrm{RegOp}(X)$ ($\mathrm{RegCl}(X)$). It is well known [61] that $\mathrm{RegOp}(X)$ and $\mathrm{RegCl}(X)$ can be made into (isomorphic) complete Boolean algebras by the operations

$$\begin{aligned} x + y &= \mathrm{int}(\mathrm{cl}(x \cup y)), & x + y &= x \cup y, \\ x \cdot y &= x \cap y, & x \cdot y &= \mathrm{cl}(\mathrm{int}(x \cap y)), \\ -x &= X \setminus \mathrm{cl}(x), & -x &= X \setminus \mathrm{int}(x), \end{aligned}$$

[5] Weak normality has been introduced as 'κ-normality' by Shchepin [96].

$$\mathbf{0} = \emptyset, \qquad \mathbf{0} = \emptyset,$$
$$\mathbf{1} = X, \qquad \mathbf{1} = X.$$

$\operatorname{RegOp}(X)$ does not fully determine the topology on X:

Proposition 2.5. *[105] If y is a dense subspace of X, then $\operatorname{RegOp}(X) \cong \operatorname{RegOp}(y)$.*

If we only want to consider the regular closed sets (or regular open sets), it suffices to look at semiregular spaces: Let us call the topology $r(\tau)$ on X which is generated by $\operatorname{RegOp}(\tau)$ the *semi–regularisation of* $\langle X, \tau \rangle$.

Proposition 2.6. *Suppose that $\langle X, \tau \rangle$ is a topological space. Then, $\langle \operatorname{RegOp}(\tau) \rangle = \langle \operatorname{RegOp}(r(\tau)) \rangle$.*

Proof: *Let $a \subseteq X$. Then,*

$$\begin{aligned}\operatorname{cl}_{r(\tau)}(a) &= -\bigcup \Big\{ m \in \operatorname{RegOp}(\tau) : m \cap \operatorname{cl}_\tau(a) = \emptyset \Big\} \\ &\supseteq -\bigcup \Big\{ m \in \tau : m \cap \operatorname{cl}_\tau(a) = \emptyset \Big\} = cl_\tau(a).\end{aligned}$$

Let $a \in \operatorname{RegOp}(\tau)$. Then,

$$\begin{aligned}\operatorname{int}_{r(\tau)} \operatorname{cl}_{r(\tau)}(a) = \ & \operatorname{int}_{r(\tau)} \Big(-\bigcup \{ m \in \operatorname{RegOp}(\tau) : m \cap \operatorname{cl}_\tau(a) = \emptyset \} \Big), \\ = \ & \bigcup \{ t \in \operatorname{RegOp}(\tau) : t \cap m = \emptyset \textit{ for all } m \in \operatorname{RegOp}(\tau) \\ & \textit{with } m \cap \operatorname{cl}_\tau(a) = \emptyset \}, \\ = \ & a,\end{aligned}$$

since a and t are regular open, and thus, $t \subseteq \operatorname{cl}_\tau(a)$ implies $t \subseteq a$.

Conversely, let $a \in \operatorname{RegOp}(r(\tau))$. Then,

$$a = \operatorname{int}_{r(\tau)} \operatorname{cl}_{r(\tau)}(a) = \bigcup \{ t \in \operatorname{RegOp}(\tau) : t \subseteq \operatorname{cl}_{r(\tau)}(a) \}.$$

Now, $\operatorname{int}_\tau \operatorname{cl}_\tau(a) \in \operatorname{RegOp}(\tau)$, and thus,

$$a \subseteq \operatorname{int}_\tau \operatorname{cl}_\tau(a) \subseteq \operatorname{int}_{r(\tau)} \operatorname{cl}_{r(\tau)}(a) = a.$$

If $a \in \operatorname{RegOp}(\tau)$, then, by the preceding consideration, $-_\tau a = -_{r(\tau)} a$, and thus, $\operatorname{cl}_\tau(a) = \operatorname{cl}_{r(\tau)}(a)$. This implies the claim. ■

2.4.3 Closure Algebras and Topologies.

The study of topologies via the closure or interior operator is sometimes called *pointless topology*, see, for example, Johnstone [54]. Already in 1944, McKinsey and Tarski [75] showed that the closure algebras (as specified by Definition 2.7) give rise to the collection of closed sets of a topological space, by proving the following representation theorem:

Proposition 2.7. *[75]*

i) *If $\langle X, \tau \rangle$ is a topological space, then $\langle 2^X, \mathrm{cl} \rangle$ is a closure algebra, called the* closure algebra over *$\langle X, \tau \rangle$.*

ii) *If $\langle B, f \rangle$ is a closure algebra, then there is some T_1 space $\langle X, \tau \rangle$ such that $\langle B, f \rangle$ is a subalgebra of $\langle 2^X, \mathrm{cl} \rangle$.*

Dual statements holds for interior algebras and the topological int operator:

Proposition 2.8.

i) *If $\langle X, \tau \rangle$ is a topological space, then $\langle 2^X, \mathrm{int} \rangle$ is an interior algebra, called the* interior algebra over *$\langle X, \tau \rangle$.*

ii) *If $\langle B, g \rangle$ is an interior algebra, then there is some T_1 space $\langle X, \tau \rangle$ such that $\langle B, g \rangle$ is a subalgebra of $\langle 2^X, \mathrm{int} \rangle$.*

2.4.4 Heyting Algebras and Topologies.

Another way of looking at these algebras is via a certain class of lattices: An algebra $\langle A, +, \cdot, \Rightarrow, \mathbf{0}, \mathbf{1} \rangle$ of type $\langle 2, 2, 2, 0, 0 \rangle$ is called a *Heyting algebra* (or *pseudo–Boolean algebra [89]*) if $\langle A, +, \cdot, \mathbf{0}, \mathbf{1} \rangle$ is a bounded lattice, and $\Rightarrow$ is the operation of *relative complementation*: If $a, b \in A$, then

(2.33) $$a \Rightarrow b \text{ is the largest } x \in A \text{ for which } a \cdot x \leq b.$$

In other words,

(2.34) $$a \cdot x \leq b \text{ if and only if } x \leq a \Rightarrow b.$$

If $a \in A$, then its *pseudocomplement* a^* is the element $a \Rightarrow \mathbf{0}$, i.e. a^* is the largest $x \in A$ for which $a \cdot x = 0$. Heyting algebras form an equational class — i.e. a collection of algebras defined by a set of universally quantified equations (for details see [89] or [6]). Furthermore, if $\langle B, g \rangle$ is an interior algebra, then the collection $O(B)$ of its open sets forms a Heyting algebra with $\Rightarrow$ defined as

(2.35) $$a \Rightarrow b = g(-a + b).$$

In view of Proposition 2.8 we now have the following Representation Theorem [75]:

Proposition 2.9. *For each Heyting algebra A, there exists a T_1 space X such that A is isomorphic to a subalgebra of the Heyting algebra of open sets of X.*

2.5 Proximity Spaces

Proximities were introduced by Efremovič [34] in the early 1950s. The intuitive meaning of a proximity Δ is that $x\Delta y$ holds for some $x, y \subseteq X$, when x *is close to* y in some sense. Their axiomatisation is very similar to that of Boolean contact algebra to be discussed in Section 4. The main source on proximity spaces is the monograph by Naimpally and Warrack [77].

From the point of view of this investigation proximity spaces play a very useful role. On the one hand, the proximity approach is close to that of point set topology, and mappings between proximity spaces and corresponding topological spaces are well established. On the other hand the formulation of proximity spaces is based on a binary relation between point sets, whose meaning can be correlated with the *contact* relation that is taken as a primitive in many axiomatic and algebraic approaches to representing topological relationships between regions (which will be considered further in Section 3 below). Hence, proximity spaces provide a link between these axiomatic or algebraic formulations and point-set topological models of space.

Formally, a binary relation Δ on the powerset of a set X is called a *proximity*, if it satisfies the following axioms for $x, y, z \subseteq X$:[6]

(**P**1) If $x \cap y \neq \emptyset$ then $x\Delta y$.

(**P**2) If $x\Delta y$ then $x, y \neq \emptyset$.

(**P**3) Δ is symmetric.

(**P**4) $x\Delta(y \cup z)$ if and only if $x\Delta y$ or $x\Delta z$.

(**P**5) If $x(-\Delta)y$ then $x(-\Delta)z$ and $y(-\Delta) - z$ for some $z \subseteq X$.

Definition 2.14. The pair $\langle X, \Delta\rangle$ is called a *proximity space*.

[6]Sometimes the term *proximity space* has been used to include structures that do not satisfy axiom **P(P5)** . Those satisfying **P(P5)** are sometimes called *Efremovič* proximity spaces [34].

Definition 2.15. A proximity is called *separated* if it satisfies

$(\mathbf{P}_{\text{sep}})$ $$\{\alpha\}\Delta\{\beta\} \text{ implies } \alpha = \beta$$

Thus, in a separated proximity space, no two distinct singleton sets are related by the proximity relation.

2.5.1 The Topology Associated with a Proximity Space.

Each proximity space determines a topology on X in the following way: we take the closure of any set x as the set of all points α, such that $\{\alpha\}$ is proximal to x:

$$\text{cl}(x) = \{\alpha \in X : \{\alpha\}\Delta x\}. \tag{2.36}$$

Proposition 2.10. *[77]*

i) *The operation of* (2.36) *defines the closure operator of a topology* $\tau(\Delta)$ *on* X *(which is not necessarily* T_1*).*

ii) $\langle X, \tau(\Delta)\rangle$ *is a completely regular space.*

iii) *If* Δ *is separated, then* $\langle X, \tau(\Delta)\rangle$ *is a* T_1 *space.*

iv) $x\Delta y$ *if and only if* $\text{cl}(x)\Delta\,\text{cl}(y)$.

A proximity which is relevant to our investigation is the *standard proximity* on a normal T_1 space X [77]: For $x, y \subseteq X$, let

$$x\Delta y \iff \text{cl}(x) \cap \text{cl}(y) \neq \emptyset. \tag{2.37}$$

Observe that Δ is separated, since X is T_1 and thus, singletons are closed.

3. Contact Relations

The relation of 'contact' is fundamental to the spatial description of configurations of objects or regions. Contact relations have been studied in the context of qualitative approaches to geometry going back as far as the work of [23, 79, 112] and subsequently of [16]. More recently the idea of considering contact[7] relation has been studied in the field of Qualitative Spatial Reasoning [13, 20, 84, 85, 88, 97] (see also Section 5.1 below. This has emerged as a significant sub-field of Knowledge

[7]In AI and Qualitative Spatial Reasoning, the contact relation is often called 'connection'. In the present work we use contact to avoid confusion with the slightly different notion of 'connection' employed in topology.

Representation, which is itself a major strand of research in Artificial Intelligence.

The contact relation can be seen as a weaker and more fine-grained cousin of the "overlap relation", which is straightforwardly defined[8] the "part of" relation. The properties of this relation were first formalised by Leśniewski [65], as the basic relation of his *Mereology* [see also 67].

Definition 3.1. *A* contact relation C *is a relation satisfying the following axioms:*

(**C**1)	$\forall x[\ xCx\ ]$	*Reflexivity,*[9]
(**C**2)	$\forall xy[\ xCy \rightarrow yCx\ ]$	*Symmetry,*
(**C**3)	$\forall xy[\ \forall z[\ zCx \leftrightarrow zCy\ ] \rightarrow x = y\ ]$	*Extensionality.*

These axioms correspond to axioms A0.1 and A0.2 given by Clarke [16] for the mereological part of his calculus of individuals.

Our main interest will be contact relations which are defined on open or closed sets of a topological space. Primary examples are collections $\mathfrak{M}$ of nonempty regular closed (or regular open) sets of some topological space X.

If we identify regions with elements of $\mathrm{RegCl}(X)$, it is natural to define C as the relation that holds just in case two regions share at least one point:

$$xCy \iff x \cap y \neq \emptyset, \tag{3.1}$$

Whereas, if our domain of regions is $\mathrm{RegOp}(X)$, it is usual to define C as holding whenever the *closures* of two regions share a point.

$$xCy \iff \mathrm{cl}(x) \cap \mathrm{cl}(y) \neq \emptyset\ . \tag{3.2}$$

It is easy to see that these interpretations fulfil the contact relation axioms **C1–3**. In the sequel, they will be called *the standard contact relations on* $\mathrm{RegCl}(X)$ *and* $\mathrm{RegOp}(X)$ *respectively.*

It is often useful to consider contact relations over other, more specific domains. Take, for example, the set D of all closed disks in the Euclidean plane, and define C by (3.1). Then, C obviously is a contact relation on D.

When describing properties of the C relation, it is often convenient to refer to the set of all regions connected to a given region. Thus, we

[8] $xOy \ \equiv_{def}\ \exists z[\ zPx \wedge zPy\ ]$.

[9] In theories whose domain includes an 'empty' region, this axiom is normally weakened to $\forall x[x = \emptyset \vee xCx]$.

define

$$C(x) \equiv_{def} \{y \mid xCy\} \tag{3.3}$$

In terms of this notation, the extensionality axiom can be stated as:

$$C(x) = C(y) \Longleftrightarrow x = y. \tag{3.4}$$

Many other useful relations can be defined in terms of contact (see [16, 88] and 5.1 below). A particularly important definable relation is that which is normally interpreted as the *part* relation:

$$xPy \equiv_{def} \forall z[\ zCx \to zCy\ ] \tag{3.5}$$

This definition (by itself) ensures that P is reflexive and transitive — i.e. it is a *pre-order*. And if we assume the extensionality of C (i.e. **C3**) it can be proved that P is antisymmetric, so that it must be a *partial order*.

The C relation is a very expressive primitive for defining topological relationships between regions. In terms of C the following useful relations can be defined. These definitions have been used to define the relational vocabulary of the well-known *Region Connection Calculus*, which will be discussed further in Section 5.1 below.

(3.6) $xPPy \equiv_{def} xPy \wedge \neg yPx$ — x is a Proper Part of y

(3.7) $xOy \equiv_{def} \exists z[zPx \wedge zPy]$ — x Overlaps y

(3.8) $xDRy \equiv_{def} \neg xOy$ — x is DiscRete from y

(3.9) $xDCy \equiv_{def} \neg xCy$ — x is disconnected from y

(3.10) $xECy \equiv_{def} xCy \wedge \neg xOy$ — x is Externally Connected to y

(3.11) $xPOy \equiv_{def} xOy \wedge \neg xPy \wedge \neg yPx$ — x Partially Overlaps y

(3.12) $xEQy \equiv_{def} xPy \wedge yPx$ — x is Equal to y[10]

(3.13) $xTPPy \equiv_{def} xPPy \wedge \exists z[zECx \wedge zECy]$ — x is a Tangential Proper Part of y

(3.14) $xNTPPy \equiv_{def} xPPy \wedge \neg\exists z[zECx \wedge zECy]$ — x is a Non-Tangential Proper Part of y

(3.15) $xTPPI\,y \equiv_{def} yTPPx$ — x is an Inverse Tangential Proper Part of y

(3.16) $xNTPPI\,y \equiv_{def} yNTPPx$ — x is an Inverse Non-Tangential Proper Part of y

It should be noted that for the defined relations to have their intuitive meaning, one should not include in the domain a ‘null’ region that is not connected to any other region. If such a null region is present, it would be be part of every other region. Consequently xOy would hold for all x and y and other relations defined in terms of O would also have un-intuitive interpretations.

Under typical interpretations of the C relation (not including a null region in the domain), the relations defined by (3.9)–(3.16) form a jointly exhaustive and pairwise-disjoint partition of possible relations between any two spatial regions (i.e. every two regions satisfy exactly one of the relations). This set of eight relations introduced in [88] is often known as RCC-8, and is widely referred to in the AI literature on Qualitative Spatial Reasoning (see also section 5.1 below).

3.1 Contact Relation Algebras

If C is taken to be a relation in a relation algebra, the properties **C1–3** of the contact relation correspond to the following relation algebraic conditions:

(**CRA1**)	$1' \leq C$,	Reflexivity,
(**CRA2**)	$C = C^{\smile}$,	Symmetry,
(**CRA3**)	$[-(C\ ;\ -C) \cap -(C\ ;\ -C)^{\smile}] \leq 1'$,	Extensionality.

Definition 3.2. *A relation algebra generated from a single relation C satisfying conditions* **CRA1–3** *will be called a* contact relation algebra *(CRA).*

Contact Relation Algebras were introduced and studied in [29], where many fundamental properties are demonstrated. CRAs provide a rich language within which many other useful topological relations can be defined. In the relation algebra setting, the part relation has the following definition:

$$P \equiv_{def} -(C\ ;\ -C) \tag{3.17}$$

Many other relations are relationally definable from C. Indeed all the relations that were defined above using first-order logic can also be

[10] In the presence of the extensionality axiom, this is equivalent to simply $x = y$.

defined using the algebraic operators of relation algebra:

(3.18)	PP	$=_{def} P \cap -1'.$	proper part of
(3.19)	O	$=_{def} P^{\smile} \,;\, P$	overlap
(3.20)	DR	$=_{def} -O$	discrete
(3.21)	DC	$=_{def} -C$	disconnected
(3.22)	EC	$=_{def} C \cap -O$	external contact
(3.23)	PO	$=_{def} O \cap -(P \cup P^{\smile})$	partial overlap
(3.24)	EQ	$=_{def} (P \cup P^{\smile}) \quad (= 1')$	equality
(3.25)	TPP	$=_{def} PP \cap (EC \,;\, EC)$	tangential proper part
(3.26)	$NTPP$	$=_{def} PP \cap -TPP$	non-tangential proper part
(3.27)	$TPPI$	$=_{def} TPP^{\smile}$	tangential proper part inv.
(3.28)	$NTPPI$	$=_{def} NTPP^{\smile}$	non-tang'l proper part inv.

In view of Proposition 2.4, this comes as no surprise, since RAs capture exactly those first order properties of C that can be expressed with up to three variables, and this is sufficient for all the definitions given above.

Depending on the base set, some of these relations might be empty or coincide. If, for example, B is a BA, and $xCy \iff x \cdot y \neq \mathbf{0}$, then C coincides with the overlap relation, and $EC = \emptyset$. A picture of some of these relations over the domain D of (non–empty) closed disks is given in Figure 3.1.

It turns out that the relations

$$(3.29) \qquad 1', DC, PO, EC, TPP, TPP^{\smile}, NTPP, NTPP^{\smile}$$

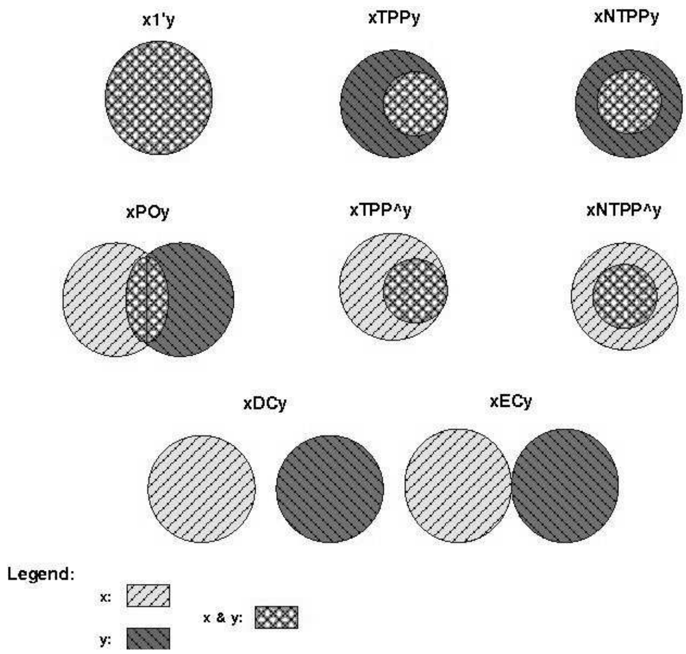
are the atoms of the relation algebra $\mathcal{D}_c$ generated by C over D, henceforth called the *(closed) disk relations.* (The 'composition table' for the RCC-8 relations over the domain $\mathcal{D}_c$ will be given in Table 6.2 below.)

4. Boolean Contact Algebras

While the contact relations of Section 3 did not assume a particular algebraic structure on the base set, we will often be interested in cases where the set of regions has further structure; and, in particular, we will often want to consider the set of regions as having the structure of a Boolean algebra.

A first order theory intended to model topological properties of regions, the *region connection calculus* (RCC), has been introduced by Randell et al. [88] in 1992, and has since gained popularity in the spatial

Figure 3.1. Topological Relations on the Domain of Closed Discs



reasoning community; we will examine the RCC more closely in Section 5.1. First, we will consider a more general class of structures:

Definition 4.1. *A* Boolean contact algebra *is a pair* $\langle B, C\rangle$*, such that* B *is a non–trivial (i.e.* $\mathbf{0} \neq \mathbf{1}$*) Boolean algebra, and* C *is a binary relation on* B^+*, called a* contact relation*, with the following properties:*

BCA0) $aCb \Rightarrow a, b \neq \mathbf{0}$

BCA1) $a \neq \mathbf{0} \Rightarrow aCa$

BCA2) C is symmetric.

BCA3) aCb and $b \leq c \Rightarrow aCc$ (The compatibility axiom)

BCA4) $aC(b+c) \Rightarrow aCb$ or aCc (The sum axiom)

While axioms **BCA0–4** characterise the properties of Boolean contact algebras in general, we shall often be interested in BCAs that satisfy additional axioms. In particular, we shall be interested in the following axioms:

BCA5) $C(a) \subseteq C(b) \Rightarrow a \leq b$ (Extensionality)

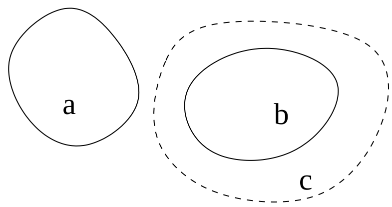


Figure 4.2. Illustration of the Interpolation Axiom

BCA6) $a(-C)b \Rightarrow (\exists c)[a(-C)c \text{ and } -c(-C)b]$ (Interpolation)

BCA7) $a \notin \{\mathbf{0}, \mathbf{1}\} \Rightarrow aC - a$ (Connection)

A BCA which satisfies **BCA5** and **BCA7** will be called an *RCC algebra*, since these axioms are satisfied by the 1st-order Region Connection Calculus theory proposed by [88] (which will be considered in further detail in Section 5.1 below).

Clearly, C is a contact relation in the sense of Section 3, and therefore, all relations specified by the definitional formulae (3.5)–(3.14) are at our disposal. It is easy to see that

(4.1) $\textbf{BCA5} \iff P$ is the Boolean order,

(4.2) $\textbf{BCA6} \iff \forall(x,y)(\exists z)[xNTPPz \wedge zNTPPy]$,

(4.3) $xOy \iff x \cdot y \neq \mathbf{0}$.

Simple structural properties include

Proposition 4.1. *Let $\langle B, C\rangle$ be a BCA.*

i) *[31] O is the smallest contact relation on B.*

ii) *[31] If B is a finite–cofinite algebra, then O is the only contact relation on B.*

iii) *[30] If C satisfies* **BCA7**, *then B is atomless.*

4.1 Interpretations of BCAs

As intended, the regions and relations of the BCA theory can be interpreted in terms of classical point-set topology. In fact, there are two dual interpretation that are equally reasonable.

Closed Interpretation:

- A *region* is identified with a regular closed set of points.
- Regions are *connected* if they share at least one point.

- Regions *overlap* if their *interiors* share at least one point.

Open Interpretation:

- A *region* is identified with a regular open set of points.
- Regions are *connected* if their *closures* share at least one point.
- Regions *overlap* if they share at least one point.

The axioms for C translate into topological properties as follows:

Proposition 4.2 (Properties of standard contact on a topological space)**.** *[32] Suppose that $\langle X, \tau \rangle$ is a topological space, and that C_τ is the standard contact relation on* $\mathrm{RegCl}(X)$*.*

i) *C_τ satisfies* **BCA0–4***.*

ii) *C_τ satisfies* **BCA5** *if and only if X is weakly regular.*

iii) *C_τ satisfies* **BCA6** *if and only if X is weakly normal.*

iv) *C_τ satisfies* **BCA7** *if and only if X is connected.*

In fact the BCA axioms are also satisfied by dense subalgebras of $\mathrm{RegCl}(X)$. Hence, proposition 4.2 can be generalised:

Proposition 4.3. *Suppose that $\langle X, \tau \rangle$ is a topological space, and that C_τ is the standard contact relation on some* dense sub-algebra *of* $\mathrm{RegCl}(X)$*; then each of the clauses i–iv of proposition 4.2 are true for C_τ.*

The preceding propositions give us many examples of BCAs. We would like to mention a countable example of a BCA which is, in some sense, one dimensional; in particular, this algebra is not complete.[11] Suppose that L is the ordered set of non–negative rational numbers enhanced by a greatest element ∞. Let B be the collection of all finite unions of left–closed, right–open intervals of L, together with the empty set. It is well known [61] that B is a Boolean subalgebra of 2^L, called the *interval algebra of* L, and that each $a \in B^+$ has a unique representation as

$$a = [x_0, y_0) \cup \ldots \cup [x_n, y_n), \tag{4.4}$$

where $x_0 \lneq y_0 \lneq x_1 \lneq y_1 \lneq \ldots \lneq x_n \lneq y_n$. The set $\{x_i : i \leq n\} \cup \{y_i : i \leq n\}$ is called the set of *relevant points of* a, denoted by $\mathrm{rel}(a)$. If we

[11] I.e. it does not contain infinite sums of its elements.

define C on B^+ by

$$aCb \iff (a \cap b) \cup (\mathrm{rel}(a) \cap \mathrm{rel}(b)) \neq \emptyset, \tag{4.5}$$

then $\langle B, C\rangle$ is a BCA which satisfies **BCA6** and **BCA7** [31]; other constructions of countable BCAs can be found in [70]. In Sections 5 and 5.1 we will present BCAs arising from spatial theories.

We now exhibit some constructions that allow us to obtain new BCAs from old (these were described in [31]):

Proposition 4.4 (Adding an *ultra-contact*)**.** *Given any atomless BCA $\langle B, C\rangle$ it is possible to augment the connection relation by picking any two ultrafilters F and G of the algebra and stipulating that $C(f, g)$ for any two regions f and g, where $f \in F$ and $g \in G$. In formal terms this means that $\langle B, C'\rangle$ is a BCA where*

$$C' = C \cup (F \times G) \cup (G \times F)\ . \tag{4.6}$$

More generally, for a contact relation C, let $R_C = \{\langle F, G\rangle : F \times G \subseteq C\}$, and, for a reflexive and symmetric relation R on $\mathrm{Ult}(B)$, set $C_R = \bigcup\{F \times G : \langle F, G\rangle \in R\}$.

Proposition 4.5. *1 [28] C_R satisfies* **BCA0–4***.*

2 [33] If R is a reflexive and symmetric relation on Ult(B) which is closed in the product topology of Ult$(B) \times$ Ult(B), then C_R satisfies **BCA0–4***.*

3 [33] The collection of all relations on B that satisfy **BCA0–4** *can be made into an atomistic complete co–Heyting algebra in which join is set union.*

Proposition 4.6 (Restriction and Extension with respect to a Dense Subalgebra)**.** *If A is a dense subalgebra of B, then the restriction of C to A is a contact relation on A which satisfies* **BCA7** *if B does.*

If B is a dense subalgebra of A, then the relation C' defined on A by

$$aC'b \iff (\forall s, t \in B)[a \leq s \text{ and } b \leq t \Rightarrow sCt]$$

is a contact relation on A, and, if C satisfies **BCA7***, so does C'. Furthermore, C' is the largest contact relation on A whose restriction to B is C.*

4.2 Representation Theorems for BCAs

Theorems that characterise the class of models of a given axiomatic theory are know as *representation theorems.* In most cases, such theorems are sought after for one (or both) of the following reasons:

a) to find an axiomatisation for a given class of structures,

b) to show that a given axiom system is complete for an intended class of models.

Famous representation results include Cayley's theorem that every group is isomorphic to a group of permutations, and Stone's theorem which shows that each Boolean algebra is isomorphic to an algebra of sets. If an axiom system has models outside an intended class of models, the existence of such non-standard models shows that the system is incomplete with respect to that intended class. In the sequel, we will exhibit both positive and negative representation results for contact relations in topological spaces.

Apart from the earlier topological representation results of Roeper [93] and Mormann [76], which do not result in the standard topological contact, the first "standard" representation result for a class of contact algebras was discovered by Vakarelov et al. [106]. It utilises the theory of proximity spaces which have been briefly described in Section 2.5. Subsequently, making use of similar techniques, topological representation results were obtained for BCAs [32].

4.2.1 Constructing a Topology to Represent a BCA.

The proof of the representation result takes a form similar to that of Stone's theorem. The plan is to devise a way to use the elements of a BCA to construct entities that can be correlated with points in a topological or proximity space. However, instead of taking ultrafilters as the base set for the topology (as is done in Stone's theorem), a somewhat different construction is required to generate suitable sets of regions that can be identified with 'points' in a proximity space or topological model.

We begin with the following definition:

Definition 4.2. *A non–empty subset* Γ *of* B *is called a* clan *if, for all* $x, y \in B$*, we have:*

CL1) *If* $x, y \in \Gamma$ *then* xCy.

CL2) *If* $x + y \in \Gamma$ *then* $x \in \Gamma$ *or* $y \in \Gamma$.

CL3) *If* $x \in \Gamma$ *and* $x \leq y$*, then* $y \in \Gamma$.

A clan can be regarded as a set of regions which share at least one point of mutual contact. The difference from a Boolean filter arises because regions may share a point of contact even though their intersection is empty. Moreover, as is illustrated in Figure 4.3, even where regions do

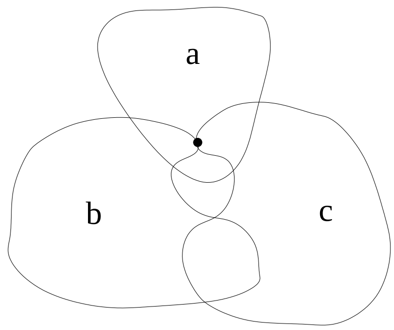


Figure 4.3. Illustration of why clans are not closed under intersection.

have a non-empty intersection, the regions may have a point of contact that is not in this intersection.

Definition 4.3. *A clan Γ that is maximal (i.e. there is no clan Γ' such that $\Gamma \subsetneq \Gamma'$) will be called a* cluster*. The set of all clusters in B will be denoted by Clust(B). Clearly, every clan is contained in some cluster.*

Since clusters will represent points in a topological space, each region will be associated with a set of clusters. Hence, to construct the topological representation of a BCA, we need to find a suitable mapping from the elements of the BCA to sets of clusters. Again the construction is similar to that used in the Stone theorem.

We define a mapping $h : B \to 2^{\mathrm{Clust}(B)}$ by

$$h(a) = \{\Gamma \in \mathrm{Clust}(B) : a \in \Gamma\}, \tag{4.7}$$

In [32] it was shown that for any BCA with domain B, we can specify a topology $\langle \mathrm{Clust}(B), \tau_B \rangle$, determined by h. This is done by taking $\{h(x) : x \in B\}$ as a basis for the closed sets of $\langle \mathrm{Clust}(B), \tau_B \rangle$. In other words the open sets τ_B are arbitrary unions of sets whose complements are in the range of h:

$$\tau_B = \left\{ \bigcup \{\mathrm{Clust}(B) \backslash h(x) : x \in S\} : S \subseteq B \right\}$$

Lemma 4.1. *The following properties of $\langle$Clust$(B), \tau_B\rangle$ were demonstrated in [32]:*

i) *The range of $h(x)$ for $x \in B$ is a dense subalgebra $\mathcal{A}_B$ of the regular closed algebra over $\langle$Clust$(B), \tau_B\rangle$.*

ii) *h preserves the Boolean structure of B in $\mathcal{A}_B$ (i.e. h is a Boolean homomorphism from B to $\mathcal{A}_B$).*

iii) For all $a, b \in B$, aCb *if and only if* $h(a) \cap h(b) \neq \emptyset$.

iv) $\langle Clust(B), \tau_B \rangle$, *is a weakly regular* T_1 *topology (which is not necessarily* T_2*),*

Together, these properties give us the following representation theorem:

Proposition 4.7. *Each BCA* $\langle B, C \rangle$ *is isomorphic to a dense substructure of some regular closed algebra* $\langle \mathrm{RegCl}(X), C_\tau \rangle$*, where* τ *is a weakly regular* T_1 *topology, and* C *is the restriction of* C_τ *to* B*.*

Moreover, from propositions 4.2 and 4.3, we immediately have the following result which tells us that the correspondence is bijective:

Proposition 4.8. *If* $\langle X, \tau \rangle$ *is a weakly regular* T_1 *space, and* B *is a dense subalgebra of* $\mathrm{RegCl}(X)$ *with* C *being the restriction of the standard contact on* $\mathrm{RegCl}(X)$*, then* $\langle B, C \rangle$ *is a BCA.*

As a consequence of this result we obtain

Proposition 4.9. *The axioms of BCAs are complete with respect to the class of substructures of regular closed algebras of weakly regular* T_1 *spaces with standard contact.*

4.2.2 The Extensionality Axiom.

The theorems stated in the last section concern BCAs satisfying the axioms **BCA0–5** — i.e. the general BCA theory together with the extensionality axiom. For certain purposes, in particular the modelling of discrete space, one may wish to remove the extensionality condition [41]. The resulting very general BCAs will not be considered further here; however, a representation theorem (in terms of atomic algebras over proximity spaces) is given in [28].

4.2.3 The Connection Axiom.

The connection axiom, **BCA7**, states that every region, except 0 and 1, is connected to its own complement:

$$a \notin \{\mathbf{0}, \mathbf{1}\} \Rightarrow aC-a$$

Suppose **BCA7** is false for an RCA with domain B; then there are regions $a, b \in B$ such that $a, b \neq 0$, $a + b = 1$ and $a - Cb$. Because the mapping h preserves Boolean identities and the contact relation, we must have regular closed regions $h(a)$ and $h(b)$ in $\langle \mathrm{Clust}(B), \tau_B \rangle$ such that $h(a) + h(b) = \mathrm{Clust}(B)$ and $h(a) \cap h(b) = \emptyset$. Therefore, $\langle \mathrm{Clust}(B), \tau_B \rangle$ must be a disconnected space.

Conversely, it can be shown that if $\langle \text{Clust}(B), \tau_B \rangle$ is a connected topological space, then the BCA, B must satisfy the axiom **BCA7**. This is a bit more difficult to demonstrate[12] but is proved in [32, 32]. Thus we have the following representation theorem for RCC algebras — i.e. BCAs satisfying axioms **BCA0–5** and **BCA7**:

Proposition 4.10. *The axioms for RCC algebras are complete with respect to the class of substructures of regular closed algebras of* connected *weakly regular T_1 spaces with standard contact.*

4.2.4 Saturated Clusters and the Interpolation Axiom.

We now consider the effect of the Interpolation axiom **BCA6**. Recall that this is the condition

$$x(-C)y \Rightarrow (\exists z)[x(-C)z \wedge \ -z(-C)y] \ .$$

This is a separation condition ensuring that for any two disconnected regions in the algebra, we can find a third region disconnected from the first and including the second as a non-tangential part.

We shall later see that we can establish a correspondence between BCAs satisfying **BCA6** and proximity spaces. In order to do this we show that in the presence of this condition, the clusters derived from the algebra exhibit a property called *saturation*, which results leads to a natural 'well-behaved' structure of the set of clusters.

Definition 4.4. *A clan is called* saturated *iff it satisfies the following condition:*

(P) If xCy for every $y \in \Gamma$, then $x \in \Gamma$.

If a clan Γ over B is saturated then for any $x \in B$ such that $x \notin \Gamma$ there is some $y \in \Gamma$ such that $\neg(xCy)$. Therefore, $\Gamma \cup \{x\}$ is not a clan. So Γ must be a maximal clan. Thus we have the following lemma [32]:

Lemma 4.2. *Every saturated clan is a cluster.*

In formulating the proximity representation theorem for BCAs, clusters corresponding to saturated clans will be will be taken as the points of a proximity space. Thus we use the following terminology:

Definition 4.5. *A cluster that is a saturated clan will be called a* proximity cluster*, or more briefly a p-cluster.*

---

[12]Since elements of the BCA form only a dense subalgebra of $\text{RegCl}(\langle \text{Clust}(B), \tau_B \rangle)$, we cannot necessarily associate an arbitrary regular closed subset of $\text{Clust}(B)$ with an element of the BCA from which the topology was constructed. This means that mapping topological constraints to BCA axioms often requires detailed analysis of the cluster construction.

Intuitively, each p-cluster can be interpreted as the set of all regions in the BCA that contain a particular point in a corresponding proximity space. However, for BCAs in general, not every cluster need be a p-cluster. The following example of a BCA which includes clusters that are not p-clusters is given in [32]:

Suppose that B is the interval algebra whose elements are finite unions of left closed, right open intervals, $[x, y)$ on the rational unit interval $[0, 1)$. Let $C(i, j)$ hold between elements just in case their closures share a point. Now, let a, b be points such that $0 < a < b < 1$. If F_a is the ultrafilter of B of all sets containing a, and F_b is the ultrafilter of B of all sets containing b, then, by Proposition 4.4, the relation $C' = C \cup (F_a \times F_b) \cup (F_b \times F_a)$ is a contact relation over B, and it can be shown that $\Gamma = F_a \cup F_b$ is a cluster. However, if $s \lneq a \lneq t \lneq b$, and $x = [s, a) \cup [t, b)$, then $\{x\} \times \Gamma \subseteq C'$ (i.e. x is connected to every member of Γ). But, neither $[s, a)$ nor $[t, b)$ is in Γ, so (because clans must satisfy $CL2$) we must have $x \notin \Gamma$.

Let us see how this anomaly arose. By adding the ultra-contact between points a and b we stipulated that every region containing point a is in the C' contact relation with every region containing point b. But, in this algebra, contact also holds between regions that do not share a point, but whose closures share a point. However, the relation C' does not necessarily hold between intervals i, j such that the closure of i includes a and the closure of j includes b. This mismatch leads to a kind of discontinuity in the contact relation C' relative to the underlying topology of the interval algebra.

Lemma 4.3. *[32] If $\langle B, C\rangle$ satisfies* **BCA6***, then each cluster is a p-cluster.*

In order to see why **BCA6** ensures that all clusters are saturated we first give another useful lemma:

Lemma 4.4. *For every region r and cluster Γ, $r \in \Gamma$ if and only if for any set of regions $S = \{r_1, \ldots, r_n\}$ such that $r \leq r_1 + \ldots + r_n$, there is a region $r_i \in S$ such that $(\forall x \in \Gamma)[r_i C x]$.*

Proof Sketch: Since clusters are maximal clans then, for any cluster Γ, if $\Gamma \cup \{r, \ldots\}$ satisfies *CL1–3* then $r \in \Gamma$. Moreover, to show that $r \in \Gamma$ it suffices to show that $\Gamma \cup \{r\}$ satisfies *CL1-2*, since then $\Gamma \cup \{x : x \geq r\}$ clearly satisfies *CL1-3*. It can be shown that $\Gamma \cup \{r\}$ satisfies *CL1-2* just in case for every sum $(r_1 + \ldots + (r_n + r_{n+1})) = r$ there is some r_i such that $(\forall x \in \Gamma)[x C r_i]$, and this implies the lemma.

Using this, we can prove Lemma 4.3 as follows:

Proof: Let $\langle B, C\rangle$ be a BCA satisfying **BCA6**. Let Γ be a cluster derived from this algebra and r a region such that $\forall x \in \Gamma[x C r]$. Suppose, in contradiction to Lemma 4.3 that $r \notin \Gamma$. Then, by Lemma 4.4, there are $r_1, \ldots, r_n$, with $r \leq r_1 + \ldots + r_n$, such that for each r_i there is some $x_i \in \Gamma$ with $r_i - Cx_i$. Then by **BCA6** there are regions $s_1, \ldots, s_n$, such that $s_i - Cx_i$ and $r_i - C - s_i$ (so each s_i contains r_i and separates it from x_i). Let $s = s_1 + \ldots + s_n$. Thus $r - C - s$. Now pick any region $y \in \Gamma$. Clearly $y = y_1 + \ldots + y_n + z$, where $y_i = y \cdot s_i$ and $z = y \cdot -s$. Because of *CL2* we must have either $z \in \Gamma$ or some $y_i \in \Gamma$. But since $y_i = y \cdot s_i$ and $s_i - Cx_i$ we have $y_i - Cx_i$; so $y_i \notin \Gamma$ (because of *CL1*). Thus we must have $z \in \Gamma$. However, since $z = y \cdot -s$, we have $z \leq -s$ and because $r - C - s$ we have $r - Cz$. But this contradicts the premiss that $\forall x \in \Gamma[xCr]$. Hence the supposition that $r \notin \Gamma$ is impossible, so we have proved Lemma 4.3.

4.2.5 Representation in Proximity Spaces.

As noted above, in Section 2.5, proximity spaces form a useful intermediary between topological spaces and axiomatic theories based on a contact relation, which has analogous properties to the proximity relation. Indeed contact can be regarded as a limiting case of proximity.

The theory of proximity spaces and their relation to topological spaces has been developed in detail in the seminal work of Naimpally and Warrak [77]. This analysis makes heavy use of a notion of *cluster*, which is very similar to (and was the inspiration for) the cluster construct for BCAs given above. Because proximity spaces satisfy axiom (**P**5) , the clusters employed in [77] are saturated. Hence, in the case of BCAs satisfying **BCA6**, many of the results of [77] can be used to demonstrate correspondences between BCAs, proximity spaces and topologies.

We first consider how we can derive a BCA from a proximity space:

Proposition 4.11. *Let $\langle X, \Delta \rangle$ be a proximity space with associated topology $\tau(\Delta)$, and* $\text{RegCl}(X)$ *be the regular closed subsets of X according to the topology $\tau(\Delta)$. Then the algebra $\langle \text{RegCl}(X), \Delta \rangle$ is a BCA called the* proximity connection algebra over $\langle X, \Delta \rangle$.

Definition 4.6. $\langle \text{RegCl}(X), \Delta \rangle$ *is called a* standard proximity connection algebra*, if*

$$x \Delta y \textit{ iff } x \cap y \neq \emptyset, \textit{for all } x, y \in \text{RegCl}(X).$$

For our purposes, it suffices to consider only standard connection algebras. This is because of the following theorem:

Proposition 4.12. *[106] Each proximity connection algebra is isomorphic to a standard proximity connection algebra.*

It follows immediately from the proximity axioms and the BCA axioms, that each standard proximity connection algebra is a BCA that satisfies the interpolation axiom **BCA6** (corresponding to the proximity axiom (**P**5)). We will demonstrate in the remainder of this Section that, conversely, each BCA which satisfies **BCA6** can be embedded into a standard proximity connection algebra.

Given a BCA, $\langle B, C \rangle$, satisfying **BCA6**, our aim is to define a proximity on Clust(B). As with the representation in a topological space, the proximity space construction will again make use of *clusters* to represent points in the proximity space. Hence, each subset of the space will correspond to a set of clusters.

Since a cluster is interpreted as the set of regions containing a given point, the intersection of two clusters is the set of regions containing two points. More generally, given a set X of clusters representing a set of points, the common intersection $\bigcap X$ will be the set of all regions that contain all those points. Using this idea, we can for any BCA define a proximity relation between pairs of cluster sets, which corresponds to the contact relation of the BCA:

Definition 4.7. *For any BCA, $\langle B, C \rangle$ that satisfies* **BCA6***, we define a proximity relation over Clust(B) in the following way:*
for each $X, Y \subseteq$ Clust(B)

$$(\Delta_{\text{rep}}) \qquad X \Delta_B Y \textit{ iff } (\forall x, y \in B)[x \in \bigcap X \textit{ and } y \in \bigcap Y \textit{ imply } xCy].$$

Using this construction, the following lemma can be proved [105]:[13]

Lemma 4.5. *[77] $\langle$Clust(B)$, \Delta_B \rangle$ is a separated proximity space.*

Thus, the construction of clusters together with the definition of a proximity relation on sets of clusters enables us to derive a proximity space from any BCA satisfying **BCA6**. The structure $\langle \text{Clust}(B), \Delta_B \rangle$ can be regarded as a canonical representation of the BCA B in terms of a (separated) proximity space.

As with the topological representation, the correspondence between the regions of the original BCA and subsets of the derived proximity space can be specified by a function $h : B \to 2^{\text{Clust}(B)}$, defined by $h(a) = \{\Gamma \in \text{Clust}(B) : a \in \Gamma\}$. This mapping both preserves the Boolean structure of the BCA and also associates the contact relation of the BCA with the proximity relation of the proximity space.

[13]The proof of this is based on [77]

We have now shown that each BCA $\langle B, C\rangle$ is isomorphic to a standard proximity algebra over the proximity space $\langle \mathrm{Clust}(B), \Delta_B\rangle$. In Section 2.5.1 we saw that each proximity space is associated with a corresponding topology and the properties of this topology were characterised by Proposition 2.36. This means that we can use the proximity space derived from a BCA to define a corresponding topological space. This gives us the following topological representation theorem for BCAs satisfying the interpolation axiom:

Proposition 4.13. *[106] Each BCA which satisfies* **BCA6** *is isomorphic to a substructure of the regular closed algebra of a completely regular* T_1 *space* X *with standard contact as defined by* (3.1)*. Furthermore,* X *is connected if and only if* C *satisfies* **BCA7***.*

It should be noted that not every completely regular T_1 space is the representation space of a BCA which satisfies **BCA6**, since these spaces must be weakly normal (see **Proposition 4.2–3**), and there are spaces that are completely regular T_1, but not weakly normal [96]. We have, however:

Corollary 4.14. *The BCA axioms* **BCA0–6** *are complete with respect to the class of substructures of regular closed algebras of weakly normal* T_1 *spaces with standard contact.*

Corollary 4.15. *The BCA axioms* **BCA0–7** *are complete with respect to the class of substructures of regular closed algebras of weakly normal* connected T_1 *spaces with standard contact.*

5. Other Theories of Topological Relations

5.1 The Region Connection Calculus

The *Region Connection Calculus* (RCC) of Randell et al. [88] is an axiomatisation of certain spatial concepts and relations in classical 1st-order predicate calculus. It has become widely known in the field of Qualitative Spatial Reasoning, a research area within the Knowledge Representation filed of Artificial Intelligence. There is some variation in the full set of axioms used for the RCC theory. The formal apparatus of the original theory is complicated by the use of the many-sorted logic LLAMA [18] and the use of a non-standard definite description operator ($\iota x[\varphi(x)]$). This makes it difficult to make a direct comparison with the algebraically based theories presented in the current paper.

The RCC theory is based on a primitive relation C, which is in this context normally called the *connection* relation. This is axiomatised to be reflexive (**C1**) and symmetric (**C2**). The extensionality axiom (**C3**) is not given in the original RCC theory [88] and does not strictly follow from the other axioms (see [10, 97]). However, the theory does contain definition 3.12 for the EQ relation; and, if (as seems to have been assumed in some subsequent development of RCC) this is taken as coinciding with logical equality, then **C3** also holds. With this assumption, we have a contact relation in the sense defined in Section 3.

The RCC theory introduces further relations by means of the definitions (3.5)–(3.16) given above (Section 3), which include of course the RCC-8 relation set. The following axiom is given stipulating that every region has a non-tangential proper part:

RCC1) $\forall x \exists y [yNTPPx]$

However, as shown in [30], this follows from the other axioms, if we assume the extensionality axiom **C3**.

RCC also incorporates a constant denoting the *universal* region, a *sum* function and partial functions giving the *product* of any two overlapping regions and the *complement* of every region except the universe. With slight modification to the original to replace the partial product and complement functions with relations, these are defined as follows:

RCCD1) $x = \mathcal{U} \;\equiv_{def}\; \forall y[xCy]$

RCCD2) $x = y + z \;\equiv_{def}\; \forall w[wCx \leftrightarrow \; [wCy \vee wCz]]$

RCCD3) $\mathsf{Prod}(x, y, z) \;\equiv_{def}\; \forall u[uCz \leftrightarrow \exists v[vPx \wedge vPy \wedge uCv]]$

RCCD4) $\mathsf{Compl}(x, y) \;\equiv_{def}\; \forall z[(zCy \leftrightarrow \neg zNTPPx) \wedge (zOy \leftrightarrow \neg zPx)]$

It should be noted that within the original RCC theory there is no such thing as a *null* (or empty) region. Thus there is no product of discrete regions or complement of the universal region. This means we do not have a full Boolean algebra of regions; but, in order that appropriate regions exist to fulfil the requirements of the *quasi-Boolean* structure suggested by the above definitions, the basic RCC theory should be supplemented with the following existential axioms:

RCC2) $\forall xy[xOy \rightarrow \exists z[\mathsf{Prod}(x, y, z)]$

RCC3) $\forall x[\neg(x = \mathcal{U}) \leftrightarrow \exists y[\mathsf{Compl}(x, y)]$

The many-sorted formalisation of RCC and the choice to exclude the 'null region' from the domain of regions was motivated partly by a desire to accord with 'commonsense' notions of spatial reality (influenced by e.g. [49]) and partly by wanting to improve the effectiveness of automated reasoning using the calculus. However, from the point of view of establishing properties of the formal system, it has been found that the lack of a null region is problematic since it considerably complicates the comparison with standard mathematical structures such as Boolean algebras. Hence, subsequent investigations (e.g. [29, 97]) have often modified the original theory by introducing a null region so that the theory can be built upon a domain that has the basic Boolean algebra structure.

Once the null region has been added, it is clear that the models of the revised RCC theory will be BCAs (as defined by **Definition 4.1**). Moreover, given the RCC axioms, it can be proved that every region is connected to its complement:

$$\forall xy[\mathsf{Compl}(x, y) \rightarrow xCy] \tag{5.1}$$

This corresponds to the BCA property **BCA7**. In fact any model of the RCC axioms modified to include the null region correspond to a BCA satisfying this property:

Lemma 5.1. *An* RCC model *is an RCC algebra, i.e. a BCA* $\langle B, C\rangle$ *which satisfies* **BCA7**.

This correspondence enables us to use connected BCAs as an algebraic counterpart to the 1st-order RCC axioms. An another algebraic analysis of the RCC theory, employing a somewhat weaker axiomatisation, is given in [97].

5.2 The 4 and 9 Intersection Representations

The 4 and 9 Intersection representations were originally described by Egenhofer and Franzosa [36, 37] as a means of representing relationships between geographic regions. The approach is based on the idea of interpreting regions as point sets and characterising binary spatial relations in terms of topological constraints on these sets. The originators suggest that the representation should be applied to Jordan curve bounded regions in the plane (i.e. regions that are homeomorphic to (closed) discs); however, there is no reason why it could not be applied more generally to regular closed subsets of a topological space.

In the 4-intersection representation the idea is to consider the intersection of the boundary and interior of one region with the boundary and interior of another. Thus, for regions A and B, we consider $\partial(A)\cap\partial(B)$, $\partial(A)\cap\text{int}(B)$, $\text{int}(A)\cap\partial(B)$, $\text{int}(A)\cap\text{int}(B)$, and we determine whether or not these intersections are empty (denoted $\emptyset$) or non-empty (denoted $\neg\emptyset$). The determined values are naturally represented by a 2x2 matrix, as shown in Table 5.1.

$\cap$	$\partial(B)$	$\mathrm{int}(B)$	$\cap$	$\partial(B)$	$\mathrm{int}(B)$
$\partial(A)$	$\neg\emptyset$	$\emptyset$	$\partial(A)$	$\neg\emptyset$	$\neg\emptyset$
$\mathrm{int}(A)$	$\emptyset$	$\neg\emptyset$	$\mathrm{int}(A)$	$\neg\emptyset$	$\neg\emptyset$
	$\mathbf{1'}$			**PO**	
$\cap$	$\partial(B)$	$\mathrm{int}(B)$	$\cap$	$\partial(B)$	$\mathrm{int}(B)$
$\partial(A)$	$\emptyset$	$\neg\emptyset$	$\partial(A)$	$\neg\emptyset$	$\neg\emptyset$
$\mathrm{int}(A)$	$\emptyset$	$\neg\emptyset$	$\mathrm{int}(A)$	$\emptyset$	$\neg\emptyset$
	NTPP			**TPP**	
$\cap$	$\partial(B)$	$\mathrm{int}(B)$	$\cap$	$\partial(B)$	$\mathrm{int}(B)$
$\partial(A)$	$\emptyset$	$\emptyset$	$\partial(A)$	$\neg\emptyset$	$\emptyset$
$\mathrm{int}(A)$	$\emptyset$	$\emptyset$	$\mathrm{int}(A)$	$\emptyset$	$\emptyset$
	DC			*EC*	
$\cap$	$\partial(B)$	$\mathrm{int}(B)$	$\cap$	$\partial(B)$	$\mathrm{int}(B)$
$\partial(A)$	$\emptyset$	$\emptyset$	$\partial(A)$	$\neg\emptyset$	$\emptyset$
$\mathrm{int}(A)$	$\neg\emptyset$	$\neg\emptyset$	$\mathrm{int}(A)$	$\neg\emptyset$	$\neg\emptyset$
	$\mathbf{NTPP}^{\smile}$			$\mathbf{TPP}^{\smile}$	

Table 5.1. Topological relations definable using the 4-intersection representation.

By reference to Figure 3.1 (in Section 3 above), it is easy to see that the base relations of the closed circle algebra can be described by this *4-intersection model* [36].

An approach which extends the 4-intersection model also takes into account the complement of the sets in question, and can be described by the following matrix:

$$\begin{pmatrix} \mathrm{int}(x) \cap \mathrm{int}(y) & \mathrm{int}(x) \cap \partial(y) & \mathrm{int}(x) \cap -y \\ \partial(x) \cap \mathrm{int}(y) & \partial(x) \cap \partial(y) & \partial(x) \cap -y \\ -x \cap -\mathrm{int}(y) & -x \cap \partial(y) & -x \cap -y \end{pmatrix}$$

While the 4–intersection model described the topological invariant relations among closed Jordan curves, the 9–intersection model is able to describe such relations for sets, which have arbitrary shaped interiors, including lines and points. Details can be found in [37].

Thus we see that the 9-intersection representation, based on a point-set interpretation of regions characterises exactly the same set of basic binary relations as the axiomatic RCC theory. This of course is not surprising given the correspondences between axiomatic algebras and topological spaces characterised by the representation theorems given in Section 4.2.

6. Reasoning about Topological Relations

The foregoing sections have defined a rich array of formal frameworks for representing topological relationships between regions. We now look as ways in which these representations can be employed to make inferences about topological configurations of regions.

6.1 Compositional Reasoning

Compositional inference may be described in general terms as a deduction, from two relational facts of the forms aRb and bSc, of a relational fact of the form aTc, involving only a and c. Such inferences may be useful in their own right or may be employed as part of a larger inference mechanism, such as a consistency checking procedure for sets of relational facts. In either case, one will normally want to deduce the strongest relation aTc that is entailed by $aRb \wedge bSc$ and which is expressible in whatever formalism is being employed.

In first-order logic we can directly express the strongest fact derivable from $aRb \wedge bSc$ by the formula $a(R;S)b$, where the ; operator is defined by:

$$x(R;S)y \ \equiv_{def} \ \exists z[xRz \wedge zSy] \tag{6.1}$$

Hence, the meaning of ';' coincides with that of composition in Binary Relation Algebras (as defined in Section 2.3). This may be called the *strong* composition operator. It is also often called the *extensional* composition, because, if we know that x and y stand in a relation equivalent to $R;S$, we can infer the existence of an entity z, such that xRz and zSy.

As a means of practical reasoning, inferring strong compositions in an expressive language such as 1st-order logic may not be very effective as the formulae generated will in general be more complex than the initial formulae and no more informative. However, if it is found that for a certain set of relation, every formula derived by compositional inference is equivalent to some relatively simple formula (preferably a single relation of the language or perhaps a disjunction of relations) then compositional inference may be a very powerful rule.

In the case of the Allen calculus based on 13 basic temporal interval relations [2], it turns out that (under a very natural interpretation in terms of intervals on the rational line) the extensional compositions of any pair of base relations correspond to some disjunction of the basic 13 relations. Hence, composition can be applied without generating more complex relations.

It has been found that in cases where the strong, extensional composition cannot be simply expressed, it is useful to generalise the notion of composition to allow weaker inferences. In particular the following notion is often used:

Definition 6.1. *Given a theory* Θ *whose vocabulary includes a set* Rels *of relations, the* weak composition*,* $WComp(R, S)$*, where* $R, S \in \mathsf{Rels}$ *is defined to be: the disjunction of all relations* $T_i \in \mathsf{Rels}$*, such that there exist individual constants* a*,* b*,* c*, where the formula* $R(a, b) \wedge S(b, c) \wedge T_i(a, c)$ *is consistent with* Θ*.*

This means that if $WComp(R, S) = T_1 \cup \ldots \cup T_n$ then

$$\Theta \models \forall x \forall y \forall z [xRy \wedge ySz \rightarrow (xT_1z \vee \ldots \vee xT_nz)] \tag{6.2}$$

and, furthermore, $T_1 \cup \ldots \cup T_n$ is the smallest subset of Rels for which such a formula is provable. Moreover, it is easy to show that $R \circ S$ is always a sub-relation (or equivalent to) $WComp(R, S)$.

Given this definition, inferences of the following form will always be valid:

$$\frac{R(a, b) \wedge S(b, c) \wedge T(a, c)}{(WComp(R, S) \cap T)(a, c)} \text{[WComp]} \tag{6.3}$$

For a finite set of relations, $WComp(R, S)$ can be pre-computed for every pair of relations and stored in a matrix known as a *composition table*. This provides a simple mechanism for computing compositional inferences by looking up compositions in the table. The typical mode by which this kind of compositional reasoning is executed is to repeatedly infer compositional inferences using table look-up until either an inconsistency is detected or no new inferences can be made. Since their introduction by [2], composition tables[14] have received considerable attention from researchers in AI and related disciplines [21, 35, 39, 87, 94, 109].

Table 6.2 is usually called *The Composition Table* for the RCC-8 relations. (The identity relation is omitted from the table since it is clear how composition with $1'$ works.) In general, the table gives the *weak* composition of the RCC-8 relations. This is because over many domains (e.g. over regular closed sets of an arbitrary topological space) the algebra is not atomic, so that compositional combinations of the RCC-8 relations generate an infinite set of different relations, many of which are not expressible as disjunctions of the RCC-8 generating set.

Nevertheless, there are some more restricted interpretations under which the RCC-8 relations are indeed the atoms of an RA (so that every relation in the algebra is a disjunction of these relations). One simple

[14] In fact Allen called his table a 'transitivity table' but 'composition table' is arguably more appropriate and it seems that this is becoming the standard term.

example is the case where the domain of regions is the set $\mathcal{D}_c$ of closed circles in the plain (with the usual interpretation of the relations) as depicted in Figure 3.1. In the case where the domain is taken as the Jordan curve bounded regions of the plane, the table also corresponds to extensional composition [68, 69].

It needs to be mentioned, that the RCC-8 relations are never the atoms of an RA generated by C over an RCC algebra [26], thus the composition table is not extensional for such algebras. For instance the composition table shows that $WComp(EC, EC) = 1' \cup DC \cup EC \cup PO \cup TPP \cup TPP^{\smile}$. If this were extensional, it would mean that for any regions a, b, such that $aECb$ we could find a third region c such that $aECc$ and $cECb$. However, suppose $a = -b$, then there can be no region in the relation EC to both a and b.

Table 6.2. The composition table of $\mathcal{D}_c$

;		C					
	DR		O				
				PP		$PP^{\smile}$	
	DC	EC	PO	TPP	$NTPP$	$TPP^{\smile}$	$NTPP^{\smile}$
DC	1	DR, PO, PP	DR, PO, PP	DR, PO, PP	DR, PO, PP	DC	DC
EC	DR, PO, $PP^{\smile}$	1', DR, PO, TPP $TPP^{\smile}$	DR, PO, PP	EC, PO, PP	PO, PP	DR	DC
PO	DR, PO, $PP^{\smile}$	DR, PO, $PP^{\smile}$	1	PO, PP	PO, PP	DR, PO, $PP^{\smile}$	DR, PO, $PP^{\smile}$
TPP	DC	DR	DR, PO, PP	PP	NTPP	1', DR, PO, TPP, $TPP^{\smile}$	DR, PO, $PP^{\smile}$
$NTPP$	DC	DC	DR, PO, PP	NTPP	NTPP	DR, PO, PP	1
$TPP^{\smile}$	DR, PO, $PP^{\smile}$	EC, PO, $PP^{\smile}$	PO, $PP^{\smile}$	1', PO, TPP, $TPP^{\smile}$	PO, PP	$PP^{\smile}$	$NTPP^{\smile}$
$NTPP^{\smile}$	DR, PO, $PP^{\smile}$	PO, $PP^{\smile}$	PO, $PP^{\smile}$	PO, $PP^{\smile}$	O	$NTPP^{\smile}$	$NTPP^{\smile}$

6.2 Equational Reasoning

In Section 2.4.3 we looked at the algebraic characterisation of topological spaces in terms of Closure Algebras and their complementary Interior Algebras. Since these algebras can be defined by purely equational axioms, this representation suggests that it should be possible to use some form of equational inference to reason about topological relationships among regions.

In general (according to Proposition 2.7) the elements of a closure can correspond to arbitrary subsets of a topological space. However, in order that the domain of regions be compatible with the topological interpretations of region-based axiomatic theories (such as the BCAs discussed in Sections 4.1 and 4.2), we will often want to identify and reason about either regular open or regular closed sets. In the first case one should assert an equation $x = \text{int}(\text{cl}(x))$ for each region variable x; and in the second case one should assert $x = \text{cl}(\text{int}(x))$. In either case we can define a large vocabulary of relations in terms of equations of an interior/closure algebra .

If we are dealing with regions corresponding to regular closed sets then the following definitions of binary topological relations can be given:

(6.4)	$xDCy$	$\Longleftrightarrow$	$-(x \cdot y) = \mathbf{1}$
(6.5)	$xDRy$	$\Longleftrightarrow$	$-(\text{int}(x) \cdot \text{int}(y)) = \mathbf{1}$
(6.6)	xPy	$\Longleftrightarrow$	$-x + y = \mathbf{1}$
(6.7)	$xP^{\smile}y$	$\Longleftrightarrow$	$x + -y = \mathbf{1}$
(6.8)	$xNTPy$	$\Longleftrightarrow$	$-x + \text{int}(y) = \mathbf{1}$[15]
(6.9)	$xNTP^{\smile}y$	$\Longleftrightarrow$	$\text{int}(x) + -y = \mathbf{1}$
(6.10)	$xEQy$	$\Longleftrightarrow$	$x = y$

But C itself (as well as many other relations, including O) cannot be defined by an interior algebraic equation. This follows from the general observation that purely equational constraints are always consistent with any purely equational theory (there must always be at least a trivial one-element model, in which all constants denote the same individual). Thus if the negation of some constraint can be expressed as an equation, then the constraint itself cannot be equationally expressible (otherwise that constraint would be consistent with its own negation).

So to define C (and O) we need to employ disequalities:

(6.11)	xCy	$\Longleftrightarrow$	$-(x \cdot y) \neq \mathbf{1}$
(6.12)	xOy	$\Longleftrightarrow$	$-(\text{int}(x) \cdot \text{int}(y)) \neq \mathbf{1}$

Moreover, all the RCC-8 relations can be defined by some combination of equations given in (6.5)–(6.10) and negations of these equations. Those not already specified, can be defined as follows:

[15]The extension of NTP coincides with $NTPP$, except that $1NTP1$ is true.

$$(6.13) \quad xECy \iff (-(x \cdot y) = \mathbf{1}) \wedge (\text{int}(-x) + \text{int}(-y) \neq \mathbf{1})$$
$$(6.14) \quad xPOy \iff (-(x \cdot y) \neq \mathbf{1}) \wedge (-x + y \neq \mathbf{1}) \wedge (x + -y \neq \mathbf{1})$$
$$(6.15) \quad xTPPy \iff (-x + y = \mathbf{1}) \wedge (x \neq y) \wedge (\text{int}(-x) + y \neq \mathbf{1})$$
$$(6.16) \quad xTPP^{\smile}y \iff (x + -y = \mathbf{1}) \wedge (x \neq y) \wedge (x + \text{int}(-y) \neq \mathbf{1})$$

For many applications we will also want to specify that certain regions are non-empty. This is easily done using the disequality $-x \neq 1$. Various other useful binary RCC relations are expressible by means of interior algebra equations. For example, $EQ(x+y, \mathbf{1})$ can be expressed by $X \cup Y = \mathbf{1}$.

The problem of reasoning with topological relations can thus be reduced to one of reasoning with algebraic equations and disequalities; and this in turn can be reduced to the problem of testing consistency of sets of equations and disequalities. Moreover, the following Lemma tells us that the consistency of such sets can be determined as long as we have a means of computing whether a given equation follows from a set of equations:

Lemma 6.1. *A set of algebraic equalities and disequalities,* $\{x_1 = y_1, \ldots x_m = y_m, z_1 \neq w_1, \ldots x_n \neq y_n\}$*, is inconsistent just in case* $x_1 = y_1, \ldots x_m = y_m \models z_i = w_i$*, where* $1 \leq i \leq n$*.*

Clearly this kind of approach could be applied to any of the other purely equationally defined algebras defined above (in Section 2).

Though equational reasoning has long been a major topic in mathematics and computer science and many general techniques are known, it seems that there has been little research direct specifically at equational reasoning in this kind of spatial algebra. However, an indirect way of implementing such reasoning is by means of an encoding into modal logic, described in the next section.

6.3 Encoding in Propositional Modal Logics

A propositional modal logic augments the classical propositional logic with one or more unary connectives. We assume familiarity with the basics of these formalisms. Full details can be found in many texts, such as [14, 53].

We first consider normal modal logics with a single modality. As usual, the modal necessity operator will be denoted by $\Box$, and its dual possibility operator by $\Diamond$ (where $\Diamond p \leftrightarrow \neg \Box \neg p$). Let $\mathfrak{F}$ be the set of all (well-formed) propositional modal formulae (defined in the usual way).

Definition 6.2. *A* normal propositional modal logic $\mathcal{ML}$ *is identified with the set of its theorems. More specifically,* $\mathcal{ML}$ *is a subset of* $\mathfrak{F}$, *satisfying the following conditions:*[16]

ML1) *All classical tautologies are in* $\mathcal{ML}$.

ML2) *If* $p \in \mathcal{ML}$ *and* $(p \to q) \in \mathcal{ML}$, *then* $q \in \mathcal{ML}$.

ML3) $\mathcal{ML}$ *is closed under substitution.*

ML4) $(\Box\, p \leftrightarrow \neg \Diamond \neg p) \in \mathcal{ML}$. *(Defn. of* $\Diamond$*)*

ML5) *If* $(p \leftrightarrow q) \in \mathcal{ML}$ *then* $(\Diamond\, p \leftrightarrow \Diamond\, q) \in \mathcal{ML}$. *(Extensionality)*

ML6) $\Diamond(p \wedge \neg p) \leftrightarrow (p \wedge \neg p)$

ML7) $(\Diamond(p \vee q) \leftrightarrow (\Diamond\, p \vee \Diamond\, q)) \in \mathcal{ML}$

Normal modal logics can be interpreted in terms of the well-known *Kripke semantics*. Specifically, a model $\mathfrak{M}$ of a normal modal logic $\mathcal{ML}$ is a structure $\langle W, R, v\rangle$, where W is a set, R a binary relation on W (i.e. $\langle W, R\rangle$ is a frame), and $v : \mathcal{V} \to 2^W$ a valuation function which is extended over $\mathcal{ML}$ as follows:

$$\begin{aligned}
v(\neg\varphi) &= W \setminus v(\varphi) = \{w \in W : w \notin v(\varphi)\},\\
v(\varphi \wedge \psi) &= v(\varphi) \cap v(\psi),\\
v(\top) &= W,\\
v(\Diamond(\varphi)) &= \{w : (\exists u)[u \in v(\varphi) \text{ and } uRw\}.
\end{aligned}$$

for all $\varphi, \psi \in \mathcal{ML}$.

The elements of W are often called *possible worlds*.

6.3.1 Modal Logics and Algebras.

There is an intimate connection between propositional logics and algebras. The set $\mathfrak{F}$ of all modal formulae can be regarded as a *term algebra* — i.e. an absolutely free algebra generated from the propositional constants by taking the connectives as (syntactic) operators on formulae.

[16]This definition of normal modal logics is chosen to make clear the connection with modal algebras. An more common approach is to take define a modal logics as a set of formulae satisfying conditions **ML1–4**, together with the 'Rule of Necessitation: if $p \in \mathcal{ML}$ then $\Box p \in \mathcal{ML}$. *Normal* modal logics are then defined as those additionally satisfying the Kripke schema, **K**: $((\Box p \wedge \Box(p \to q)) \to \Box q) \in \mathcal{ML}$. The two specifications are known to be equivalent (see e.g. [14, chapter 4]).

To obtain an algebraic perspective on the structure of a particular modal logic $\mathcal{ML}$ we can construct a *quotient algebra*[17] of $\mathfrak{F}$ relative to the logical equivalence relation of $\mathcal{ML}$. Each element of this algebra, will thus correspond to a semantically distinct proposition expressible in the logic.

Definition 6.3. *Given* $\mathcal{ML} \subseteq \mathfrak{F}$, *the* Lindenbaum–Tarski algebra $\mathfrak{F}_{\mathcal{ML}}$ *of* $\mathcal{ML}$ *is the quotient algebra of* $\mathfrak{F}$ *by the equivalence relation*

$$x \approx_{\mathcal{ML}} y \quad \textit{if and only if} \quad x \leftrightarrow y \in \mathcal{ML}. \tag{6.17}$$

The resulting algebras are modal algebras in the sense of Definition 2.6. The Lindenbaum-Tarski construction can also be used to characterise the equational class of all modal algebras: each equivalence $x \approx_{\mathcal{ML}} y$ corresponds to a universally quantified equation $\forall v_1, \ldots, v_n[x = y]$, where $v_1, \ldots, v_n$ are all the propositional variables occurring in either x or y.

There is a direct correspondence between modal algebras and modal logics. The rule **ML5** ensures that $\Diamond$ (and hence $\Box$ is functional); **ML6** corresponds to the algebraic normality condition (2.3) and **ML7** to additivity (2.2). The generality of the correspondence is expressed by the following proposition:

Proposition 6.1. *[56] Let* $\mathcal{V}(\mathcal{ML})$ *be the equational class generated by* $\mathfrak{F}_{\mathcal{ML}}$. *The mapping* $\mathcal{ML} \mapsto \mathcal{V}(\mathcal{ML})$ *is a dual isomorphism from the lattice of all normal modal logics to the lattice of equational classes of modal algebras.*

6.3.2 $S4$ and Interior Algebras.

One of the better known modal logics is **S4**. This can be defined as a normal modal logic that also satisfies the following axiom schemas:[18]

$$(p \vee \Diamond p) \leftrightarrow \Diamond p \tag{6.18}$$

$$\Diamond \Diamond p \leftrightarrow \Diamond p \tag{6.19}$$

Clearly, in the Lindenbaum-Tarski algebra generated from the set of theorems of **S4**, these schema will generate equations of the form of 2.4 and 2.5 characterising a closure operator. From a semantic point of view, the class of models of **S4** consists of all Kripke frames whose accessibility

[17]Given an algebra $\mathfrak{A} = \langle A, f_1, \ldots, f_n \rangle$, and an equivalence relation $\approx$, the quotient algebra of $\mathfrak{A}$ relative to $\approx$ is the structure $\langle A^{\approx}, f_1^{\approx}, \ldots, f_m^{\approx} \rangle$. Let $x^{\approx} = \{y \mid y \approx x\}$. Then $A^{\approx} = \{x^{\approx} \mid x \in A\}$, and $f_i^{\approx}(x_1^{\approx}, \ldots, x_n^{\approx}) = y^{\approx}$ if $f_i(x_1, \ldots, x_n) = y$.

[18]S4 is more often defined by the schemas $\Box\varphi \to \varphi$ (**T**) and $\Box\varphi \to \Box\Box\varphi$ (**4**), which are equivalent to those given here.

relation is reflexive and transitive; thus according to Proposition 2.3, their complex algebras are exactly the closure algebras.

This correspondence is the basis of the encoding of topological relationships into **S4** proposed in [9] and [10] and also investigated in [80].[19]

Let $\tau \rightleftharpoons \varphi$ denote the one-to-one mapping from terms of closure algebra to syntactically isomorphic formulae of **S4**: specifically φ is obtained from τ by replacing $-$ by $\neg$, $\vee$ by $+$, $\cdot$ by $\wedge$, cl by $\Diamond$ and int by $\Box$. The following Lemma enables us to use deduction in **S4** to determine entailment among equations in closure algebra:

Lemma 6.2. *[10] Let $\tau_1 = 1, \ldots, \tau_n = 1$ be equations of closure algebra and $\tau_i \rightleftharpoons \varphi_i$. Then*

$$(6.20) \qquad \tau_1 = 1, \ldots, \tau_n = 1 \vdash \tau_0 = 1 \quad \textit{iff} \quad \Box\varphi_1, \ldots, \Box\varphi_n \vdash_{S4} \varphi_0$$

Because of Lemma 6.1, we can also use the modal encoding for testing inconsistency of sets of equations and disequations of closure algebra, and hence for reasoning about topological properties and relationships among spatial regions.

6.3.3 A Bi-Modal Spatial Logic.

We now give a brief overview of a somewhat more expressive modal encoding of topological relationships proposed in [10]. This is obtained by employing a bi-modal logic incorporating both **S4** and the 'universal' modal operator [46], here denoted by $\blacksquare$ (with its dual being denoted, $\blacklozenge$, where $\blacklozenge\varphi \leftrightarrow \neg\blacksquare\neg\varphi$). As before, the $S4$ operators $\Box$ and $\Diamond$ correspond respectively to the as interior and closure operators, int and cl. The interpretation of $\blacksquare\varphi$ is that φ holds at all possible worlds.[20] The axiomatisation of $\blacksquare$ is the same as that as $\Box$, with the addition of the following two schemata [46, 116]:

$$(6.21) \qquad \blacklozenge p \rightarrow \blacksquare\blacklozenge\, p$$

$$(6.22) \qquad \blacksquare p \rightarrow \Box p$$

As before, the $S4$ operators $\Box$ and $\Diamond$ correspond respectively to the as interior and closure operators, int and cl. A formula of the form

[19] A similar method had previously been used in [8] to encode modal formulae into intuitionistic propositional logic. This is based on the relation of intuitionistic logic to Heyting algebras, which in turn can be interpreted over topological spaces (2.4.4 above).

[20] The universal modal operator is closely related to the better known **S5** modality, which is the logic semantically determined by taking the accessibility relation to be reflexive, symmetric and transitive (i.e. an equivalence relation). However, this allows the possibility that the set of worlds is partitioned into several sets of worlds which are not accessible to each other. Thus, if $\blacksquare$ were an **S5** modality, $\blacksquare\varphi$ it would be true at world w as long as φ holds in all worlds in the same equivalence class as w — not necessarily in all worlds.

$\blacksquare\varphi$ ensures that the topological condition encoded by φ holds at every point in space. Similarly, $\blacklozenge\varphi$ means that there is some point p satisfying the condition represented by φ. p can be thought of as a *sample point*, which bears *witness* to some topological constraint. For instance, where two regions x and y overlap, the corresponding modal formula $\blacklozenge(\Box(x) \wedge \Box(y))$ ensures the existence of a point which is in the interior of both x and y.

In terms of the bi-modal logic $S5/S4$, a set of key RCC relations are represented as as follows:

$$\mathsf{C}(x,y) \iff \blacklozenge(x \wedge y) \tag{6.23}$$

$$\mathsf{DC}(x,y) \iff \blacksquare(\neg x \vee \neg y) \tag{6.24}$$

$$\mathsf{O}(x,y) \iff \blacklozenge(\Box(x) \wedge \Box(y)) \tag{6.25}$$

$$\mathsf{DR}(x,y) \iff \blacksquare\Diamond(\neg x \vee \neg y) \tag{6.26}$$

$$\mathsf{P}(x,y) \iff \blacksquare(x \rightarrow y) \tag{6.27}$$

$$\neg\mathsf{P}(x,y) \iff \blacklozenge(x \wedge \neg y) \tag{6.28}$$

$$\mathsf{TP}(x,y) \iff \blacksquare(x \rightarrow y) \wedge \blacklozenge(x \wedge c(\neg y)) \tag{6.29}$$

$$\mathsf{NTP}(x,y) \iff \blacksquare(x \rightarrow \Box(y)) \tag{6.30}$$

$$\mathsf{Non\text{-}Empty}(x) \iff \blacklozenge x \tag{6.31}$$

$$\mathsf{Regular}(x) \iff \blacksquare(\Box(\neg x) \vee \Diamond(\Box(x))) \tag{6.32}$$

All the RCC-8 relations can be expressed in terms of these formula by using conjunction and negation. Further details of how this logic can be used for topological reasoning are given in [10, 116].

6.4 A Proof System for Contact Relation Algebras

In this Section we will describe a sound and complete logic for *contact relation algebras* within which general facts about CRAs can be proved. The semantics of this logic are relational, introduced by Orłowska [81, 82], and the proof system is in the style of Rasiowa and Sikorski [89].

Our language $\mathcal{L}$ consists of the disjoint union of the following sets:

1 A set $\{C, 1'\}$ of constants, representing the generating contact relation and the identity.

2 An infinite set $\mathcal{V}$ of individuum variables.

3 A set $\{+, \cdot, -, \;;\; , \breve{}\;\}$ of names for the relational operators.

4 A set $\{(,)\}$ of delimiters.

With some abuse of language, we use the same symbols for the actual operations. The terms of the language are defined recursively:

1 C and $1'$ are terms.

2 If R and S are terms, so are $(R+S)$, $(R\cdot S)$, $(-R)$, $(R\ ;\ S)$, $(R^{\smile})$.

3 No other string is a term.

The set of all terms will denoted by $\mathcal{T}$. In the sequel, we will follow the usual conventions of reducing brackets. The set $\mathcal{F}$ of $\mathcal{L}$–formulae is

$$\{xRy : R \in \mathcal{T}, x, y \in \mathcal{V}\}.$$

A model of $\mathcal{L}$ is a pair $M = \langle W, m\rangle$, where W is a nonempty set, and $m : \mathcal{T} \to W \times W$ is a mapping such that

(6.33) $m(C)$ is a contact relation.

(6.34) $m(1')$ is the identity relation on W.

(6.35) m is a homomorphism from the algebra of terms to $\langle Rel(W), \cup, \cap, -,\ ;\ , {}^{\smile}\rangle$

A valuation v is a mapping from $\mathcal{V}$ to W. If xRy is a formula, then we say that M *satisfies* xRy *under* v, written as $M, v \models xRy$, if $\langle v(x), v(y)\rangle \in m(R)$. xRy is called *true in the model* M, if $M, v \models xRy$ for all valuations v. xRy is called *valid*, if it is true in all models.

The proof system consists of two types of rules: With *decomposition rules* we can decompose formulae into an equivalent sequence of simpler formulae. The decomposition rules are the same for every system of relation algebras. The *specific rules* are tailored towards the concrete situation; they modify a sequence of formulae and have the status of structural rules. The role of axioms is played by *axiomatic sequences.*

Proofs have the form of trees: Given a formula xRy, we successively apply decomposition or specific rules; in this way we obtain a tree whose root is xRy, and whose nodes consist of sequences of formulae. A branch of a tree is *closed* if it contains a node which contains an axiomatic sequence as a subsequence. A tree is called *closed* if all its branches are closed.

Rasiowa–Sikorski systems are, in way, dual to tableaux: Whereas in the latter one tries to refute the negation of a formula, the Rasiowa–Sikorski systems attempt to verify a formula by closing the branches of a decomposition tree with axiomatic sequences.

The decomposition rules of the system are given in Table 6.3, and the specific rules for the system are given in Table 6.4. There, a variable z is called *restricted* in a rule, if it does not occur in the upper part of that

Table 6.3. Decomposition rules

$(\cup)$	$\dfrac{K, x(R \cup S)y, H}{K, xRy, xSy, H}$	$(\neg\cup)$	$\dfrac{K, x-(R \cup S)y, H}{K, x(-R)y, H \mid K, x(-S)y, H}$
$(\cap)$	$\dfrac{K, x(R \cap S), H}{K, xRy, H \mid K, xSy, H}$	$(\neg\cap)$	$\dfrac{K, x-(R \cap S)y, H}{K, x(-R)y, x(-S)y, H}$
$(\breve{\ })$	$\dfrac{K, xR\breve{\ }y, H}{K, yRx, H}$	$(\neg\breve{\ })$	$\dfrac{K, x(-R\breve{\ })y, H}{K, y(-R)x, H}$
$(\neg-)$	$\dfrac{K, x(--R)y, H}{K, xRy, H}$		
$(\,;\,)$	$\dfrac{K, x(R\,;\,S)y, H}{K, xRz, H, x(R\,;\,S)y \mid K, zSy, H, x(R\,;\,S)y}$	where z is any variable	
$(\neg\,;\,)$	$\dfrac{K, x-(R\,;\,S)y, H}{K, x(-R)z, z(-S)y, H}$	where z is a restricted variable	

rule. K and H are finite, possibly empty, sequences of $\mathcal{L}$ formulae. The axiomatic sequences are

$$xRy,\ x(-R)y, \tag{6.36}$$
$$x1'x, \tag{6.37}$$

where $R \in \mathcal{T}$.

The following result shows that the logic is sound and complete:

Proposition 6.2. *[27]*

1 All decomposition rules are admissible.

2 All specific rules are admissible.

3 The axiomatic sequences are valid.

4 If a formula is valid then it has a closed proof tree.

An example in [27] shows that there is a CRA with infinitely many atoms below $1'$, and thus, by a result of Andréka et al. [3], the equational logic of CRAs is undecidable.

Table 6.4. Specific rules

(sym 1′) $$\frac{K, x1'y, H}{K, y1'x, H}$$

(tran 1′) $$\frac{K, x1'y, H}{K, x1'z, H, x1'y \mid K, z1'y, H, x1'y},$$ where z is any variable

($1'_1$) $$\frac{K, xRy, H}{K, x1'z, H, xRy \mid K, zRy, H, xRy},$$ where z is any variable

($1'_2$) $$\frac{K, xRy, H}{K, xRz, H, xRy \mid K, z1'y, H, xRy},$$ where z is any variable

(refl C) $$\frac{K, xCy, H}{K, x1'y, xCy, H}$$

(sym C) $$\frac{K, xCy, H}{K, yCx, H}$$

(ext C) $$\frac{K}{K, x(-C)z, yCz \mid K, y(-C)t, xCt \mid K, x(-1')y}$$ where z and t are restricted variables

(cut C) $$\frac{K}{K, xCy \mid K, x(-C)y}$$

7. Conclusion

In this chapter we have examined the topic of region-based spatial representation from a number of perspectives. We have looked at the relationships between algebraic models, point-set topology and axiomatic theories of spatial regions. The approach of modelling space in terms of a Boolean algebra, supplemented with additional operations and/or relations provides very general and adaptable analysis. Moreover, such algebraic formalisms provide a powerful tool for establishing correspondences between relational axiomatic theories and the models of point-set topology. Specifically, we have seen that Boolean Contact Algebras, have essentially the same expressive capabilities as theories such as the Region Connection Calculus [88] (developed as a knowledge representation formalism for Artificial Intelligence) and have presented representation theorems that characterise topological models of BCAs.

It is interesting to note that the properties of topological spaces that characterise the topological representations of relational theories (according to the representation theorems of Section 4.2) do not coincide with those most familiar to point-set topologists. This is primarily because the elements of region-based theories are modelled as regular subsets of a topological space. Thus, relevant properties for spaces models are typically weaker than better known separation properties, in that they impose conditions on regular subsets of the space rather than on points or on open or closed sets in general. Although we believe that these representations are particularly natural, it is worth noting that there may be alternative topological representations where the embedding of the algebraic structure in the topology takes a different form (as with the representations of [93] and [76]).

As well as considering algebras of regions, we have also seen how the formalism of Relation Algebra provides an algebraic treatment of the relational concepts of a theory. This proves to be well suited to representing spatial relations, in that a large vocabulary of significant spatial relations can be equationally defined from just the contact relation, C. The Relation Algebraic analysis also serves to provide a foundation for the technique of compositional inference, which has been found to be effective in a number of AI applications, for reasoning with both temporal [2] and spatial relations [87, 92].

A technique that has proved particularly useful for reasoning about topology has been the encoding into modal logic. Again, algebra provides a bridging formalism, since propositional logics have a direct correspondence to Boolean algebras with operators. Because the principal function of logical languages is to describe mechanisms of valid inference, much is known about how such inferences can be automated, and about the computational complexity of reasoning algorithms using these systems. Thus the encodings have lead to the development of decision procedures and establishment of complexity results for reasoning about topological relations [10, 91, 92, 115, 116]. Modal encodings have also be applied to encode relations in projective geometry [7, 108].

Another promising avenue for extending the use of modal encodings is by the use of multi-dimensional modal logics [40, 73, 95], which are multi-modal logics, with different modalities ranging over orthogonal dimensions of their model structures. These have been used to capture both multiple spatial dimensions and the combination of space with time. Yet another approach is to employ modal logics in which spatial relations are associated with the accessibility relation associated with the modal operators [19, 71].

The current chapter has focused on purely topological aspects of spatial information. However, other geometrical properties have also been treated in terms of region-based relational and algebraic theories. An early paper of Tarski [98] showed how the whole of Euclidean geometry could be re-constructed by taking regions (rather than points) as the basic spatial entities, and the relation of parthood and the property of sphericity as the conceptual primitives. A simpler axiomatisation of a theory of this kind is given in [11]. Though from a theoretical point of view such formalisms are highly interesting, it is less clear whether they could provide useful mechanisms for computing inferences. It seems that by adding only a little more than topology to a representation one easily obtains a computationally intractable theory. For instance, in [22] it is shown that reasoning with the RCC-8 relations together with a convexity predicate is already massively intractable. The question of the expressive power of region based theories has been the subject of much research (e.g. [85]). [Davis] gives some rather general results demonstrating the very high expressive power of theories that allow quantification over regions.

One potentially very useful development of spatial logics, which has yielded positive results regarding tractability, is the combination of spatial and temporal concepts into a combined spatio-temporal calculus. Certain restricted syntax fragments of modal logics that can encode spatial and temporal information can express a significant range of spatio-temporal relationships whilst remaining tolerably amenable to automated reasoning [12, 114].

An important aspect of space that has not been explicitly considered in this chapter is dimensionality. The formalisms presented in this chapter do not explicitly constrain the dimensionality either of the regions or the embedding space. However, the interpretation of regions as regular sets of a space means that in such models, regions will all have the same dimension as the whole space. The dimensionality of the space could be fixed by appropriate axioms constraining the connection relation, but the dimensionality of regions would still be uniform. For many applications it would be useful to have a richer theory incorporating regions of different dimensionality into its domain. Axiomatic topological theories that can handle diverse dimensionalities have been proposed in [42, 43, 47]. However, the relationship between axiomatic theories of this kind and topological models has not been fully established and is certainly a rich area for further work.

The more computationally amenable region-based calculi also suffer from inexpressivity regarding self-connectedness of regions — i.e. the domain can include multi-piece regions, but single and multi-piece regions cannot be distinguished within the theory.[21] This is closely related to their inexpressibility in regard to dimensionaliy. It was shown in [90] that any consistent set of RCC-8 relations has a model in which the regions

[21]The distinction can easily be made in 1st-order theories such as the full RCC theory, where we can define $\forall x \mathsf{Self\text{-}Connected}(x) \leftrightarrow \forall yz[(x = y + z) \rightarrow yCz$, but the full RCC theory is undecidable [25].

are self-connected regular subsets of a three-dimensional space. However, as explained in [48] an interpretation over self-connected regions of two-dimensional space may not be possible, despite the existence of a higher dimensional model. Hence, in this case, enforcing self-connectedness of regions would have no affect on consistency unless we also had some means of enforcing planarity (or linearity) of the space. Developing any kind of computationally effective calculus for reasoning about topological relations between self-connected regions in two-dimensional space has proved elusive. Some results developed from a graph-theoretic viewpoint suggest that this is at least NP-hard, and may well be undecidable [62, 63].

Another constraint on the structure of space, which has received attention is discreteness. There are many applications, such as describing or reasoning about video images, where one is dealing with a discrete spatial structure. Axiomatic theories which allow atomic regions have been investigated in [74] and [41]; and [28] presents a generalisation of BCAs, in which the extensionality axiom is dropped, and proves a representation theorem in terms of discrete proximity spaces.

The diversity of spatial formalisms is testament to the richness and depth of spatial concepts. Indeed Tarski [100] suggested that geometrical primitives may provide a conceptual basis from which all precise concepts can be defined. Although axiomatic, region-based theories of topology are increasingly well understood and integrated with related areas of mathematics and knowledge representation, many directions for further research remain open.

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