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Algebras of approximating regions*

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1. Introduction

It is rarely the case that spatial regions can be determined up to their true boundaries, if, indeed, they have such boundaries; in most cases, we can only observe regions up to a certain granularity. Often, this is a desirable feature, since too much detail can disturb the view, and we will not be able to see the forest for the trees, if our desire is to see the forest.

Having as our basic assumption that regions can (or need to) be observed only approximately, we want to find an operationalisation of the domain of regions, which is broad enough to express the properties

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which we want to study, and, at the same time, has enough mathematical structure to serve as a reasoning mechanism without being overly restrictive to our intuition.

We make three model assumptions:

1. The first assumption is that there is a partial order P, called the "part-of" relation (or relation of "part"), defined on the collection of regions. We say that two regions x, y *overlap*, if they have a common part, in other words, if there is some z such that zPx and zPy.
2. The second assumption is that there is a collection B of *crisp* or *definable* regions, which forms a Boolean algebra with natural order $\leq$, such that P restricted to B coincides with $\leq$. The crisp regions delineate the bounds up to the granularity of which other regions can be observed.

   The power of observation is expressed by pairs of the form $\langle a, b\rangle,\ a \leq b$, where a, b are definable regions. In other words, to each (unknown) region x there is associated a lower bound $i(x) = a$ and an upper bound $h(x) = b$, both of which are crisp, up to which x is discernible. If $i(x) = h(x)$, then x itself is definable. The pair $\langle i(x), h(x)\rangle$ is called an *approximating region*.
3. The final assumption is that the bounds $\langle a, b\rangle$ are best possible; in other words, if x is a region approximated by $\langle a, b\rangle$, then

   (1.1) No definable region c with $a \lneq c$ is a part of x,

   (1.2) If $c \lneq b$, then x overlaps with $-c$.

   This implies that for each approximating region $z = \langle a, b\rangle$ there is a collection $m(z)$ of regions each of which is approximated by z, and for which (1.1) and (1.2) hold. Furthermore, if y is an approximating region different from z, then $m(z) \cap m(y) = \emptyset$.

The example below shows that these assumptions are fulfilled in an important area of application, namely, screen resolution: Consider the region X in the Euclidean plane, depicted in Figure 1. We suppose in our example that granularity in the plane is determined by an equivalence relation on the points, the classes of which are the atoms of the Boolean algebra B of definable regions; these are drawn as squares. We can, for example, think of the squares as pixels on a computer screen. The region X can only be discerned up to the bounds given by its lower and upper approximation, each of which is a union of squares, i.e.

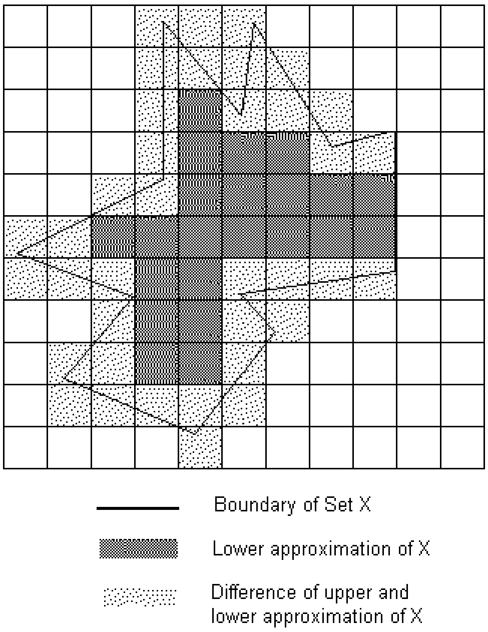
$$i(X) = \{x \in U : \theta x \subseteq X\}, \tag{1.3}$$

$$h(X) = \{x \in U : \theta x \cap X \neq \emptyset\} \tag{1.4}$$

is the *lower*, resp. *upper approximation of* X. Here, $\theta x = \{y : x\theta y\}$ is the equivalence class containing x. It is obvious that our three model assumptions are fulfilled. This is related to the rough set approach to data analysis of [16]; similar paradigms have been put forward in the field of spatial reasoning by Lehmann & Cohn [12] aand Worboys [20].

This paper is organised as follows: Firstly, we will define the class of *approximation algebras* which is then shown to be equipollent to a class of well known algebraic structures. Secondly, we suggest a way to generalise contact of regions to the approximate case. Finally,we will give an outlook to future work.

Figure 1. An approximated region



2. Approximation algebras

In this Section, we will translate the intuitive assumptions of the previous Section into a formal system which will reflect the ontology that we have in mind. Even though we will appeal to the spatial intuition, we will, however, put no formal restrictions on the kind of object that we want to approximate. We will then proceed to show that this system is term – equivalent to a well known equational class of algebras.

We assume a basic knowledge of the theory of ordered structures, lattices, and Boolean algebras, and we invite the reader to consult [3] for terminology and concepts not explained here. If no confusion can arise, we shall identify algebras with their base set, and, with some abuse of language, we will describe algebras by the name of their classes. For example, BA will be the class of Boolean algebras, and we also call a Boolean algebra a BA. If L is an ordered structure with smallest element 0, we set $L_0 = L \setminus \{0\}$.

Throughout, $\langle B, +, \cdot, -, 0, 1\rangle$ will denote a Boolean algebra (BA), with the base set possibly indexed. We may think of B as an algebra of definable (or crisp) objects within some domain as mentioned in the introduction. Since we intend to identify approximate objects with pairs of definable objects from below and above , we start by setting

$$B^{[2]} = \{\langle a, b\rangle : a \leq b\}. \tag{2.1}$$

We regard $B^{[2]}$ as a sublattice of $B \times B$, so that

$$\langle a,b\rangle + \langle c,d\rangle = \langle a+c, b+d\rangle,$$
$$\langle a,b\rangle \cdot \langle c,d\rangle = \langle a\cdot c, b\cdot d\rangle.$$

Lower and upper approximation are defined by

$$i(\langle a,b\rangle) = \langle a,a\rangle,$$
$$h(\langle a,b\rangle) = \langle b,b\rangle.$$

We observe that

$$h(i(\langle a,b\rangle)) = i(\langle a,b\rangle),\ i(h(\langle a,b\rangle)) = h(\langle a,b\rangle). \tag{2.2}$$

We can recover B by identifying B with $\{\langle a,a\rangle : a \in B\}$. Thus, an approximating region is definable, if it is equal to its lower and upper approximation. The operators i and h are a co-normal multiplicative interior, respectively, a normal additive closure operator, i.e. for $x, y \in B^{[2]}$,

(2.3)	$i(1) = 1,$	Co-normal
(2.4)	$i(x \cdot y) = i(x) \cdot i(y),$	Multiplicative
(2.5)	$x \leq y \Rightarrow i(x) \leq i(y),$	Interior
(2.6)	$i(x) \leq x,$	"
(2.7)	$i(i(x)) = i(x),$	"

and

(2.8)	$h(0) = 0,$	Normal
(2.9)	$h(x+y) = h(x) + h(y),$	Additive
(2.10)	$x \leq y \Rightarrow h(x) \leq h(y),$	Closure
(2.11)	$x \leq h(x),$	"
(2.12)	$h(h(x)) = h(x).$	"

Furthermore, we see that for $x, y \in B^{[2]}$,

$$i(x) = i(y) \text{ and } h(x) = h(y) \text{ imply } x = y. \tag{2.13}$$

This expresses the intuition that approximating regions are uniquely determined by their lower and upper bound. The algebra $B^{[2]}$ may be too large for certain situations. It describes the situation that for each $x = \langle a,b\rangle$ with $a \lneq b$ there are "true" regions which are approximated by $\langle a,b\rangle$; however, this may not be always the case. Thus, a less restrictive notion is required, and we generalise $B^{[2]}$ as follows:

An *approximating algebra* (AA) $\langle L, +, \cdot, 0, 1, i, h\rangle$ is a structure of type $\langle 2, 2, 0, 0, 1, 1\rangle$ such that for all $x, y \in L$,

(2.14) $\langle L, +, \cdot, 0, 1\rangle$ is a bounded distributive lattice.

(2.15) i is a co-normal multiplicative interior operator on L.

(2.16) h is a normal additive closure operator on L.

(2.17) $i(h(x)) = h(x),\ h(i(x)) = i(x)$.

(2.18) $i(x) = i(y)$ and $h(x) = h(y)$ imply $x = y$.

(2.19) Each closed element has a complement.

It is not hard to see that $B^{[2]}$ is an AA, and one can show that each AA is a subalgebra of some $B^{[2]}$ [4]. We will denote by $B(L)$ – or just by B if no confusion can arise – the set of closed elements of L. By (2.17), B is also the set of interior elements of L.

Our first aim is to show that B does what we want it to do:

Proposition 2.1. *B is a subalgebra of L and a Boolean algebra.*

Proof:
By (2.17), the functions i and h are the identity on B, and thus, B is closed under i and h. Furthermore, $0, 1 \in B$ since h is normal and i is co–normal. If $h(x), h(y) \in B$, then,

$$h(x) + h(y) = h(x + y),$$

since h is additive, and thus, B is closed under $+$. Now,

$$\begin{aligned} h(x) \cdot h(y) &= i(h(x)) \cdot i(h(y)), && \text{by (2.17)} \\ &= i(h(x) \cdot h(y)) && \text{since } i \text{ is multiplicative} \\ &= h(i(h(x) \cdot h(y))). && \text{by (2.17).} \end{aligned}$$

Thus, B is a $0, 1$–sublattice of L. Since L, and thus B, is distributive, the complement whose existence is guaranteed by (2.19) is unique, and it follows, that B is a Boolean algebra. □

2.1. AA and regular double Stone algebras

In this section we will show that the class AA is definitionally equivalent to the well known class of regular double Stone algebras (RDSA) which have been investigated, among others, in [11, 19]. Many other classes definitionally equivalent to RDSA are known, for example, the class of three valued Łukasiewicz algebras; for more details we invite the reader to consult [2, 15].

For each $x \in L$ we define

$$x^{+} = -i(x), \tag{2.20}$$

$$x^{*} = -h(x), \tag{2.21}$$

where $-$ denotes the complement operation in B. Observe that $i(x) = x^{++}$ and $h(x) = x^{**}$ for all $x \in L$, and that for $x \in B$ we have $-x = x^{*} = x^{+}$.

Proposition 2.2. *Let $x \in L$.*

(2.22) x^{+} *is the smallest* $z \in L$ *for which* $x + z = 1$.

(2.23) x^{*} *is the largest* $z \in L$ *for which* $x \cdot z = 0$.

(2.24) $B(L) = \{x^{+} : x \in L\} = \{x^{*} : x \in L\}$.

Proof:
(2.22): Since $i(x) \leq x$, and $i(x) + -i(x) = 1$, we have $x + x^{+} = 1$. Suppose that $x + z = 1$.Since i is additive, normal, and order preserving, we have $i(x) + z = 1$ which implies that $z \geq -i(x) = x^{+}$.

(2.23): This is shown analogously.

(2.24): This follows from $x^{+} = x^{+++}$ and $x^{*} = x^{***}$, respectively. □

The operation * is called *pseudocomplementation*, and the operation $^{+}$ is called *dual pseudocomplementation*.

An algebra $\langle L, +, \cdot, ^{*}, ^{+}, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ is called a *regular double Stone algebra* (RDSA), if

1. $\langle L, +, \cdot, 0, 1\rangle$ is a bounded distributive lattice.
2. * satisfies (2.23), and $^{+}$ satisfies (2.24).
3. ** distributes over $+$, and $^{++}$ distributes over $\cdot$.
4. $x^{+} = y^{+}$ and $x^{*} = y^{*}$ imply $x = y$.

In the sequel, $1'$ is the identity relation on the set under consideration. The following result is now straightforward to prove:

Proposition 2.3. *Let AA be as above, and RDSA be the class of regular double Stone algebras, regarded as categories with the appropriate homomorphisms. Define mappings γ and δ in the following way:*

Let $\langle L, +, \cdot, i, h, 0, 1\rangle \in \mathrm{AA}$, and let * and $^{+}$ be defined as in (2.22) *and* (2.23). *Set $\gamma(L) = \langle L, +, \cdot, ^{*}, ^{+}, 0, 1\rangle$, and $\gamma(f) = f$ for any AA morphism. Conversely, let $\langle L, +, \cdot, ^{*}, ^{+}, 0, 1\rangle \in \mathrm{RDSA}$, and set $i(x) = x^{++}$ and $h(x) = x^{**}$. Define $\delta(L) = \langle L, +, \cdot, i, h, 0, 1\rangle$, and $\delta(f) = f$ for any RDSA morphism.*

Then, $\gamma : \mathrm{AA} \to \mathrm{RDSA}$ and $\delta : \mathrm{RDSA} \to \mathrm{AA}$ are covariant functors, and $\gamma ; \delta = \delta ; \gamma = 1'$. □

We call $B(L)$ the *centre of* L, and the set $D(L) = \{x \in L : x^{**} = 1\}$ its *dense set*. We shall usually just write D instead of $D(L)$, if no confusion can arise. The name of D derives from the property

$$h(x) = x^{**} = 1, \tag{2.25}$$

i.e.

$$x \cdot y \gneq 0 \tag{2.26}$$

for all $x \in D,\ y \in L_0$. Note that $L = B$ if and only if $D = \{1\}$.

We can regard D as a crude measure of the granularity level: If $D = \{1\}$, then $L = B$, and all regions under consideration are definable. If D_L is isomorphic to B_L, then for all $a, b \in B$ with $a \leq b$ there is a unique $x \in L$ with $i(x) = a$ and $h(x) = b$; $B^{[2]}$ is an example for this situation.

3. Contact relations and their approximations

A *contact relation* C *on a set* W *(of regions)* has the properties

(3.1) C is reflexive.

(3.2) C is symmetric.

(3.3) If $C(x) = C(y),$ then $x = y$.

Here, for $z \in W$, we let $C(z) = \{t \in W : zCt\}$. C is called *proper*, if it is not the identity.

By the extensionality axiom (3.3), a region is determined by all regions to which it is in contact. It was shown by [8] that, in the presence of (3.1) and (3.2), the extensionality axiom (3.3) is equivalent to

(3.4) $-(C; -C)$ is a partial order.

Here, the complement is taken with respect to the set of all binary relations on W, and ; is relational composition. This partial order will usually be denoted by P, and we call it the *part – of relation of* C. It is well known that P is the *right residual of* C, i.e. that P is the largest relation R on W such that $C; R \subseteq C$ [10]. For later use, we also let $Rel(U)$ be the set of all binary relations on a set U.

If $\langle W, \leq \rangle$ is an ordered structure, and C a contact relation on W, we say that C *is compatible with* W *(or* $\leq$*)* if for all $x, y \in W$,

$$x \leq y \Longleftrightarrow xPy. \tag{3.5}$$

It is not hard to see that (3.5) is equivalent to

$$x \leq y \Longleftrightarrow C(x) \subseteq C(y). \tag{3.6}$$

It was shown in [7] that on each atomless BA there are at least two compatible proper contact relations. The next result shows that on an AA L which is not a Boolean algebra, there is no proper compatible contact relation:

Proposition 3.1. *If L is an AA, and C a proper compatible contact relation on L_0, then $L = B$.*

Proof:
Let $x \in D$. Then, by (2.26), $x \cdot y \gneq 0$ for all $y \in L_0$. Then, $x \cdot yCy$ and $C(x \cdot y) \subseteq C(x)$ show that xCy.It follows that $C(x) = C(1)$, and therefore, $x = 1$ by (3.3). □

Thus, if $L \neq B$, we cannot define a proper compatible contact relation on L. Since each AA is obtained from its Boolean algebra B of definable elements via the approximation functions, we suppose that we have a contact relation C on B, which we want to approximate in a similar way. It seems natural for $x, y \in L$ to say that

$$x \text{ and } y \text{ are certainly connected } \iff i(x)Ci(y),$$
$$x \text{ and } y \text{ are possibly connected } \iff h(x)Ch(y).$$

Thus, given $\langle B, C\rangle$, we let

$$xC^iy \iff i(x)Ci(y), \tag{3.7}$$
$$xC^hy \iff h(x)Ch(y). \tag{3.8}$$

For formal reasons, however, we do not want to start with C on B, but with the approximation relations C^i, C^h on L, whose restriction to B is a contact relation C such that (3.7) and (3.8) hold. Recall that for a relation R its *converse* $R^\smile$ is defined as

$$R^\smile = \{\langle y, x\rangle : xRy\}.$$

An *approximate contact algebra* (ACA) is a structure $\langle L, i, h, C^i, C^h\rangle$ such that $\langle L, i, h\rangle$ is an AA and

$$C^i = C^{i\smile},\ C^h = C^{h\smile} \tag{3.9}$$
$$1' \subseteq C^i \cap C^h, \tag{3.10}$$
$$xC^iy \iff i(x)C^ii(y), \tag{3.11}$$
$$xC^hy \iff h(x)C^hh(y), \tag{3.12}$$
$$h(x)C^ih(y) \iff h(x)C^hh(y), \tag{3.13}$$
$$C^h(h(x)) \subseteq C^h(h(y)) \iff h(x) \leq h(y). \tag{3.14}$$

Proposition 3.2. *1. Let $\langle L, i, h, C^i, C^h \rangle$ be an ACA, and set $C = C^h \restriction B^2$. Then, C is a compatible contact relation on B, and* (3.7) *and* (3.8) *hold. Furthermore, $C^h \restriction B^2 = C^i \restriction B^2$.*

2. If $\langle L, i, h \rangle$ is an AA and C a contact relation on B, then, C^i and C^h, defined by (3.7) *and* (3.8) *satisfy* (3.9) – (3.14)*. Furthermore, $C = C^h \restriction B^2$.*

Proof:

1. The facts that C is symmetric and reflexive follow immediately from (3.9) and (3.10). Let $h(x)Ch(t) \iff h(y)Ch(t)$, and assume $h(x) \neq h(y)$. Then, by (3.14), there is some $z \in L$ such that $h(x)C^h z$, but not $h(y)C^h z$. It follows from the fact that h is idempotent and from (3.12), that $h(x)C^h h(z)$, and not $h(y)C^h h(z)$, contradicting our hypothesis.

Condition (3.8) follows immediately from (3.12), and the definition of C. Now,

$$
\begin{aligned}
xC^i y &\iff i(x)C^i i(y), && \text{By (3.11)}\\
&\iff h(i(x))C^i h(i(y)), && \text{By (2.17)}\\
&\iff h(i(x))C^h h(i(y)), && \text{By (3.13)}\\
&\iff i(x)C^h i(y), && \text{By (2.17)}\\
&\iff i(x)Ci(y). && \text{By definition of } C
\end{aligned}
$$

The fact that C is compatible with the Boolean order follows from (3.14), and $C^h \restriction B^2 = C^i \restriction B^2$ follows from (2.17).

2. The proofs are straightforward computations. By way of example, we show (3.13):

$$
\begin{aligned}
h(x)C^i h(y) &\iff i(h(x))Ci(h(y)), && \text{By (3.7)}\\
&\iff h(x)Ch(y), && \text{By (2.17)}\\
&\iff h(h(x))C^h h(h(y)), && \text{By (3.8)}\\
&\iff h(x)C^h h(y). && \text{By (2.12)}
\end{aligned}
$$

$C = C^h \restriction B^2$ now follows from (3.8). □

In the sequel, we let $\langle L, i, h, C^i, C^h \rangle$ be a generic ACA, and $C = C^h \restriction B^2$ be the associated contact relation on B.

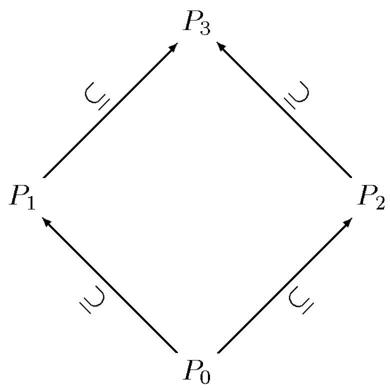
Let

$$x\theta^i y \iff i(x) = i(y), \tag{3.15}$$

$$x\theta^h y \iff h(x) = h(y).. \tag{3.16}$$

Then, θ^i is an $\langle L, \cdot, 0, 1, i \rangle$ congruence, θ^h is an $\langle L, +, 0, 1, h \rangle$ congruence, and $\theta^i \cap \theta^h = 1'$ by (2.13). The following is easy to prove, observing that $C \in Rel(B)$:

Figure 2. Approximate "part-of" relations



Proposition 3.3. *$C^i = \theta^i; C; \theta^i$ and $C^h = \theta^h; C; \theta^h$.* □

The question arises, how the part-of relation P belonging to C (in the sense of (3.4)) can be sensibly approximated, in other words, what does it mean to say that the approximating region x is (approximately) part of the approximating region y?

We know that the part-of relation P on B generated by C is the Boolean order. Besides the lattice ordering $\leq$ on L which extends P, there are, on first glance, several possibilities to generalise P to a relation on L:

(3.17) $$xP_0y \iff h(x) \leq i(y),$$

(3.18) $$xP_1y \iff h(x) \leq h(y),$$

(3.19) $$xP_2y \iff i(x) \leq i(y),$$

(3.20) $$xP_3y \iff i(x) \leq h(y).$$

A sketch of the $\subseteq$ relationships among these part-of relations is given in Figure 2. Note that $P_1 \cap P_2 = \leq$.

Following our assumptions (1.1) and (1.2), we think of lower bound as certainty and upper bound as possibility, where both bounds are best possible. With this in mind, we see that an approximated part-of P^a relation on an ACA must satisfy

(3.21) The restriction of P^a to B is equal to P.

(3.22) $xP^ay \Rightarrow i(x) \leq i(y)$ and $h(x) \leq h(y)$.

While the first condition is clear, we should explain the second one: If x is approximately a part of y, and if z is certainly part of x, i.e. part of all regions which x approximates, then z should be certainly part of anything which is approximated by y. If $h(x) \not\subseteq h(y)$, then no region approximated by y has a common part with $h(x) \cdot -h(y)$, but every region approximated by x has. Thus, there can be no region in $h(x)$ which is part of some region in $h(y)$.

With these observations, there can be only two suitable ordering relations, namely, the lattice ordering $\leq$, and the ordering P_0, denoted by $\preceq$, which is defined by

$$x \preceq y \iff h(x) \leq i(y). \tag{3.23}$$

The lattice ordering can be interpreted as "possible part-of", while $x \preceq y$ says that anything which x approximates is certainly a part of anything which y approximates. Finally, we present the following compatibility Theorem:

Proposition 3.4. *For all* $x, y \in L$,

$$x \leq y \iff C^i(x) \subseteq C^i(y) \text{ and } C^h(x) \subseteq C^h(y). \tag{3.24}$$

Proof:
Let x, y in L; then,

$$\begin{aligned} x \leq y &\iff i(x) \leq i(y) \text{ and } h(x) \leq h(y) \\ &\iff C(i(x)) \subseteq C(i(y)) \text{ and } C(h(x)) \subseteq C(h(y)) \\ &\iff (\forall z)[i(x)Ci(z) \Rightarrow i(y)Ci(z)] \text{ and } (\forall z)[h(x)Ch(z) \Rightarrow h(y)Ch(z)], \\ &\iff (\forall z)[xC^i z \Rightarrow yC^i z] \text{ and } (\forall z)[xC^h z \Rightarrow yC^h z] \\ &\iff C^i(x) \subseteq C^i(y) \text{ and } C^h(x) \subseteq C^h(y). \end{aligned}$$

□

4. Summary and outlook

In this paper we have provided an algebraic foundation for the intuitive notion of "approximating region". In a second step, we have suggested a way to define "approximately in contact" and "approximately part-of" on these algebras. For the construction, we have made three model assumptions which are fulfilled in the important example of screen resolution.

Currently, we are developing a logic for AA and ACA using the relational semantics introduced in [13] which seems to us most appropriate for the spatial context. Many constraints which are not modally expressible receive an explicit representation in the form of a relational rule or a relational axiomatic sequence. Pertaining to our present context, it was shown in [6] that the extensionality axiom (3.2) of contact relations is not expressible in a classical modal logic, nor, as shown in [5], in its sufficiency counterpart defined in [9]. It is, however, expressible in a mixed logic with relational semantics.

A further task is the comparison of "approximate part-of" with the notion of "rough inclusion" described in connection with rough mereology [17, 18].

References

[1] Andréka, H., Monk, J. D. & Németi, I. (Eds.) (1991). Algebraic Logic, vol. 54 of *Colloquia Mathematica Societatis János Bolyai*. Amsterdam: North Holland.

[2] Boicescu, V., Filipoiu, A., Georgescu, G. & Rudeanu, S. (1991). Łukasiewicz–Moisil Algebras, vol. 49 of *Annals of Discrete Mathematics*. Amsterdam: North Holland.

[3] Davey, B. A. & Priestley, H. A. (1990). Introduction to Lattices and Order. Cambridge University Press.

[4] Düntsch, I. (1983). On free or projective Stone algebras. *Houston J. Math.*, **9**, 455–463.

[5] Düntsch, I. & Orłowska, E. (2000a). Beyond modalities: Sufficiency and mixed algebras. In E. Orłowska & A. Szałas (Eds.), *Relational Methods in Computer Science Applications*, 277–299, Berlin. Springer–Verlag.

[6] Düntsch, I. & Orłowska, E. (2000b). A proof system for contact relation algebras. *Journal of Philosophical Logic*, **29**, 241–262.

[7] Düntsch, I., Wang, H. & McCloskey, S. (1999). Relation algebras in qualitative spatial reasoning. *Fundamenta Informaticae*, **39**, 229–248.

[8] Düntsch, I., Wang, H. & McCloskey, S. (2000). A relation algebraic approach to the Region Connection Calculus. *Theoretical Computer Science*. To appear.

[9] Gargov, G., Passy, S. & Tinchev, T. (1987). Modal environment for Boolean speculations. In D. Skordev (Ed.), *Mathematical Logic and Applications*, 253–263, New York. Plenum Press.

[10] Jónsson, B. (1991). The theory of binary relations. In [1], 245–292.

[11] Katriňák, T. (1974). Construction of regular double p–algebras. *Bull. Soc. Roy. Sci. Liège*, **43**, 294–301.

[12] Lehmann, F. & Cohn, A. G. (1994). The EGG/YOLK reliability hierarchy: Semantic data integration using sorts with prototypes. In *Proc. Conf. on Information Knowledge Management*, 272–279. ACM Press.

[13] Orłowska, E. (1991). Relational interpretation of modal logics. In [1], 443–471.

[14] Orłowska, E. (Ed.) (1997). Incomplete Information – Rough Set Analysis. Heidelberg: Physica – Verlag.

[15] Pagliani, P. (1997). Rough sets theory and logic-algebraic structures. In [14], 109–190.

[16] Pawlak, Z. (1982). Rough sets. *Internat. J. Comput. Inform. Sci.*, **11**, 341–356.

[17] Polkowski, L. (2001). On connection synthesis via rough mereology. *Fundamenta Informaticae*. to appear.

[18] Polkowski, L. & Skowron, A. (1996). Rough mereology: A new paradigm for approximate reasoning. *International Journal of Approximate Reasoning*, **15**, 333–365.

[19] Varlet, J. C. (1972). A regular variety of type $\langle 2,2,1,1,0,0\rangle$. *Algebra Universalis*, **2**, 218–223.

[20] Worboys, M. (1998). Imprecision in finite resolution spatial data. *Geoinformatica*, **2**, 257–280.