# Algebraic aspects of attribute dependencies


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**Abstract**

We exhibit some new connections between structure of an information system and its corresponding semilattice of equivalence relations. In particular, we investigate dependency properties and introduce a partial ordering of information systems over a fixed object set $U$ which reflects the sub–semilattice relation on the set of all equivalence relations on U.


## 1 Introduction

Rough set theory has been developed by Pawlak and his co–workers since the early 1980s as a means to handle uncertain or incomplete information, and we invite the reader to consult Pawlak (1991) and Słowiński (1992) for the theoretical foundations and the practical aspects of the rough set model.

A major tool in the rough set model are *indiscernibility relations* – equivalence relations on the universe up to which distinction of objects is possible. Knowledge representation in the model is done via the notion of an *information system* which, roughly speaking, is like a table in a relational database. A dependency $P \to Q$ in an information system is a relation between sets of attributes: If objects agree on all attributes in $P$, then they agree on all attributes in $Q$.

Numerous articles in the literature are concerned with algebraic aspects of dependencies in information systems, for example Pawlak & Rauszer (1985), Rauszer (1991), Novotný & Novotný (1992), Novotný & Pawlak (1992), Novotný (1997).

While Novotný & Pawlak (1991) and Novotný (1997) concentrate on dependence spaces, and Novotný & Novotný (1992) on dependence relations – both of which are primarily concerned with the set of attributes of an information system –, we investigate in the present paper the connections to

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the corresponding semilattice of equivalence relations on the object set in some detail. Our approach is related to the algebras of dependency of Pawlak & Rauszer (1985).

The paper is organized as follows: Section 2 will introduce our notation and give the basic definitions and conversions. Section 3 will discuss several reductions of information systems, and Section 4 investigates dependency in our algebraic systems. Finally, Section 5 will introduce a partial ordering of information systems over a fixed object set which reflects the subsemilattice relation on the set of all equivalence relations on this set.

# 2 Notation and definitions

## 2.1 Information systems

An information system $\mathcal{I}$ is a tuple $\langle U, \Omega, V_q, f_q \rangle_{q \in \Omega}$, where

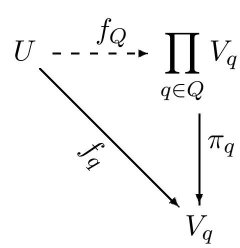
1. $U,\ V_q$ and $\Omega$ are a finite nonempty sets.
2. Each $f_q : U \to V_q$ is a mapping whose range contains at least two elements.

The elements of $\Omega$ are called *attributes*, and $V_q$ is the *domain* of attribute $q$. The *descriptor functions* $f_q$ assign to each object $u \in U$ its value under attribute $q$.

For each $q \in Q \subseteq \Omega$, let $\pi_q$ be the projection of $\prod_{p \in Q} V_p$ to $V_q$, and $f_Q$ be the extension of the mappings $f_q$ to $\prod_{q \in Q} V_q$, i.e.

$$\pi_q \circ f_Q = f_q$$

for each $q \in Q$:[1]




For each $\emptyset \neq Q \subseteq \Omega$ we let $\theta_Q$ be the kernel of $f_Q$, i.e.

$$x \theta_Q y \Longleftrightarrow f_Q(x) = f_Q(y).$$

We also set $\theta_\emptyset \stackrel{\text{def}}{=} U \times U$, and usually write $\theta_q$ instead of $\theta_{\{q\}}$.

The relations $\theta_Q$ are the *indiscernibility relations* on $U$ with respect to $\mathcal{I}$: Two elements in the same $\theta_Q$–class cannot be distinguished by the attributes in $Q$.

Of particular interest in rough set dependency theory are those attribute sets $Q$ for which – given an attribute set $P$ – we have $\theta_Q \subseteq \theta_P$. If $Q$ is minimal with respect to set inclusion $\subseteq$, it is called *minimal*

[1]Diagrams are drawn using Paul Taylor's *Commutative Diagrams in TEX* macro package.

*determining set for* $P$. If $Q$ is minimal determining for $P$ and $Q \subseteq P$, then we call $Q$ a *reduct of* $P$. In other words, $Q$ is a reduct of $P$ if and only if $Q \subseteq P$ and the following conditions hold:

$$\theta_Q \subseteq \theta_P, \tag{2.1}$$

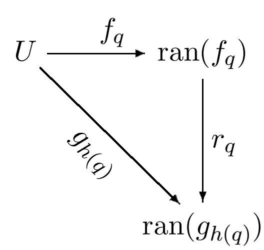
$$\theta_R \not\subseteq \theta_P \text{ for all } R \subsetneq Q. \tag{2.2}$$

If $Q$ is minimal determining for $\Omega$, we call $Q$ a *reduct of* $\mathcal{I}$, or simply, a *reduct*. The *core* of $\mathcal{I}$ – denoted by core$(\mathcal{I})$ – is the intersection of all reducts of $\mathcal{I}$.

We call two information systems $\mathcal{I} = \langle U, \Omega, V_q, f_q \rangle_{q \in \Omega}$ and $\mathcal{H} = \langle U, \Delta, W_p, g_p \rangle_{p \in \Delta}$ *isomorphic*, written as $\mathcal{I} \cong \mathcal{H}$, if there are bijective mappings

$$\begin{cases} h: & \Omega \to \Delta, \\ r_q: & \operatorname{ran}(f_q) \to \operatorname{ran}(g_{h(q)}) \end{cases}, \tag{2.3}$$

such that the following diagram commutes:




(Here, if $f : D \to W$ is a function, we let $\operatorname{ran}(f) \stackrel{\text{def}}{=} \{f(x) : x \in D\}$ be the range of $f$.)

Isomorphic systems differ only by renaming attributes and their values (and possibly unused attribute values).

## 2.2 Properties of semilattices

Throughout this section, let $\mathfrak{A} = \langle A, \wedge, 1 \rangle$ be a finite meet–semilattice with largest element 1. The finiteness of $\mathfrak{A}$ implies that $\mathfrak{A}$ is in fact a lattice via

$$a \vee b \stackrel{\text{def}}{=} \bigwedge \{c \in A : a \leq c \text{ and } b \leq c\}.$$

We denote the smallest element $\bigwedge A$ of $A$ by 0 which exists, since $A$ is finite.

An element $a$ of $A$ is called *dense*, if

$$x \wedge a = 0 \text{ implies } x = 0$$

for all $x \in A$.

The *pseudocomplement* of $a$ is the largest element $b$ of $A$ for which $a \wedge b = 0$. We denote it by $a^*$, if it exists.

A subset $X$ of $\mathfrak{A}$ is called *independent*, if for every semilattice $\mathfrak{B} = \langle B, \wedge, 1\rangle$ and every mapping $f : X \to B$ there is a homomorphic extension $\overline{f} : [X]_{\mathfrak{A}} \to \mathfrak{B}$. Here, $[X]_{\mathfrak{A}}$ is the subalgebra of $\mathfrak{A}$ generated by $X$.

A characterization of independence in semilattices was given in Szász (1963):

**Proposition 2.1.** *$X \subseteq A$ is independent if and only if*

$$\text{If } \bigwedge Y \leq p, \text{ then } p \in Y$$

*for all $Y \subseteq X,\ p \in X$.* □

Using 2.1 the following is not hard to show:

**Proposition 2.2.** *Let $X \subseteq A$. Then, the following are equivalent:*

1. *$X$ is independent.*
2. *For all $p \in X$, $\bigwedge(X \setminus \{p\}) \not\leq p$,*
3. *$[X]_{\mathfrak{A}}$ is a Boolean algebra.* □

$a \in A$ is called (meet–) *irreducible*, if

$$x \wedge y = a \text{ implies } x = a \text{ or } y = a$$

for all $x, y \in A$. We denote the set of all irreducible elements of $A$ less than $1$ by $\mathsf{Irr}(\mathsf{A})$. $\mathsf{Irr}(\mathsf{A})$ is the smallest generating set of $\mathfrak{A}$.

### 2.3 Semilattices of equivalence relations

Let $\mathcal{I} = \langle U, \Omega, V_q, f_q\rangle_{q\in\Omega}$ be an information system, and $\mathrm{Eq(U)}$ be the set of all equivalence relations on $U$.

Our main consideration is the connection between information systems and subsemilattices of the semilattice $\mathbf{Eq}(\mathrm{U}) \stackrel{\text{def}}{=} \langle \mathrm{Eq(U)}, \cap, U \times U\rangle$ of equivalence relations on $U$. The relational system $\langle U, \theta_Q\rangle_{Q\subseteq\Omega}$ may be regarded as the essence of the informational content of the information system, when one abstracts from the concrete nature of the attributes and their values, and only considers whether objects can be distinguished by a given set of attributes.

The following connections have been established:

Define a mapping $\mathbf{E} : \wp(\Omega) \to \mathrm{Eq(U)}$ by

$$\mathbf{E}(Q) \stackrel{\text{def}}{=} \theta_Q. \tag{2.4}$$

Then,

**Proposition 2.3.** *Pawlak & Rauszer (1985), Comer (1991)*
*$\mathbf{E}$ is a homomorphism from $\langle \wp(\Omega), \cup, \emptyset\rangle$ to $\mathbf{Eq}(\mathrm{U})$ where $\mathbf{E}(\emptyset) = U \times U$.* □

In particular, $\langle \mathrm{ran}(\mathbf{E}), \cap, U \times U\rangle$ is a 1–subsemilattice of $\mathbf{Eq}(\mathrm{U})$ generated by $\{\theta_q : q \in \Omega\}$. By abuse of language, we shall usually denote this structure by $\mathbf{E}(\mathcal{I})$.

Conversely, if $S$ is a subalgebra of $\mathbf{Eq}(\mathrm{U})$ with its set of meet–irreducible generators $\mathsf{Irr(S)} = \{\theta_\mathsf{q} : \mathsf{q} \in \Omega\}$, we define an information system $\mathbf{F}(S) = \langle U, V_q, f_q\rangle_{q\in\Omega}$ by

$$\begin{cases} V_q & \stackrel{\mathrm{def}}{=} U/\theta_q, \\ f_q(x) & \stackrel{\mathrm{def}}{=} x/\theta_q, \end{cases} \tag{2.5}$$

for each $q \in Q,\ x \in U$. This is somewhat different from the corresponding constructions in Comer (1991) or Novotný & Novotný (1992) in that we take the smallest generating set of $S$ as the set of attributes.

We now have

**Proposition 2.4.** *Comer (1991), Novotný & Novotný (1992)*

*If $S$ is a subalgebra of $\mathbf{Eq}(\mathrm{U})$ then $\mathbf{E}(\mathbf{F}(S)) = S$.* □

# 3 Removing overhead

Given an information system $\mathcal{I} = \langle U, \Omega, V_q, f_q\rangle_{q\in\Omega}$, a natural question is whether there is a set $\Delta$ of attributes which does not change the information which $\mathcal{I}$ provides with regard to the (in–) discernibility of objects. The first step we can take is to remove or aggregate attributes which lead to the same partition of $U$: Two attributes $p, q \in \Omega$ are called *equivalent*, if $\theta_p = \theta_q$. An information system $\mathcal{I}$ over $U$ is called *essential*, if there are no equivalent attributes. In other words,

$$\mathcal{I} \text{ is essential} \iff \mathbf{E} \text{ is one–one on singletons.}$$

For each $\mathcal{I}$ there is an essential $\mathcal{I}'$ such that $\mathbf{E}(\mathcal{I}) = \mathbf{E}(I')$: Just choose one attribute from each set $M_q = \{p \in \Omega : \theta_p = \theta_q\}$, and use only the resulting attributes. Observe that this choice is usually not unique.

Alternatively, we can collect attributes into a new one as follows: Let $\emptyset \neq Q \subseteq \Omega$; define an information system $\mathcal{I}' = \langle U, \Delta, W_r, g_r\rangle_{r\in\Delta}$ by

$$\begin{gathered} \Delta \stackrel{\mathrm{def}}{=} (\Omega \setminus Q) \cup \{t_Q\}, \\ V_t \stackrel{\mathrm{def}}{=} \begin{cases} V_t, & \text{if } t \neq t_Q, \\ \prod_{q\in Q} V_q, & \text{otherwise,} \end{cases} \\ g_t \stackrel{\mathrm{def}}{=} \begin{cases} f_t, & \text{if } t \neq t_Q, \\ f_Q, & \text{otherwise.} \end{cases} \end{gathered} \tag{3.1}$$

If all attributes in $Q$ are equivalent, then $\theta_{t_Q} = \theta_q$ for all $q \in Q$. Having done this for all classes of equivalent attributes will give us an essential information system $\mathcal{J}$ for which $\mathbf{E}(\mathcal{I}) = \mathbf{E}(\mathcal{J})$.

From now on, we shall suppose that this first reduction has been done, i.e.

**All information systems considered are essential**,

unless indicated otherwise. This assumption may be too restrictive for certain applications, since we may lose some content. Indiscernibility in information systems does not distinguish between *yes* and *no*: If there are decision attributes $p, q$ such that for all $x \in U$,

$$f_p(x) = \text{ yes } \iff f_q(x) = \text{ no }, \tag{3.2}$$

then $\theta_p = \theta_q$. In this respect, information systems are 'coarser' than related structures such as contexts or knowledge structures.

The second step is to remove from an (essential) information system those attributes which are exactly a combination of others: Since $S \stackrel{\text{def}}{=} \mathbf{E}(\mathcal{I})$ is a finite meet–semilattice, it is generated by the set $\mathsf{Irr}(\mathsf{S})$ of its meet–irreducible elements. $\mathsf{Irr}(\mathsf{S})$ is the smallest generating set for $S$, and hence it is contained in $\{\theta_q : q \in \Omega\}$. These observations lead to the following definitions:

Let $\Delta \stackrel{\text{def}}{=} \{q \in \Omega : \theta_q \in \mathsf{Irr}(\mathsf{S})\}$. The information system $\mathcal{J} = \langle U, V_q, f_q \rangle_{q \in \Delta}$ is called the *attribute reduct* of $\mathcal{I}$, denoted by $\mathcal{I}^{red}$. We use the term *attribute reduct*, since $\mathcal{I}^{red}$ corresponds to an algebraic reduct of the knowledge approximation algebra associated with $\mathcal{I}$ defined by Comer (1991); on the other hand, as we have seen above, in rough set theory *reduct* has a different meaning. Note that, once we have chosen an essential system $\mathcal{I}$ from a given system, $\mathcal{I}^{red}$ is uniquely determined.

The following observation, which, in a way, is a converse to Theorem 2.4 now follows immediately from the definitions:

**Proposition 3.1.** $\mathcal{I}^{red} \cong \mathbf{F}(\mathbf{E}(\mathcal{I}))$. □

We call $\mathcal{I}$ *reduced*, if $\mathcal{I} = \mathcal{I}^{red}$. An easily proved characterization of reduced information systems is given by

**Proposition 3.2.** *The following are equivalent:*

1. *$\mathcal{I}$ is reduced.*

*For all $Q \subseteq \Omega,\ p \in \Omega$,*

2. *If $\theta_Q = \theta_p$ then $p \in Q$,*
3. *$\bigcap (\{p\}/\mathsf{ker}(\mathbf{E})) \neq \emptyset$.* □

Here, $\mathsf{ker}(\mathbf{E})$ is the kernel of **E**, i.e. the equivalence relation on $\wp(\Omega)$ induced by **E**.

In a way, $\mathcal{I}^{red}$ is the smallest information system over $U$ which has the same information on indiscernibility as $\mathcal{I}$:

**Proposition 3.3.** *Let $\mathcal{I}^{red} = \langle U, V_q, f_q \rangle_{q \in \Delta}$. Then, $\Delta$ is the smallest subset $Q$ of $\Omega$ for which*

$$[\{\theta_q : q \in Q\}]_{\mathbf{Eq}(\mathrm{U})} = \mathbf{E}(\mathcal{I}).$$

*Proof.* This follows from the fact that $\mathsf{Irr}(\mathbf{E}(\mathcal{I}))$ is the smallest set of generators of $\mathbf{E}(\mathcal{I})$. ☐

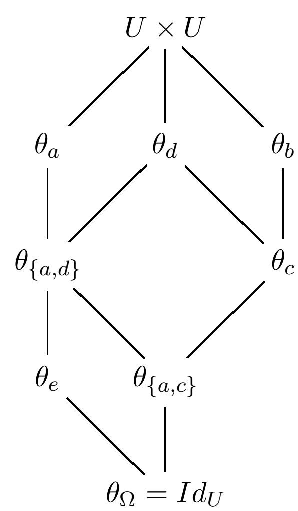
When analyzing an information system it is advisable to consider its attribute reduct; otherwise, information may be misinterpreted and possibly important connections may be lost, as the following example shows:

**Example 1.** Rauszer (1991)

Consider the system $\mathcal{I}$ in the following Table 1 with its corresponding diagram of $\mathbf{E}(\mathcal{I})$. The reducts of $\mathcal{I}$ are $\{b, e\}$ and $\{c, e\}$, and $\mathsf{core}(\mathcal{I}) = \{e\}$. On the other hand, $\Delta = \{a, b, d, e\}$, and $\theta_\Omega = \theta_e \cap \theta_b$. Thus, the only reduct of $\mathcal{I}^{red}$ is $\{e, b\}$, and hence it is also the core of $\mathcal{I}^{red}$. The fact that attribute $b$ is necessary for $\theta_\Omega$ is revealed when we consider $\mathcal{I}^{red}$, while in $\mathcal{I}$ it is hidden by the fact that $c$ is reducible via $b$ and $d$. This is an instance of the situation that the core is enlarged, and we still retain the original information.

**Table 1:** An information system

| U | a | b | c | d | e |
|---|---|---|---|---|---|
| $t_1$ | 0 | 2 | 1 | 3 | 1 |
| $t_2$ | 1 | 0 | 2 | 1 | 2 |
| $t_3$ | 1 | 2 | 1 | 3 | 3 |
| $t_4$ | 1 | 0 | 2 | 1 | 0 |
| $t_5$ | 0 | 0 | 3 | 3 | 1 |




## 4 Dependencies

Throughout this section, we again let $\mathcal{I} = \langle U, \Omega, V_q, f_q \rangle_{q \in \Omega}$, and $S = \mathbf{E}(\mathcal{I})$, which we regard as a subsemilattice of $\mathbf{Eq}(\mathrm{U})$.

At the centre of rough set theory is the notion of (in–) dependence of attributes. An attribute $p$ is dependent on a set $Q$ of attributes, written as $Q \to p$, if whenever two objects $x, y$ agree on all attributes in $Q$, then they agree on $p$. In other words,

$$\begin{aligned} Q \to p \text{ iff } & (\forall x, y \in U)[(\forall q \in Q) f_q(x) = f_q(y) \Rightarrow f_p(x) = f_p(y)] \\ \text{iff } & \theta_Q \subseteq \theta_p. \end{aligned}$$

It is easy to see that $\to$ has the following properties, cf Düntsch & Gediga (1998):

| | | |
|---|---|---|
| (4.1) | If $q \in Q,$ then $Q \to q$ | (Reflexivity) |
| (4.2) | If $Q \to q,$ then $(Q \cup P) \to q$ | (Monotony) |
| (4.3) | If $Q \to p$ for all $p \in P$ and $(Q \cup P) \to q,$ then $Q \to q$ | (Cut). |

These are the properties of a consequence relation of a monotone logic. We extend $\to$ over $\wp(\Omega) \times \wp(\Omega)$ by setting

$$Q \to^{\Omega} P \text{ if and only if } Q \to p \text{ for all } p \in P.$$

The relation $\to^{\Omega}$ is the dependency relation of Novotný & Novotný (1992), and we say that $P$ *is dependent on* $Q$ just in case $Q \to^{\Omega} P$. The relation $\to^{\Omega}$ has the following properties:

| | |
|---|---|
| (4.4) | $\to^{\Omega}$ contains $\supseteq$, |
| (4.5) | $\to^{\Omega}$ is transitive, |
| (4.6) | $\langle A, C_i \rangle \in \to^{\Omega}$ for all $i \in I$ implies $\langle A, \bigcup_{i \in I} C_i \rangle \in \to^{\Omega}$ |

If no confusion can arise we shall usually write $\to$ instead of $\to^{\Omega}$. More generally, we call a binary relation $R$ on $\wp(\Omega)$ which satisfies (4.4), (4.5), and (4.6) a *dependence relation* on $\Omega$; these relations have been studied in detail in Novotný (1997).

Apart from information systems and logical consequence, dependence relations arise in the context of querying experts to obtain knowledge structures, see for example Koppen & Doignon (1990), Düntsch & Gediga (1998) or Düntsch & Gediga (1996):

Let $Rel(\wp(\Omega))$ be the set of all binary relations on $\wp(\Omega)$, partially ordered by set inclusion. We define an operator $Rel(\wp(\Omega)) \xrightarrow{\rho} \wp(\wp(\Omega))$ by

$$R^{\rho} \stackrel{\text{def}}{=} \{A \subseteq \Omega : (\forall \langle P, Q \rangle \in R)[P \subseteq A \Rightarrow Q \subseteq A]\}.$$

The action of the operator $^{\rho}$ on dependence relations is described by

**Proposition 4.1.** *If $R$ is a dependence relation on $\Omega$, then*

1. $A \in R^{\rho} \iff (\forall B \subseteq \Omega)(\langle A, B \rangle \in R \Rightarrow B \subseteq A).$
2. *Let* $S \stackrel{\text{def}}{=} 1' \setminus [(R \cap \subsetneq) \circ (R \cap \subsetneq)^{-1}]$, *where* $1'$ *is the identity relation on* $\wp(\Omega)$. *Then,*

$$A \in R^{\rho} \iff A \in \operatorname{dom}(S).$$

*Proof.* 1. "$\Rightarrow$": If $\langle A, B \rangle \in R$, then $A \in R^{\rho}$ implies $B \subseteq A$.

"$\Leftarrow$": Let $\langle P, Q \rangle \in R$, and $P \subseteq A$. Then, $\langle A, P \rangle \in R$, since $R$ contains $\supseteq$. Transitivity of $R$ gives $\langle A, Q \rangle \in R$, and the condition now implies that $Q \subseteq A$.

2. Let $A \subseteq \Omega$. Then,

$$\begin{aligned} A \notin \mathrm{dom}(S) &\iff \langle A, A\rangle \in [(R\cap \subsetneq) \circ (R\cap \subsetneq)^{-1}] \\ &\iff (\exists B \subseteq \Omega)[\langle A, B\rangle \in R \text{ and } A \subsetneq B] \\ &\iff A \notin R^{\rho}, \end{aligned}$$

by 1. □

Condition 2. shows an efficient way to compute $R^{\rho}$ from $R$.

Conversely, we define an operator $\wp(\wp(\Omega)) \xrightarrow{\sigma} Rel(\wp(\Omega))$ by

$$K^{\sigma} = \{\langle A, B\rangle \in \wp(\Omega)^2 : (\forall C \in K)(A \subseteq C \Rightarrow B \subseteq C)\}.$$

**Proposition 4.2.** *Koppen & Doignon (1990)*

*$K^{\sigma}$ is a dependence relation on $\Omega$ and the pair $\langle \rho, \sigma\rangle$ is a Galois connection between $\langle \wp(\wp(\Omega)), \subseteq\rangle$ and $\langle Rel(\wp(\Omega)), \subseteq\rangle$, in which the Galois closed sets are the closure systems on $\Omega$ (i.e. the substructures of $\langle \wp(\Omega), \cap, \Omega\rangle$), respectively the dependence relations on $\Omega$.*

Another source of dependence relations are Galois connections (see e.g. Grätzer (1978) for a definition):

**Proposition 4.3.** *Let $\langle \varphi, \psi\rangle$ be a Galois connection between $\langle \wp(A), \subseteq\rangle$ and $\langle \wp(B), \subseteq\rangle$. Then $R \in Rel(\wp(A))$ defined by*

$$XRY \iff X^{\varphi} \subseteq Y^{\varphi}$$

*is a dependence relation on $A$.*

*Proof.* Clearly, $R$ is reflexive, transitive, and contains $\supseteq$, the latter by $X \subseteq Y \Rightarrow Y^{\varphi} \subseteq X^{\varphi}$. If $XRY$ and $XRZ$, then $X^{\varphi} \subseteq Y^{\varphi}$, $X^{\varphi} \subseteq Z^{\varphi}$. Thus, $X^{\varphi} \subseteq Y^{\varphi} \cap Z^{\varphi} = (Y \cup Z)^{\varphi}$, and it follows that $XR(Y \cup Z)$. □

In Düntsch & Gediga (1996) we have described all Galois connections between $\langle \wp(\wp(\Omega)), \subseteq\rangle$ and $\langle Rel(\wp(\Omega), \subseteq\rangle$ in which the Galois closed sets of the latter structure are the dependence relations.

We now turn to (in–)dependent subsets. Each dependency which holds in $\mathcal{I}$ corresponds to an equation[2] in $\mathbf{E}(\mathcal{I})$; vice versa, if $Q \longrightarrow P$ is true in $\mathcal{I}$, then $\bigcap_{q\in Q} \theta_q \subseteq \bigcap_{p\in P} \theta_p$ holds in $\mathbf{E}(\mathcal{I})$. Conversely, if $\theta, \psi \in \mathbf{E}(\mathcal{I})$, and $\bigcap_{q\in Q} \theta_q = \theta \subseteq \psi = \bigcap_{p\in P} \theta_p$, where $P, Q \subseteq \Omega$, then $Q \longrightarrow P$ is true in $\mathcal{I}$. Hence, the dependency structure of $\mathcal{I}$ is essentially the same as the equational structure of $\mathbf{E}(\mathcal{I})$.

Pawlak & Rauszer (1985) call a nonempty subset $Q$ of $\Omega$ *independent* in $\Omega$, if $\theta_{Q\setminus\{q\}} \neq \theta_Q$ for all $q \in Q$, otherwise, $Q$ is called *dependent*.

This notion of (in–)dependence coincides with the algebraic one:

[2] We use *equation* not in the universal algebraic sense, but as a relation between concrete elements of $\mathbf{E}(\mathcal{I})$.

**Proposition 4.4.** *Let $\emptyset \neq Q \subseteq \Omega$. Then,*

$$Q \textit{ is independent in } \Omega \Longleftrightarrow \{\theta_q : q \in Q\} \textit{ is independent in } \mathbf{E}(\mathcal{I}).$$

*Proof.*

$$\begin{aligned} Q \text{ is independent in } \Omega &\Longleftrightarrow \theta_{Q\setminus\{q\}} \neq \theta_Q, && q \in Q \\ &\Longleftrightarrow \theta_{Q\setminus\{q\}} \not\subseteq \theta_Q && q \in Q, \\ &\Longleftrightarrow \{\theta_q : q \in Q\} \text{ is independent in } \mathbf{E}(I), \end{aligned}$$

the last equivalence by Proposition 2.2 □

We say that $\mathcal{I}$ has *only trivial dependencies* if for all $P \subseteq \Omega,\ q \in \Omega$

$$\text{If } P \longrightarrow q, \text{ then } q \in P.$$

In view of the correspondence between dependencies of $\mathcal{I}$ and the equational structure of $\mathbf{E}(\mathcal{I})$, the following comes as no surprise; an equivalent result can be found in Pagliani (1993) with a different proof.

**Proposition 4.5.** *The following are equivalent:*

1. *$\mathcal{I}$ has only trivial dependencies,*
2. *$\mathbf{E}(\mathcal{I})$ is freely generated by $X$,*
3. *$\mathbf{E}(\mathcal{I})$ is a Boolean algebra,*
4. *$\longrightarrow^{\Omega} = \supseteq$.*
5. *$\mathsf{core}(\mathcal{I}) = \Omega$*

*Proof.* "1. ⇒ 2.": Since $X$ generates $\mathbf{E}(\mathcal{I})$, we only need to show independence. If $p, q \in \Omega,\ \theta_q \in X$ and $\bigcap(X \setminus \{\theta_q\}) \subseteq \theta_p$, then $\mathcal{I}$ satisfies the nontrivial dependency $(\Omega \setminus \{q\}) \longrightarrow \{p\}$, contradicting our hypothesis.

"2. ⇒ 1.": Let $Q \subseteq \Omega,\ p \in \Omega$, and $Q \longrightarrow \{p\}$. Then, $\bigcap\{\theta_q : q \in Q\} \subseteq \theta_p$. Independence implies $p \in Q$.

The equivalence of 2. and 3. is just Lemma 2.2 (see also 3.10 of Pawlak & Rauszer (1985)), and the equivalence of 1., 4., and 5. is obvious. □

Even though no proper subset of $\mathsf{Irr}(\mathbf{E}(\mathcal{I}))$ generates $\mathbf{E}(\mathcal{I})$ – i.e. no proper subset of $\mathsf{Irr}(\mathbf{E}(\mathcal{I}))$ gives us the same information as the attributes belonging to the elements in $\mathsf{Irr}(\mathsf{A})$ –, the set $P$ associated with $\mathsf{Irr}(\mathbf{E}(\mathcal{I}))$ need not be a reduct as Example 1 shows.

This example also shows that the $\mathbf{E}$–image of a reduct need not generate $\mathbf{E}(\mathcal{I})$. Thus, a reduct carries, in general, less information than the smallest base. This is not surprising, since, by definition, reducts are only concerned with $\theta_\Omega$.

**Proposition 4.6.** *Let $Q \subseteq \Omega$ and and $T \stackrel{\text{def}}{=} \mathbf{E}(Q)$ such that $\theta_Q = \theta_\Omega$. Then,*

$$Q \textit{ is a reduct } \iff T \textit{ is maximally independent.}$$

*Proof.* First, note that

$$\text{If } Y \subseteq \mathbf{E}(\mathcal{I}) \text{ is independent and } \bigcap Y = \theta_\Omega, \text{ then } Y \text{ is maximally independent.}$$

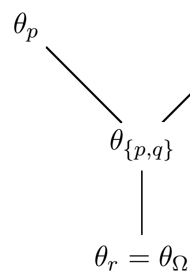
Indeed, this follows immediately from $\bigcap Y = \theta_\Omega \subseteq \theta_p$ for any $p \in \Omega$.

Thus, it is enough to show that $Q$ is a reduct $\iff$ $T$ is independent:

$$\begin{aligned}
Q \text{ is a reduct } &\iff \theta_\Omega \subsetneq \theta_{Q \setminus \{q\}}, && \text{for any } q \in Q \\
&\iff \theta_{Q \setminus \{q\}} \not\subseteq \theta_q && \text{for any } q \in Q \\
&\iff \bigcap_{\theta_p \in T \setminus \{q\}} \theta_p \not\subseteq \theta_q && \text{for any } \theta_q \in T \\
&\iff T \text{ is independent,}
\end{aligned}$$

which proves our claim. □

Above, we require the condition that $\theta_Q = \theta_\Omega$ as the following example shows:

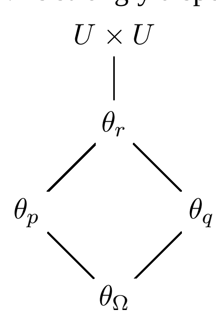



$Q = \{p, q\}$ corresponds to a maximally independent subset of $\mathbf{E}(\mathcal{I})$, but it is not a reduct.

**Corollary 4.7.** *If $Q$ is a reduct of $\mathcal{I}$ and $\mathcal{J} = \langle U, V_q, f_q \rangle_{q \in Q}$, then $\mathcal{J}$ fulfills only trivial dependencies.*

*Proof.* $\mathbf{E}(\mathcal{J})$ has an independent set of generators by 4.6, and thus $\mathcal{J}$ fulfils only trivial dependencies by 4.5 □

Rauszer (1991) calls $p \in \Omega$ *strongly dispensable* if it is not in any reduct. The terminology seems somewhat narrow in the light of Example 1: Some $r \in \Omega$ for which $\theta_Q = \theta_r$, $Q \neq \{r\}$ – as the attribute $c$ in the Example – surely is superfluous, though it may be contained in some reduct. Conversely, $\theta_r$ may be irreducible, while $r$ is strongly dispensable:




An algebraic characterization of strongly dispensable elements is given by

**Proposition 4.8.** *$p$ is strongly dispensable if and only if $\theta_p$ is dense in $\mathbf{E}(\mathcal{I})$.*

*Proof.* "⇒": Let $\theta_Q \cap \theta_p = \theta_\Omega$. Then, there is a reduct $P$ below $Q \cup \{p\}$. If $\theta_Q \neq \theta_\Omega$, then the same is true for every subset of $Q$, and it follows that $p \in P$, contradicting our assumption.

"⇐": Let $P \subseteq \Omega$ be a reduct, and assume that $p \in P$. Then, since $P$ is minimal with $\theta_P = \theta_\Omega$, we have

$$\theta_\Omega = \theta_{P \setminus \{p\}} \cap \theta_p, \text{ and } \theta_{P \setminus \{p\}} \neq \theta_\Omega.$$

It follows that $\theta_p$ is not dense in $\mathbf{E}(\mathcal{I})$. □

This result is equivalent to 3.7.2. of Pagliani (1993).

Our final result of this section gives an easy proof of the (known) fact that to find the core of $\mathcal{I}$ it is not necessary to find all the reducts:

**Proposition 4.9.** *$p \in \mathsf{core}(\mathcal{I})$ if and only if $\bigcap_{q \in \Omega \setminus \{p\}} \theta_q \neq \theta_\Omega$.*

*Proof.* "⇒" Let $\theta_\Omega = \bigcap_{q \in \Omega \setminus \{p\}} \theta_q$. Then, there is some $Q \subseteq \Omega,\ p \notin Q$, such that $Q$ is minimal with the property that $\theta_Q = \theta_\Omega$. Now, $Q$ is a reduct which does not contain $p$, and it follows that $p \notin \mathsf{core}(\mathcal{I})$.

"⇐": If $p \notin \mathsf{core}(\mathcal{I})$, there is a reduct $Q$ such that $p \notin Q$. Hence, $\theta_\Omega = \theta_Q \supseteq \theta_{\Omega \setminus \{p\}}$. □

# 5 Ordering information systems

Given an information system $\mathcal{I} = \langle U, \Omega, V_q, f_q \rangle_{q \in \Omega}$, there are two ways of reducing its granularity: One possibility is to disregard attributes by looking at the systems $\langle U, \Delta, V_q, f_q \rangle_{q \in \Delta}$ where $\Delta \subseteq \Omega$. Another way is to collect attributes into a single one as in 3.1

These constructions are fundamentally different: In the first one, we lose information which cannot be reconstructed once we have made the reduction (and forgotten what we had before). The second one is a shift of information from the number of attributes to the domains; structure which is lost from the attribute set is added to the attribute domains, and can (partially) be recovered, if the new domain stays recognizable as a product of sets.

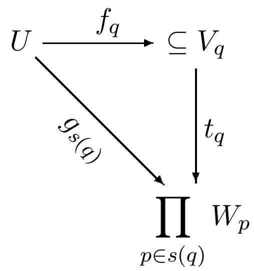
Our aim in defining a quasi–ordering on information systems over $U$ is twofold: on the one hand we want to capture both constructions mentioned above, and on the other hand, we want to imitate the ordering on Sub($\mathbf{Eq}$(U)). This is achieved by the following construction:

If $\mathcal{I}$ is as above, and $\mathcal{H} = \langle U, \Delta, W_p, g_p \rangle_{p \in \Delta}$, we say that $\mathcal{H}$ *extends* $\mathcal{I}$, and write $\mathcal{I} \leq \mathcal{H}$, if there are mappings

$$\text{(5.1)} \qquad \begin{cases} s: & \Omega \to \wp(\Delta), \\ t_q: & \operatorname{ran}(f_q) \subseteq V_q \hookrightarrow \prod_{p \in s(q)} W_p \end{cases}$$

such that for each $q \in \Omega$ the mapping $t_q$ is one–one, and the following diagram commutes:

(5.2)




We also define $\overline{s} : \wp(\Omega) \to \wp(\Delta)$ by

$$\overline{s}(Q) \stackrel{\text{def}}{=} \bigcup s[Q].$$

Observe that the mapping $t_q$ is unique if it exists. Each attribute of $\mathcal{I}$ corresponds to a set of attributes of $\mathcal{H}$, but $\mathcal{H}$ may have more attributes than are captured via $s$. Thus, we can say that the information given by $\mathcal{H}$ is finer than that given by $\mathcal{I}$.

Our next result gives a necessary and sufficient condition for the mappings $t_q$ to exist:

**Proposition 5.1.** *Let $\mathcal{I}$ and $\mathcal{H}$ be as above and $s : \Omega \to \wp(\Delta)^+$, $q \in \Omega$. Then, there is a $t_q$ which satisfies condition* (5.2) *if and only if*

(5.3) $$\theta_q = \theta_{s(q)}.$$

*Proof.* "⇒": Suppose that $t_q$ satisfies (5.2). It is enough to show that every class of $\theta_q$ is a class of $\theta_{s(q)}$. Each class of $\theta_q$ is of the form $f_q^{-1}$ for some $a \in V_q$. Thus, let $x \in f_q^{-1}(a)$, and $M$ be the class of $\theta_{s(q)}$ containing $x$. We need to show that $M = f_q^{-1}(a)$:

"$\subseteq$": Let $y \in M$. Then,

$$g_{s(q)}(x) = g_{s(q)}(y),$$

and hence

$$t_q(f_q(x)) = t_q(f_q(y)).$$

Injectivity of $t_q$ gives us $f_q(x) = f_q(y) = a$.

"$\supseteq$": Let $f_q(y) = q$. Then, $f_q(x) = f_q(y)$, and therefore

$$g_{s(q)}(x) = t_q(f_q(x)) = t_q(f_q(y)) = g_{s(q)}(y).$$

Hence, $x\theta_{s(q)}y$, and it follows that $f_q^{-1} \subseteq M$.

"⇐": Let $a \in V_q$, and $M = f_q^{-1}(a)$. Then, by our hypothesis, $g_{s(q)}(x) = g_{s(q)}(y)$ for all $x, y \in M$. Set $t_q(a) \stackrel{\text{def}}{=} g_{s(q)}(x)$ for $x \in M$. By our previous remark, $t_q$ is well defined, and it clearly satisfies (5.2). It remains to show that $t_q$ is injective:

Let $a, b \in \operatorname{ran}(f_q)$, $a \neq b$, and assume that $t_q(a) = t_q(b)$. Then, by definition of $t_q$, there are $x \in f_q^{-1}(a), y \in f_q^{-1}(b)$ such that $g_{s(q)}(x) = g_{s(q)}(y)$. This implies $x\theta_{s(q)}y$ which contradicts the fact that by our hypothesis, $f_q^{-1}(a)$ and $f_q^{-1}(b)$ are different classes of $\theta_{s(q)}$. □

Proposition 5.1 immediately shows that the relation $\leq$ captures disregarding as well as collecting attributes, and thus we have achieved our first aim:

**Proposition 5.2.** *Let $\mathcal{I} = \langle U, \Omega, V_q, f_q \rangle_{q \in \Omega}$ be an information system, and $\emptyset \neq \Delta, Q \subseteq \Omega$. If $\mathcal{I}'$ results from $\mathcal{I}$ by using just $\Delta$ as the attribute set, or by collecting the attributes in $Q$ as in* (3.1)*, then $\mathcal{I}' \leq \mathcal{I}$.* □

Condition (5.3) easily extends to subsets of $\Omega$:

**Corollary 5.3.** *Let $\mathcal{I}$ and $\mathcal{H}$ be as above, and $\mathcal{I} \leq \mathcal{H}$ via $s$. Then,*

$$\theta_Q = \theta_{\overline{s}(Q)}$$

*for all $Q \subseteq \Omega$.*

*Proof.* If $Q = \emptyset$, then $\overline{s}(Q) = \emptyset$. Otherwise,

$$\begin{aligned} \theta_Q &= \bigcap_{q \in Q} \theta_q &&= \bigcap_{q \in Q} \theta_{s(q)} &&= \bigcap_{q \in Q} \bigcap_{r \in s(q)} \theta_r \\ &= \bigcap_{r \in \bigcup_{q \in Q} s(q)} \theta_r &&= \bigcap_{r \in \overline{s}(Q)} \theta_r &&= \theta_{\overline{s}(Q)}. \end{aligned}$$

□

It also follows that $\leq$ is compatible with the ordering on $Sub\,\mathrm{Eq(U)}$:

**Corollary 5.4.** *Let $\mathcal{I}$ and $\mathcal{H}$ be as above. Then,*

$$\mathcal{I} \leq \mathcal{H} \Longleftrightarrow \mathbf{E}(\mathcal{I}) \leq \mathbf{E}(\mathcal{H}).$$

*Proof.* "$\Rightarrow$": If $q \in \Omega$, then $\theta_q = \bigcap_{p \in s(q)} \theta_p \in \mathbf{E}(\mathcal{H})$ by Proposition 5.1

"$\Leftarrow$": For each $q \in \Omega$, choose some $M_q \subseteq \Delta$ such that $\theta_q = \bigcap_{t \in M} \theta_t$, and define

$$q \overset{s}{\mapsto} M_q$$

Then, $\theta_q = \theta_{s(q)}$, and $\mathcal{I} \leq \mathcal{H}$ by 5.1 □

The relation $\leq$ is clearly reflexive and transitive, but it need not be antisymmetric, even up to isomorphism. This is due to the fact that an information system may carry redundant information which is not taken into consideration. However, if we only consider reduced systems, then $\leq$ is a partial order on the isomorphism classes:

**Proposition 5.5.** *Let $\mathcal{I}, \mathcal{J}$ be reduced information systems. Then,*

$$\mathcal{I} \leq \mathcal{J} \text{ and } \mathcal{J} \leq \mathcal{I} \Rightarrow I \cong J.$$

*Proof.* $\mathcal{I} \leq \mathcal{J}$ and $\mathcal{J} \leq \mathcal{I}$ implies that $\mathbf{E}(\mathcal{I}) = \mathbf{E}(\mathcal{J})$ by Corollary 5.4 By Proposition 3.1 we now obtain

$$\mathcal{I} = \mathcal{I}^{red} \cong \mathbf{F}(\mathbf{E}(\mathcal{I})) = \mathbf{F}(\mathbf{E}(\mathcal{J})) \cong \mathcal{J}^{red} = \mathcal{J},$$

which is our desired result. □

Finally, we show that within the realm of reduced systems the ordering $\leq$ is the adequate notion to capture the sub–semilattice relation on $\mathbf{Eq}(\mathrm{U})$. In other words, the sub–semilattices of $\mathbf{Eq}(\mathrm{U})$ correspond exactly to the (isomorphism classes of) reduced information systems over $U$:

**Proposition 5.6.** *The mapping* $\mathbf{F}$ *is an order isomorphism between* $Sub(\mathbf{Eq}(\mathrm{U}))$ *and the isomorphism classes of reduced information systems over* $U$*, ordered by* $\leq$.

*Proof.* If $L \leq \mathbf{Eq}(\mathrm{U})$, then clearly $\mathbf{F}(L)$ is reduced by Proposition 3.2, and by Proposition 3.1 each weakly reduced system is isomorphic to some $\mathbf{F}(L)$.

Let $L \leq M \leq \mathbf{Eq}(\mathrm{U})$, $\mathsf{Irr(L)} = \{\theta_\mathsf{q} : \mathsf{q} \in \Omega\}$, $\mathsf{Irr(M)} = \{\theta_\mathsf{r} : \mathsf{r} \in \Delta\}$. For each $q \in \Omega$ choose some $s(q) \subseteq \Delta$ such that $\theta_q = \bigcap_{r \in s(q)} \theta_r$. Such $s(q)$ exists since $L \leq M$; by 5.1 there are mappings $t_q$ which satisfy (5.2), and thus $\mathbf{F}(L) \leq \mathbf{F}(M)$.

Let $\theta \in M \setminus L$; then there is some $\theta_r \in \mathsf{Irr(M)} \setminus \mathsf{L}$. Assume that $\mathbf{F}(L) \cong \mathbf{F}(M)$. Then, in particular, $\mathbf{F}(M) \leq \mathbf{F}(L)$, which is exhibited w.l.o.g. by $s : \Delta \to \wp(\Omega)$. By 5.1 we have

$$\theta_r = \theta_{s(r)} \in \mathbf{E}(\mathbf{F}(L)) = L,$$

a contradiction. ☐

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