Relational Properties of Sequential Composition of Co-Algebras

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Abstract. In this paper we define a sequential composition for arbitrary co-algebras in a Dedekind category. We show some basic algebraic properties of this operation up to bisimulation. Furthermore, we consider certain recursive equations and provide an explicit solution, i.e., a solution not based on an iterative process.

1 Introduction

Co-algebras, bisimulation, and co-induction play an important role in mathematics and computer science. One of the first examples of bisimulations appear in process calculi such as CSP or CCS [4, 6]. Such processes can be modeled by co-algebras, and the behavioral equivalence is based on bisimulation. These calculi also introduce the basic operations of parallel composition, summation, and prefixing on processes.

In addition, models of modal logics give naturally rise to co-algebras based on the underlying transition relation of □ and ◤ [9]. Bisimulation and co-induction is used to reason about models, property preserving constructions, and relationship to other logics. For example, Van Benthem’s characterization theorem shows that modal logic is the fragment of first-order logic that is invariant under bisimulation. Another example is given by the method of filtration that is used to show that the satisfiability problem of certain modal logics is decidable. It has been shown that filtration is based on a bisimilarity relation [12].

Recently there has been great interest in using coinduction and bisimulation to reason about lazy functional programming languages. The first motivation for this approach was given in [1]. In this paper the lazy lambda calculus has been defined and it was shown a bisimulation, called applicative bisimulation, is a congruence on the terms of this calculus. The interest in lazy language also started intensive research on co-algebraic specification. These specifications are

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based on observation operations instead of constructors, an approach that leads
to algebraic specifications. For an intensive study of examples of co-algebraic
specifications we refer to [5].

For additional examples and a more detailed overview of methods and appli-
cations of co-algebras, bisimulation, and co-induction we refer to [10, 11].

In this paper we want to consider co-algebras in the context of Dedekind cat-
ergyes. These categories provide a suitable abstraction to reason about relation
[7, 8]. Therefore, a co-algebra is a relation $Q : S \to F(S)$ with an appropri-
ate functor $F$. This generalizes processes, labeled transition systems, or other
co-algebraic structures such as Kripke structures, in multiple ways. First of all,
relations in Dedekind category need not to be relations in the classical sense,
i.e., subsets of the cartesian product of sets. Lattice-valued or $L$-fuzzy relations
as well as probabilistic relations are models of Dedekind categories. Among the
huge class of additional examples there are even non-representable Dedekind
categories, i.e., categories where the morphisms are not equivalent to any notion
of a relation based on sets of pairs. The second generalization is given by the
functor $F$. A co-algebra $Q : S \to F(S)$ has two aspects. It provides a transition
from a state to a successor state together with an additional effect encoded by
$F$. In this paper we will make no assumption on this additional behavior.

Relational co-algebras have been studied before in [16, 17]. These paper mainly
studied parallel composition of processes and its relational properties. It was
shown, for example, that equivalence classes of certain co-algebras form an or-
dered category based on parallel composition. In the current paper we want to
concentrate on sequential composition.

This paper is organized as follows. Section 2 introduce the basic mathemati-
cal notions such as Dedekind categories, relators, co-algebras, rooted co-algebras,
and bisimulations. In Section 3 we define the sequential composition of two
rooted co-algebras. After investigating some basic properties of this operation
we define the sum of rooted co-algebras in Section 4. This is corresponds to the
disjoint union of co-algebras by taking their roots into account. Furthermore,
some basic property of this operation are shown. The main theorem of this sec-
tion shows that the sequential composition distributes from the right over the
the sum. In Section 5 we consider simple recursion on a co-algebra by providing
an explicit solution to the equation $X = P.X$ where $P$ denotes the sequential com-
position. Finally, Section 6 outlines how to solve general equations in a language
based on the construction defined in this paper. As in Section 5 the solution is
given explicitly.

2 Mathematical Preliminaries

Throughout this paper, we use the following notation. To indicate that a mor-
phism $R$ of a category $\mathcal{R}$ has source $A$ and target $B$ we write $R : A \to B$. The
collection of all morphisms $R : A \to B$ is denoted by $\mathcal{R}[A, B]$ and the compo-
sition of a morphism $R : A \to B$ followed by a morphism $S : B \to C$ by $R; S$.
Last but not least, the identity morphism on $A$ is denoted by $I_A$. 
In this section we recall some fundamentals on Dedekind categories [7, 8].
This kind of category is called locally complete division allegories in [3].

**Definition 1.** A Dedekind category \( \mathcal{R} \) is a category satisfying the following:

1. For all objects \( A \) and \( B \) the collection \( \mathcal{R}[A,B] \) of morphisms/relations is a complete distributive lattice. Meet, join, the induced ordering, the least and the greatest element are denoted by \( \sqcap, \sqcup, \sqsubseteq, \sqsupseteq \) and \( \top, \bot \), respectively.
2. There is a monotone operation \( \sim \) (called converse) mapping a relation \( Q : A \rightarrow B \) to a relation \( Q \sim : B \rightarrow A \) such that \( (Q;R) \sim = R \sim ; Q \sim \) and \( (Q \sim) \sim = Q \) for all relations \( Q : A \rightarrow B \) and \( R : B \rightarrow C \).
3. For all relations \( Q : A \rightarrow B, R : B \rightarrow C \) and \( S : A \rightarrow C \) the modular law \( Q; R \sqcap S \sqsubseteq Q ; (R \sqcap Q \sim ; S) \) holds.
4. For all relations \( R : B \rightarrow C \) and \( S : A \rightarrow C \) there is a relation \( S/R : A \rightarrow B \) (called the left residual of \( S \) and \( R \)) such that for all \( Q : A \rightarrow B \) the following holds: \( Q; R \sqsubseteq S \iff Q \sqsubseteq S/R \).

Throughout this paper we will use several basic properties of Dedekind categories such as \( I \sim A = I A \), the monotonicity of composition in both parameters, or the distributivity of \( ; \) over \( \sqcap \) without mentioning. For details we refer to [2, 3, 13–15].

An important class of relations within Dedekind categories are mappings.

**Definition 2.** Let \( Q : A \rightarrow B \) be a relation. Then we call

1. \( Q \) univalent iff \( Q \sim; Q \sqsubseteq I_B \);
2. \( Q \) total iff \( I_A \sqsubseteq Q; Q \sim \);
3. \( Q \) a map iff \( Q \) is univalent and total.

In the next lemma we recall an important properties of mappings that we will use in this paper.

**Lemma 1.** Suppose \( Q : A \rightarrow B \) and \( R : D \rightarrow C \) are relations, and \( f : B \rightarrow C \) is a map. Then we have

\[ Q; f \sqsubseteq R \iff Q \sqsubseteq R; f \sim. \]

A proof may be found in [13–15].

The notion of a unit in a category of relations corresponds to terminal objects in categories of functions.

**Definition 3.** A unit \( I \) of a Dedekind category \( \mathcal{R} \) is an object of \( \mathcal{R} \) so that \( I I = \Pi_I \) and \( \Pi_{AI}; \Pi_{IA} = \Pi_{AA} \), i.e., \( \Pi_{AI} \) is total, for all objects \( A \).

A unit is a terminal object in the subcategory of mapping, and, therefore, unique up to isomorphism. In the Dedekind category of sets and relations a unit is any singleton set.

Another important construction is based on forming the disjoint union of sets.
Definition 4. Let $A$ and $B$ be objects of a Dedekind category. An object $A + B$ together with relations $\iota : A \to A + B$ and $\kappa : B \to A + B$ is called a relational sum of $A$ and $B$ if

$\iota; \iota \sim = \iota_A$,  $\kappa; \kappa \sim = \kappa_B$,  $\iota; \kappa \sim = \iota_{AB}$,  $\iota \sqcup \kappa; \kappa = \iota_{A+B}$.

$\mathcal{R}$ has (binary) relational sums iff for every pair of objects the relational sum does exist.

Notice that a relational sum is a biproduct of the Dedekind category. We will use the following abbreviations for relation $Q : A \to C, R : B \to C$ and $S : B \to D$:

$[Q, R] := \iota; Q \sqcup \kappa; R$,  
$Q + S := \iota; Q \sqcup \kappa; S; \kappa$.

The relations $[Q, R] : A + B \to C$ and $Q + T : A + B \to C + D$ satisfy the following properties:

$\iota; [Q, R] = Q$,  $\kappa; [Q, R] = R$,  $[Q, R]; T = [Q; T, R; T]$,  
$Q + S = [Q; \iota, S; \kappa]$,  $(Q + S); \iota \sim = \iota; Q$,  $(Q + S); \kappa \sim = \kappa; R$,  
$(Q + S); [U, V] = [Q; U, S; V]$.

It is well-known that morphisms on biproducts can be represented by matrices. Each row (column) of the matrix corresponds to one factor of the relational sum of the source (target) of the relation. For example, if $Q : A \to C, R : B \to D, S : B \to C$ and $T : B \to D$, then the matrix representation of $X = \iota; Q \sqcup \kappa; R; \kappa \sqcup \kappa; S; \iota \sqcup \kappa; T; \kappa$ is

$X = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$.

Similar, if $Y : A + B \to C + D$, then we have

$Y = \begin{pmatrix} \iota; Y; \iota & \iota; Y; \kappa \sim \\ \kappa; Y; \iota & \kappa; Y; \kappa \sim \end{pmatrix}$.

In particular, for $[Q, R]$ and $Q + S$ we obtain

$[Q, R] = \begin{pmatrix} Q \\ P \end{pmatrix}$,  
$Q + S = \begin{pmatrix} Q & \iota_{AB} \\ \iota_{BC} & S \end{pmatrix}$.

The operations on relations correspond to matrix operations on their matrix representation. In most papers, nested biproducts, and, hence, nested matrices, are flattened, i.e., intermediate brackets are ignored due to the associativity of biproducts. We will not adapt this approach here because we want to make the corresponding isomorphisms, and, hence, all bisimulation relations, explicit. However, the relation $\text{assoc} : S_1 + (S_2 + S_3) \to (S_1 + S_2) + S_3$ defined by

$\text{assoc} = [\iota; \iota, \kappa + \iota_{S_3} = \iota; \iota \sqcup \kappa; \iota \sqcup \kappa; \kappa; \iota \sqcup \kappa; \kappa \sim ; \kappa$.

Notice that relational sums correspond to the lower triangular part of nested biproducts.
is the isomorphism induced by the associativity of the relational sum. As a consequence we have

$$\text{assoc} ; ((Q + R) + S) = (Q + (R + S)) ; \text{assoc}$$

for all suitable $Q, R$ and $S$.

A relator $F : \mathcal{R} \to \mathcal{S}$ between two Dedekind categories $\mathcal{R}$ and $\mathcal{S}$ is a functor that is monotonic on relations and preserves converse. Notice that a relator preserves the properties of being total, univalent, and a map, i.e., it maps mappings to mappings.

### 2.1 Co-Algebras and Bisimulation

As mentioned in the introduction co-algebras play an important role in multiple areas of mathematics and computer science. We are especially interested in co-algebras based on Dedekind categories. Given an endorelator $F : \mathcal{R} \to \mathcal{R}$ a relation $Q : S \to F(S)$ is called an $F$ co-algebra (with states space $S$).

**Definition 5.** A relation $\Phi : S_1 \to S_2$ is called a bisimulation between the two $F$ co-algebras $P : S_1 \to F(S_1)$ and $Q : S_2 \to F(S_2)$ iff

$$\Phi ; Q \sqsubseteq P ; F(\Phi) \land \Phi^\frown ; P \sqsubseteq Q ; F(\Phi^\frown).$$

![Diagram](image)

Notice that by using residuals the two inclusion above can be transformed into a single incusion

$$\Phi \sqsubseteq (P ; F(\Phi)) / Q \sqcap P^\frown \setminus (F(\Phi) ; Q^\frown).$$

Given two co-algebras it is possible to define a co-algebra on the relational sum of their states.

**Definition 6.** Let $P : S_1 \to F(S_1)$ and $Q : S_2 \to F(S_2)$ be co-algebras. Then define

$$P \oplus Q := [P ; F(\iota), Q ; F(\kappa)] : S_1 + S_2 \to F(S_1 + S_2).$$

In matrix representation we obtain

$$P \oplus Q = \begin{pmatrix} P ; F(\iota) \\ Q ; F(\kappa) \end{pmatrix}. $$
Notice that we also have
\[
(Q + R); (S ⊕ T); F(U + V)
= (Q + R); [S; F(\iota); T; F(\kappa)]; F(U + V)
= [Q; S; F(\iota); R; T; F(\kappa)]; F(U + V)
= \iota^-; Q; S; F(\iota; (U + V)) \sqcup \kappa^-; R; T; F(\kappa; (U + V))
= \iota^-; Q; S; F(U; \iota) \sqcup \kappa^-; R; T; F(V; \kappa)
= (Q; S; F(U)) \oplus (R; T; F(V))
\]

and with respect to the isomorphism assoc
\[
\text{assoc; } ((P \oplus Q) \oplus U) = ((\iota^-; \iota \sqcup \kappa^-; \iota^-; \kappa); (P \oplus Q); F(\iota) \sqcup \kappa^-; \iota^-; \kappa^-; \kappa; \iota \sqcup \kappa^-; \iota^-; \kappa^-; \kappa; (P \oplus Q) \oplus U)
= (\iota^-; \iota \sqcup \kappa^-; \iota^-; \kappa); (P \oplus Q); F(\iota) \sqcup \kappa^-; \iota^-; Q; F(\kappa; \iota) \sqcup \kappa^-; \kappa^-; \kappa^-; \kappa; \iota \sqcup \kappa^-; \iota^-; \kappa^-; \kappa; (P \oplus Q) \oplus U)
= (P \oplus (Q \oplus U)); F(\iota^-; \iota; \iota \sqcup \kappa^-; (\iota^-; \kappa; \iota \sqcup \kappa^-; \kappa)
= (P \oplus (Q \oplus U)); F(\text{assoc})
\]

For the examples in this paper we will always use the Dedekind category of sets and relations together with the functor \(F(S) = L \times S\). We will choose a particular representation of the unit \(I\) as the set that contains the element \(*\). A co-algebra \(Q : S \rightarrow F(S)\) in this category is a labeled transition system with state space \(S\) and labels from the set \(L\). For example, if \(S = \{1, 2, 3\}, L = \{a, b\}\) and \(Q = \{(1, (a, 2)), (1, (b, 3)), (2, (a, 3))\}\), then this system can be visualized as a labeled graph by

```
1
 /\ \
/  a \__
\ /  \ \
2   b
\ /  \ \
\ /  \ \
\  a  \ / \
3
```

Notice that labeled transition systems as \(F\) co-algebras with \(F\) defined above do not have an explicit initial state. Even though it seems natural to claim that a source node of the graph is an initial state of the labeled transition system, this is not the case. Actually any state may serve as the initial (or current) state. All of this can be illustrated by the fact that the following two labeled graphs are bisimilar, i.e., each state of each co-algebra is related by a bisimulation to at least one state of the other co-algebra. The dotted lines indicate the largest bisimulation between the graphs which is indeed total and surjective.
The state 2 in the left graph is not a source of the graph. However, 2 is bisimilar to the state 0 of the second graph which is a source of that graph. As a second example consider the following two infinite graphs:

The left graph does not have any sources but each state is bisimilar to source in the right graph. Notice that those graphs are neither smaller, greater, nor bisimilar to \{(0, (a, 0))\}, i.e., the graph:

Since this paper studies sequential composition of \( F \) co-algebras we need to make an initial state explicit. Therefore, we will consider rooted \( F \) co-algebras. A rooted co-algebra is a pair \((P, i)\) consisting of a \( F \) co-algebra \( P : S \rightarrow F(S) \) and a point \( i : I \rightarrow S \); its initial state. We will use the notation \((P, i) : S \rightarrow F(S)\) to indicate that \((P, i)\) is a rooted \( F \) co-algebra with states from \( S \).

To call two rooted co-algebras bisimilar it is not longer necessary that all states of one co-algebra correspond to at least one state of the other co-algebra.
It is sufficient that the all the states that are reachable from the initial state are covered, i.e., that there is a bisimulation that is total and surjective with respect to the reachable states. Equivalently, we may require that the initial states are related.

**Definition 7.** Two $F$ co-algebras $(P, i_P) : S_1 \rightarrow F(S_1)$ and $(Q, i_Q) : S_2 \rightarrow F(S_2)$ are called bisimilar, denoted by $(P, i_P) \sim (Q, i_Q)$, iff there is a bisimulation $\Phi : S_1 \rightarrow S_2$ so that $i_P^{\circ}; i_Q \sqsubseteq \Phi$.

Notice that the property above is equivalent to either of the two inclusions

$$i_Q \sqsubseteq i_P; \Phi, \quad i_P \sqsubseteq i_Q; \Phi^{\circ}.$$

On the other hand, an arbitrary co-algebra $P$ may have terminal states or nodes. These are described by the vector $\perp \perp F(S)/P$ since $v \sqsubseteq \perp \perp F(S)/P \iff v; P \sqsubseteq \perp \perp F(S)$, i.e., this vector describes those states from which no transition is possible. We will denote this vector also by $t_P$, i.e., $t_P = \perp \perp F(S)/P$.

**Lemma 2.** Suppose $\Phi : S_1 \rightarrow S_2$ is a bisimulation between $P_1 : S_1 \rightarrow F(S_1)$ and $P_2 : S_2 \rightarrow F(S_2)$. Then we have $t_{P_1}; \Phi \sqsubseteq t_{P_2}$ and $t_{P_2}; \Phi^{\circ} \sqsubseteq t_{P_1}$.

**Proof.** From the computation

$$(\perp \perp F(S_1)/P_1); \Phi; P_2 \sqsubseteq (\perp \perp F(S_1)/P_1); P_1; F(\Phi) \quad \Phi \text{ bismilation}$$

$\sqsubseteq \perp \perp F(S_2) \quad \text{Def. residual}$$

we immediately conclude the first inclusion. The second inclusion follows analogously using the second property of bisimulations. \qed

### 3 Sequential Composition of Co-Algebras

On of the main purposes of this paper is to define a sequential composition of co-algebras and show its basic properties. The idea of the sequential composition $P.Q$ is that if reach a terminal state of $P$ we continue at the initial state of $Q$.

**Definition 8.** Let $\mathcal{R}$ be a Dedekind category with (binary) relational sums and subobjects. Furthermore, let $F : \mathcal{R} \rightarrow \mathcal{R}$ be an endorelator, $(P, i_P) : S_1 \rightarrow F(S_1)$ and $(Q, i_Q) : S_2 \rightarrow F(S_2)$ be rooted co-algebras, and $R_{P,Q} : S_{P,Q} \rightarrow S_1 + S_2$ the splitting of the relation $\Xi_{P,Q} = \perp S_1 + S_2 \sqcup (t_P; i \sqcup i_Q; \kappa) ; (t_P; i \sqcup i_Q; \kappa)$. Then we define a rooted co-algebra $(P.Q, i_{P,Q}) : S' \rightarrow F(S')$ by

$$(P.Q, i_{P,Q}) : S' \rightarrow F(S')$$

$$i_{P,Q} := i_P^{\circ}; i; R_{P,Q}^{\circ}.$$
The typing of the relations in the previous definition can be visualized by:

![Diagram](image)

It is easy to verify that $i_{P,Q}$ is indeed a point. This follows immediately from the fact that $i_P, i$ and $R_{P,Q}$ are all maps.

The equivalence relation $\Xi_{P,Q}$ identifies the terminal states of $P$ with the initial state of $Q$. The following computation presents this relation in matrix form:

\[
\Xi_{P,Q} = \mathbb{I}_{S_1 + S_2} \sqcup (t_P; \iota \sqcup i_Q; \kappa) \quad (t_P; \iota \sqcup i_Q; \kappa)
\]

\[
= i_\sim; \iota \sqcup \kappa; \kappa \sqcup i_\sim; t_P; \iota \sqcup i_\sim; t_P; \iota \sqcup i_\sim; i_Q; \kappa \sqcup i_\sim; i_Q; \kappa \sqcup i_\sim; i_Q; \kappa \sqcup i_\sim; i_Q; \kappa \sqcup i_\sim; i_Q; \kappa \sqcup i_\sim; i_Q; \kappa \sqcup i_\sim; i_Q; \kappa
\]

\[
= i_\sim; (\mathbb{I}_{S_1} \sqcup t_P; t_P); \iota \sqcup i_\sim; t_P; \iota \sqcup i_\sim; i_Q; \kappa \sqcup i_\sim; i_Q; \kappa \sqcup i_\sim; i_Q; \kappa
\]

\[
= i_\sim; (\mathbb{I}_{S_1} \sqcup t_P; t_P); \iota \sqcup i_\sim; t_P; \iota \sqcup i_\sim; i_Q; \kappa \sqcup i_\sim; i_Q; \kappa
\]

\[
= \begin{pmatrix}
\mathbb{I}_{S_1} & t_P & t_P; i_Q \\
i_Q; t_P & t_P; i_Q & \mathbb{I}_{S_2}
\end{pmatrix}
\]

Notice that $R_{P,Q}$ is a surjective map because it splits an equivalence relations.

Before we show properties about sequential composition we want to mention that one can define the sequential composition with respect of certain terminal states of $P$ only, i.e., with respect to a vector $r \subseteq t_P$. This composition continues with the initial state in $Q$ only if a terminal state in $r$ is reached. We denote this composition by $P \triangledown Q$. Formally, this definition only requires to replace $t_0$ in the definition of $\Xi$ by $r$. All properties (or their obvious generalization) in this paper remain true because their proofs only use that terminal states are terminal, i.e., $t_P; P = \mathbb{I}_{SF(S)}$.

As a first property we want to show that the definition of a sequential composition respects the equivalence relation on rooted co-algebras induced by the notion of bisimilarity, i.e., it defines an operation on equivalence classes of bisimilar co-algebras.

**Lemma 3.** Let $(P_1, i_{P_1}) : S_1 \to F(S_1), (P_2, i_{P_2}) : S_2 \to F(S_2), (Q_1, i_{Q_1}) : T_1 \to F(T_1)$ and $(Q_2, i_{Q_2}) : T_2 \to F(T_2)$ be rooted $F$ co-algebras with $(P_1, i_{P_1}) \sim \ldots$
(P₂, iₚ₂) and (Q₁, iₚ₁) ∼ (Q₂, iₚ₂). Then we have

(P₁, Q₁, iₚ₁, Q₁) ∼ (P₂, Q₂, iₚ₂).

Proof. Suppose Φ : S₁ → S₂ and Ψ : T₁ → T₂ are bisimulations with iₚ₁ ; iₚ₂ ⊆ Φ and iₚ₂ ; iₚ₁ ⊆ Ψ. Then we compute using the matrix representation

\[
(\Phi + \Psi) ; \Xi_{P₂, Q₂} ; (P₂ ⊕ Q₂)
\]

\[
= \begin{pmatrix}
\Phi & \sqcup S₂ T₂
\\
\sqcap P₂ S₂ & \Psi
\end{pmatrix} ;
\begin{pmatrix}
\| S₂ ⊔ \| t\overset{\sim}{p₂} ; t P₂ & t\overset{\sim}{p₂} ; iₚ₂ \\
\sqcap T₂ & \| iₚ₂ ; t P₂
\end{pmatrix} ;
\begin{pmatrix}
P₂ ; F(ι) \\
Q₂ ; F(κ)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\Phi ; t\overset{\sim}{p₂} ; t P₂ & t\overset{\sim}{p₂} ; iₚ₂ \\
\Psi & iₚ₂ ; t P₂
\end{pmatrix} ;
\begin{pmatrix}
P₂ ; F(ι) \\
Q₂ ; F(κ)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\Phi ; P₂ ⊔ Φ ; t\overset{\sim}{p₂} ; t P₂ ; iₚ₂ ; Q₂ ; F(κ) \\
Ψ ; iₚ₂ ; t P₂ ; P₂ ; F(ι) ⊔ Ψ ; Q₂ ; F(κ)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\Phi ; P₂ ; F(ι) ⊔ Φ ; t\overset{\sim}{p₂} ; iₚ₂ ; Q₂ ; F(κ) \\
Ψ ; Q₂ ; F(κ)
\end{pmatrix}
\]

\[
\]
we first compute

\[ P.Q; (P \oplus Q) \]

we immediately conclude the inclusion

\[ (P_1, Q_1) \subseteq (P_1, Q_1) \]

Proof. From the computation

\[ t_Q; \kappa; R_{P,Q}^{-}; (P, Q) \]

\[ = t_Q; \kappa; R_{P,Q}^{-}; (P \oplus Q); F(R_{P,Q}^{-}) \]

\[ = t_Q; \kappa; \left( \frac{\mathbb{I}_{S_1} \cup i_Q}{t_P}; t_P \right) \left( \frac{i_Q}{t_P}; i_Q \right) \left( \frac{P; F(\iota)}{Q; F(\kappa)} \right) \]

\[ = t_Q; \left( \frac{i_Q}{t_P}; t_P \right) \left( \frac{P; F(\iota)}{Q; F(\kappa)} \right) \]

\[ = t_Q; \left( \frac{P; F(\iota)}{Q; F(\kappa)} \right) \]

\[ = \downarrow \]

we immediately conclude the inclusion \( t_Q; \kappa; R_{P,Q}^{-} \subseteq t_{P,Q} \). For the other inclusion we first compute

\[ \iota; R_{P,Q}^{-}; (P \oplus Q) \]

\[ = \iota; \left( \frac{\mathbb{I}_{S_1} \cup i_Q}{t_P}; t_P \right) \left( \frac{i_Q}{t_P}; i_Q \right) \left( \frac{P; F(\iota)}{Q; F(\kappa)} \right) \]

The second inclusion is shown analogously. Furthermore, we have

\[ i_{P_2, Q_2} = i_{P_2} \ni R_{P_2, Q_2}^{-} \]

\[ i_{P_2} \ni \Phi \ni R_{P_2, Q_2}^{-} \]

\[ i_{P_2} \ni \Xi_{P_1, Q_1}^{-}; (\Phi + \Psi); R_{P_2, Q_2}^{-} \]

which verifies that \((P_1, Q_1, i_{P_1, Q_1})\) and \((P_2, Q_2, i_{P_2, Q_2})\) are bisimilar. \( \square \)

As a second property we want to investigate the terminal states of \( P.Q \). They are essentially given by the terminal states of \( Q \) modulo the equivalence relation \( \Xi_{P,Q} \).

Lemma 4. Let \((P, i_P) : S_1 \rightarrow F(S_1)\) and \((Q, i_Q) : S_2 \rightarrow F(S_2)\) be rooted F co-algebras. Then \( t_{P,Q} = t_Q; \kappa; R_{P,Q}^{-} \).

Proof. From the computation

\[ t_Q; \kappa; R_{P,Q}^{-}; (P, Q) \]

\[ = t_Q; \kappa; R_{P,Q}^{-}; (P \oplus Q); F(R_{P,Q}^{-}) \]

\[ = t_Q; \kappa; \left( \frac{\mathbb{I}_{S_1} \cup i_Q}{t_P}; t_P \right) \left( \frac{i_Q}{t_P}; i_Q \right) \left( \frac{P; F(\iota)}{Q; F(\kappa)} \right) \]

\[ = t_Q; \left( \frac{i_Q}{t_P}; t_P \right) \left( \frac{P; F(\iota)}{Q; F(\kappa)} \right) \]

\[ = t_Q; \left( \frac{P; F(\iota)}{Q; F(\kappa)} \right) \]

\[ = \downarrow \]

we immediately conclude the inclusion \( t_Q; \kappa; R_{P,Q}^{-} \subseteq t_{P,Q} \). For the other inclusion we first compute

\[ \iota; R_{P,Q}^{-}; (P \oplus Q) \]

\[ = \iota; \left( \frac{\mathbb{I}_{S_1} \cup i_Q}{t_P}; t_P \right) \left( \frac{i_Q}{t_P}; i_Q \right) \left( \frac{P; F(\iota)}{Q; F(\kappa)} \right) \]
The last two properties together imply conclude suppose from which \( v; R_{P,Q}^\frown; (P.Q) = (P; F(\iota) \sqcup t_P^\frown; i_Q; Q; F(\kappa)); F(R_{P,Q}^\frown) \) follows. Now, suppose \( v : I \to S_1 \) is an arbitrary relation and compute

\[
v \sqsubseteq t_{P,Q}; R_{P,Q}; \iota^\frown
\]

\[
\iff v; \iota \sqsubseteq t_{P,Q}; R_{P,Q}
\]

\[
\iff v; \iota; R_{P,Q}^\frown \sqsubseteq t_{P,Q}
\]

\[
\iff v; \iota; R_{P,Q}^\frown; (P.Q) \sqsubseteq \sqcup_{F(S_{P,Q})}
\]

\[
\iff v; (P; F(\iota) \sqcup t_P^\frown; i_Q; Q; F(\kappa)); F(R_{P,Q}^\frown) \sqsubseteq \sqcup_{F(S_{P,Q})}
\]

\[
\iff v; (P; F(\iota) \sqcup t_P^\frown; i_Q; Q; F(\kappa)) \sqsubseteq \sqcup_{F(S_{P,Q})}
\]

\[
\iff v; P; F(\iota) \sqsubseteq \sqcup_{F(S_{P,Q})} \text{ and } v; t_P^\frown; i_Q; Q \sqsubseteq \sqcup_{F(S_{P,Q})}
\]

\[
\iff v \sqsubseteq t_P \text{ and } v; t_P^\frown; i_Q \sqsubseteq t_Q
\]

\[
\iff v \sqsubseteq t_P \text{ and } v; t_P^\frown; i_Q \sqsubseteq t_Q
\]

\[
\text{The last two properties imply } v \sqsubseteq v; v^\frown; \text{ so that we conclude } t_{P,Q}; R_{P,Q}; \iota^\frown \sqsubseteq t_Q; i_Q^\frown; t_P. \text{ In addition, we obtain}
\]

\[
t_{P,Q}; (P.Q) \sqsubseteq \sqcup_{F(S_{P,Q})}
\]

\[
\iff t_{P,Q}; R_{P,Q}; (P \oplus Q); F(R_{P,Q}^\frown) \sqsubseteq \sqcup_{F(S_{P,Q})}
\]

\[
\iff t_{P,Q}; R_{P,Q}; (P \oplus Q) \sqsubseteq \sqcup_{F(S_{P,Q})}
\]

\[
\iff t_{P,Q}; R_{P,Q}; \kappa^\frown; Q; F(\kappa^\frown) \sqsubseteq \sqcup_{F(S_{P,Q})}
\]

\[
\iff t_{P,Q}; R_{P,Q}; \kappa^\frown; Q \sqsubseteq \sqcup_{F(S_{P,Q})}
\]

\[
\iff t_{P,Q}; R_{P,Q}; \kappa^\frown \sqsubseteq t_Q.
\]

The last two properties together imply

\[
t_{P,Q} = t_{P,Q}; R_{P,Q}; R_{P,Q}^\frown
\]

\[
= t_{P,Q}; R_{P,Q}; (\iota^\frown; \iota \sqcup \kappa^\frown; \kappa); R_{P,Q}^\frown
\]

\[
= (t_{P,Q}; R_{P,Q}; \iota^\frown; \iota \sqcup t_{P,Q}; R_{P,Q}; \kappa^\frown; \kappa); R_{P,Q}^\frown
\]

\[
\sqsubseteq (t_Q; i_Q^\frown; t_P; \iota \sqcup t_Q; \kappa); R_{P,Q}^\frown
\]

see above
Proof.

Lemma 5. This completes the proof.

The following lemma gives sufficient properties when the sequential composition of two co-algebras result in just the first co-algebra.

**Lemma 5.** Let \((P, i_P) : S_1 \to F(S_1)\) and \((Q, i_Q) : S_2 \to F(S_2)\) be rooted F co-algebras. Then we have

1. If \(Q, i_Q \sim (\sqcup_{\mathcal{F}(I)}, \sqcup_I)\) or \((P, i_P) \sim (P', i_{P'})\) with \(t_{P'} = \sqcup_{I_S'}\), then \((P, i_P) \sim (P.Q, i_{P.Q})\).
2. If \((Q, i_Q) \sim (\sqcup_{\mathcal{F}(I)}, \sqcup_I)\), then \((P, i_P) \sim (Q.P, i_{Q.P})\).

**Proof.** 1. Since the sequential composition preserves bisimulation, it is sufficient to show that \(Q = \sqcup_{S_2F(S_2)}\) (with \(S_2 = I\)) or \(t_P = \sqcup_{I_S'}\) implies \((P, i_P) \sim (P.Q, i_{P.Q})\). In both cases we want to show that \(\nu; R^-_{P,Q}\) is the required bisimulation between \(P\) and \(P.Q\). The second inclusion of a bisimulation is obtained independent of the case by

\[
\begin{align*}
  (\nu; R^-_{P,Q}) & ; P \\
  = R_{P,Q} ; \nu^- ; P \\
  = R_{P,Q} ; \nu^- ; P ; F(\nu; \nu^-) \\
  \sqsubseteq R_{P,Q} ; (\nu^- ; P ; F(\nu) \sqcup \kappa^- ; Q ; F(\kappa)) ; F(\nu^-) \\
  = R_{P,Q} ; (P \sqcup Q) ; F(\nu^-) \\
  \sqsubseteq R_{P,Q} ; (P \sqcup Q) ; F(\Xi_{P,Q}) ; F(\nu^-) \\
  = R_{P,Q} ; (P \sqcup Q) ; F(R^-_{P,Q}) ; F(R_{P,Q} ; \nu^-) \\
  = (P.Q) ; F((\nu; R^-_{P,Q})^-).
\end{align*}
\]

In addition, we always have \(i_{P,Q} = i_P ; (\nu; R^-_{P,Q})\) by definition. For the first and remaining inclusion of a bisimulation we distinguish the two cases:

\(Q = \sqcup_{S_2F(S_2)}\): In this case the first inclusion follows from

\[
\begin{align*}
  \nu; R^-_{P,Q} ; (P.Q) \\
  = \nu ; R^-_{P,Q} ; R_{P,Q} ; (P \sqcup Q) ; F(R^-_{P,Q}) \\
  = \nu ; \Xi_{P,Q} ; (P \sqcup Q) ; F(R^-_{P,Q})
\end{align*}
\]
\[
= \nu; \left( \begin{array}{c}
\mathbb{I}_{S_1} \cup t_P^-; t_P^- ; t_P^- ; i_Q^- \\
i_Q^- ; t_P^- ; \mathbb{I}_{S_2}
\end{array} \right); \left( \begin{array}{c}
P; F(\kappa) \\
Q; F(\kappa)
\end{array} \right); F(R_{P,Q}^-)
\]

\[
= \left( \mathbb{I}_{S_1} \cup t_P^-; t_P^- ; t_P^- ; i_Q^- \right); \left( \begin{array}{c}
P; F(\kappa) \\
Q; F(\kappa)
\end{array} \right); F(R_{P,Q}^-)
\]

\[
= \left( (P \cup t_P^-; t_P^-; P); F(\kappa) \cup t_P^-; i_Q^-; Q; F(\kappa) \right); F(R_{P,Q}^-)
\]

\[
P; F(\kappa); F(R_{P,Q}^-)
\]

\[
t_P = \downarrow t_{S_1}; \text{ In this case } \Xi_{P,Q} = \mathbb{I}_{S_1 + S_2} \text{ follows immediately. As a consequence } R_{P,Q} \text{ is an isomorphism. This time we obtain the first inclusion by}
\]

\[
\nu; R_{P,Q}^-; (P;Q)
\]

\[
\nu; R_{P,Q}^-; R_{P,Q}; (P \oplus Q); F(R_{P,Q}^-)
\]

\[
\nu; (P \oplus Q); F(R_{P,Q}^-) \quad \text{RP,Q isomorphism}
\]

\[
P; F(\kappa); F(R_{P,Q}^-)
\]

\[
P; F(\kappa; R_{P,Q}^-)
\]

2. As in the previous case it is sufficient to show the assertion if \( Q = \downarrow t_{S_2}F(S_2) \). In this case we have \( t_Q = \mathbb{I}_I \) since \( \downarrow I; \downarrow t_{F(I)} = \downarrow t_{F(I)} \). Therefore, we have

\[
\Xi_{Q,P} = \left( \begin{array}{c}
\mathbb{I}_I \\
i_P \\
\mathbb{I}_{S_1}
\end{array} \right).
\]

We want to show that \( \Phi = \kappa; R_{Q,P}^- \) is the required bisimulation. The first inclusion follows from

\[
\Phi; (Q;P) = \kappa; R_{Q,P}^-; R_{Q,P}; (Q \oplus P); F(R_{Q,P}^-)
\]

\[
= \kappa; \left( \begin{array}{c}
\mathbb{I}_I \\
i_P \\
\mathbb{I}_{S_1}
\end{array} \right); \left( \begin{array}{c}
Q; F(\kappa) \\
P; F(\kappa)
\end{array} \right); F(R_{Q,P}^-)
\]

\[
= \kappa; \left( \begin{array}{c}
i_P; P; F(\kappa) \\
P; F(\kappa)
\end{array} \right); F(R_{Q,P}^-)
\]

\[
= P; F(\kappa; R_{Q,P}^-)
\]

\[
= P; F(\kappa).
\]

The second inclusion is shown by

\[
\Phi^-; P = R_{Q,P; \kappa^-}; P
\]

\[
= R_{Q,P; (Q \oplus P); F(\kappa^-)}
\]

\[
\sqsubseteq R_{Q,P; (Q \oplus P); F(R_{Q,P; R_{Q,P; \kappa^-}})}
\]

\[
= (Q;P); F(\Phi^-).
\]
Finally from the computation
\[
\begin{align*}
\llbracket S_1 \rrbracket \subseteq i_P; i_P & \quad i_P \text{ total} \\
= i_P; \kappa; \Xi_{Q,P}; i_Q & \\
= i_P; \kappa; R_{Q,P}; R_{Q,P}; i_Q & \quad i_Q = \llbracket I \rrbracket \\
= i_P; \Phi; i_{Q,P} & \\
\end{align*}
\]
we conclude \( i_{Q,P} \subseteq i_P; \Phi. \)
\square

The main theorem of this section shows that the sequential composition of co-algebras is associative.

**Theorem 1.** Let \((P, i_P) : S_1 \to F(S_1), (Q, i_Q) : S_2 \to F(S_2)\) and \((U, i_U) : S_3 \to F(S_3)\) be rooted \(F\) co-algebras. Then \( (P, (Q,U), i_{P,(Q,U)}) \sim ((P,Q),U, i_{P,(Q,U)}) \).

**Proof.** We want to show that
\[
\Phi = R_{P,(Q,U)}; (\llbracket S_1 + R_{Q,U} \rrbracket); \text{assoc}; (R_{P,Q}^- + \llbracket S_1 \rrbracket); R_{(P,Q,U)}^-
\]
is the required bisimulation from \(P, (Q,U)\) to \((P,Q),U\). First we compute
\[
P, (Q,U) = R_{P,(Q,U)}; (P \oplus (Q,U)); F(R_{P,(Q,U)}^-)
= R_{P,(Q,U)}; (P \oplus (R_{Q,U} + (Q \oplus U); F(R_{Q,U}^-))); F(R_{P,(Q,U)}^-)
= R_{P,(Q,U)}; (\llbracket S_1 + R_{Q,U} \rrbracket); (P \oplus (Q \oplus U)); F((\llbracket S_1 + R_{Q,U} \rrbracket); R_{P,(Q,U)}^-)
\]
Analogously we obtain
\[
(P,Q),U = R_{(P,Q,U)}; (R_{P,Q} + \llbracket S_1 \rrbracket); ((P \oplus Q) \oplus U); F((R_{P,Q} + \llbracket S_1 \rrbracket); R_{(P,Q,U)}^-).
\]
Now we want to analyze the structure of the equivalence relation \(\Xi_1\) on \(S_1 + (S_2 + S_3)\) induced by the two equivalence relations \(\Xi_{Q,U}\) and \(\Xi_{P,(Q,U)}\).
\[
(\llbracket S_1 + R_{Q,U}^- \rrbracket); R_{P,(Q,U)}^; R_{P,(Q,U)}; (\llbracket S_1 + R_{Q,U} \rrbracket)
\]
\[
= (\llbracket S_1 \rrbracket \sqcup \llbracket S_1 + R_{Q,U} \rrbracket _{Q,U} ; R_{Q,U}^-) ; (\llbracket S_1 \rrbracket \sqcup \llbracket S_1, S_3, R_{Q,U} \rrbracket ; R_{Q,U}^-)
\]
\[
= (\llbracket S_1 \rrbracket \sqcup \llbracket S_1, S_3, R_{Q,U} \rrbracket ; R_{Q,U}^-) ; (\llbracket S_1 \rrbracket \sqcup \llbracket S_1 + S_3, R_{Q,U} \rrbracket)
\]
\[
= (\llbracket S_1 \rrbracket \sqcup \llbracket S_1 + S_3, R_{Q,U} \rrbracket ; R_{Q,U}^-) ; (\llbracket S_1 \rrbracket \sqcup \llbracket S_1, S_3, R_{Q,U} \rrbracket ; R_{Q,U}^-)
\]
\[
= (\llbracket S_1 \rrbracket \sqcup \llbracket S_1 + S_3, R_{Q,U} \rrbracket ; R_{Q,U}^- ; R_{Q,U}^-)
\]
\[
= (\llbracket S_1 \rrbracket \sqcup \llbracket S_1 + S_3, R_{Q,U} \rrbracket ; R_{Q,U}^- ; R_{Q,U}^-)
\]
\[
= (\llbracket S_1 \rrbracket \sqcup \llbracket S_1 + S_3, R_{Q,U} \rrbracket ; R_{Q,U}^- ; R_{Q,U}^-)
\]
Next we focus on the equivalence relation \( \Xi_2 \) on \((S_1 + S_2) + S_3\) induced by \( \Xi_{P,Q} \) and \( \Xi_{(P,Q),U} \). We obtain

\[
\begin{align*}
\left( \left( \left( R_{P,Q}^{-} + \mathbb{1}_{S_3} \right) ; R_{(P,Q),U}^{-} ; R(P,Q) ; U ; \left( R_{P,Q}^{-} + \mathbb{1}_{S_3} \right) \right) \right) ;
\left( \left( R_{P,Q}^{-} \downarrow_{S_1 + S_2 + 3} \mathbb{1}_{S_3} \right) ; \left( \mathbb{1}_{S_1} \cup t_{P} ; t_{P} \right) ; t_{P} ; \left( \begin{array}{c}
\mathbb{1}_{S_1} \cup t_{P} ; t_{P} \\
\mathbb{1}_{S_1} \cup t_{P} ; t_{P}
\end{array} \right) \right) ;
\left( \left( \left( R_{P,Q}^{-} \downarrow_{S_1 + S_2 + 3} \mathbb{1}_{S_3} \right) ; \left( \mathbb{1}_{S_1} \cup t_{P} ; t_{P} \right) ; \left( \begin{array}{c}
\mathbb{1}_{S_1} \cup t_{P} ; t_{P} \\
\mathbb{1}_{S_1} \cup t_{P} ; t_{P}
\end{array} \right) \right) \right) \right).
\end{align*}
\]
In order to verify that the matrix representation of $\Xi_2$ has the same entries (but a different hierarchy of matrices) as the matrix representation of $\Xi_1$ we have to show that

$$R_{P,Q}^\sim; (\mathbb{I}_{S_1} \sqcup t_P^\sim; t_P; t_P^\sim; i_Q; (\mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q)) = (\mathbb{I}_{S_1} \sqcup t_P^\sim; t_P; t_P^\sim; i_Q; (\mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q)).$$

Therefore, we investigate the term on the left-hand side as follows

$$R_{P,Q}^\sim; (\mathbb{I}_{S_1} \sqcup t_P^\sim; t_P; t_P^\sim; i_Q; (\mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q)) = R_{P,Q}^\sim; (\mathbb{I}_{S_1} \sqcup t_P^\sim; t_P; t_P^\sim; i_Q; (\mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q)) \quad \text{Lemma 4}$$

$$= R_{P,Q}^\sim; (\mathbb{I}_{S_1} \sqcup t_P^\sim; t_P; t_P^\sim; i_Q; (\mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q)) = R_{P,Q}^\sim; (\mathbb{I}_{S_1} \sqcup t_P^\sim; t_P; t_P^\sim; i_Q; (\mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q)) = R_{P,Q}^\sim; (\mathbb{I}_{S_1} \sqcup t_P^\sim; t_P; t_P^\sim; i_Q; (\mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q))$$

Starting with the term in the left-upper corner of this composition we get

$$= \mathbb{I}_{S_1} \sqcup t_P^\sim; t_P; t_P^\sim; i_Q; (\mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q); i_Q; t_P \quad \text{Lemma 4}$$

where the last equation follows because $i_Q; (\mathbb{I}_{S_2} \sqcup t_Q^\sim; t_Q); i_Q \sqsubseteq \top_1 = \mathbb{I}_1$. The term in the right-upper corner computes to

$$= t_P^\sim; i_Q \sqcup t_P^\sim; i_Q; (\mathbb{I}_{S_2} \sqcup t_Q^\sim; t_Q) \quad \text{since } t_P^\sim; t_P; t_P^\sim; i_Q \sqsubseteq t_P^\sim; \top_1; i_Q = t_P^\sim; i_Q$$

Analogously, we obtain $(\mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q); i_Q; t_P$ in the left-lower corner. Finally, we obtain in the right-lower corner

$$= \mathbb{I}_{S_1} \sqcup t_Q^\sim; t_Q \quad \text{since } i_Q^\sim; t_P; t_P; i_Q \sqsubseteq i_Q^\sim; \top_1; i_Q = i_Q^\sim; i_Q \sqsubseteq \mathbb{I}_{S_2}$$

From the fact that all entries in $\Xi_1$ and $\Xi_2$ are identical and the definition of the isomorphism assoc we conclude that $\text{assoc}^\sim; \Xi_1; \text{assoc} = \Xi_2$ or, equivalently, $\Xi_1; \text{assoc} = \text{assoc}; \Xi_2$. Using this equation and the additional properties of assoc
shown earlier we obtain
\[
\Phi; ((P.Q).U) \\
= R_{P.(Q.U)}; ([I_{S_1} + R_{Q.U}]; \text{assoc}; \Xi_1; ((P \oplus Q) \oplus U); F((R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim) \\
= R_{P.(Q.U)}; ([I_{S_1} + R_{Q.U}]; \Xi_1; \text{assoc}; ((P \oplus Q) \oplus U); F((R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim) \\
= R_{P.(Q.U)}; ([I_{S_1} + R_{Q.U}]; \Xi_1; (P \oplus (Q \oplus U)); F(\text{assoc}; (R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim) \\
= R_{P.(Q.U)}; ([I_{S_1} + R_{Q.U}]; (P \oplus (Q \oplus U)); F(\text{assoc}; \Xi_2; (R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim) \\
= R_{P.(Q.U)}; ([I_{S_1} + R_{Q.U}]; (P \oplus (Q \oplus U)); F(\Xi_1; \text{assoc}; (R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim) \\
= (P.(Q.U)); F(\Phi).
\]
The inclusion \(\Phi^\sim; (P.(Q.U)) \sqsubseteq ((P.Q).U); F(\Phi^\sim)\) follows analogously. Finally, we obtain
\[
i_{P.(Q.U)}; \Phi \\
= i_P; \iota; R_{P.(Q.U)}^\sim; R_{P.(Q.U)}; ([I_{S_1} + R_{Q.U}]; \text{assoc}; (R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim) \\
= i_P; \iota; \Xi_1; \text{assoc}; (R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim \quad \text{ since } \iota = \iota; ([I_{S_1} + R_{Q.U}]) \\
= i_P; \iota; \Xi_2; (R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim \\
= i_P; \iota; \text{assoc}; (R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim \\
= i_P; \iota; (R_{P,Q}^\sim + I_{S_1}); R_{(P.Q),U}^\sim \\
= i_P; \iota; R_{P,Q}^\sim; i_{P,Q}; R_{(P.Q),U}^\sim \\
= i_P; \iota; R_{(P.Q),U}^\sim \\
= i_{(P.Q),U}.
\]
This completes the proof. \(\square\)

4 Sum of Rooted Co-Algebras

The operation \(\oplus\) defines a co-algebra on the sum of the state spaces. In terms of labeled transition systems this co-algebra corresponds to two alternative branches of execution. However, \(P \oplus Q\) does not provide sufficient means to define an initial state. This can be done by adding a new state essentially combining the initial state of \(P\) and \(Q\).

**Definition 9.** Let \((P, i_P) : S_1 \to F(S_1)\) and \((Q, i_Q) : S_2 \to F(S_2)\) be rooted \(F\) co-algebras. Then we define a rooted co-algebra \((P \boxplus Q, i_{P\oplus Q}) : I + (S_1 + S_2) \to F(I + (S_1 + S_2))\) by
\[
P \boxplus Q = [(i_P; \iota \sqcup i_Q; \kappa), I_{S_1 + S_2}]; (P \oplus Q); F(\kappa),
i_{P\oplus Q} := \iota.
\]
Notice that simply identifying the initial state of $P$ and $Q$ in $P \oplus Q$ would not work because any initial state can also be state reachable state later, i.e., there might be a loop in the co-algebra leading back to the initial state. For example, if we denote by $(P \oplus Q)_\Xi$ the co-algebra that is obtained from $P \oplus Q$ by identifying the initial states we get

$$
P \quad \quad Q \quad \quad (P \oplus Q)_\Xi \quad \quad U
$$

The sequence $ab$ is possible in the labeled transition system $(P \oplus Q)_\Xi$ but neither possible in $P$ nor $Q$. Notice that $P \boxplus Q$ is bisimilar to $U$.

**Lemma 6.** Let $(P_1, i_{P_1}) : S_1 \to F(S_1), (P_2, i_{P_2}) : S_2 \to F(S_2), (Q_1, i_{Q_1}) : T_1 \to F(T_1)$ and $(Q_2, i_{Q_2}) : T_2 \to F(T_2)$ be rooted $F$ co-algebras with $(P_1, i_{P_1}) \sim (P_2, i_{P_2})$ and $(Q_1, i_{Q_1}) \sim (Q_2, i_{Q_2})$. Then we have

$$(P_1 \boxplus Q_1, i_{P_1} \boxplus i_{Q_1}) \sim (P_2 \boxplus Q_2, i_{P_2} \boxplus i_{Q_2}).$$

**Proof.** Suppose $\Phi : S_1 \to S_2$ and $\Psi : T_1 \to T_2$ are bisimulations with $i_{P_1}^\ast : i_{P_2} \sqsubseteq \Phi$ and $i_{Q_1}^\ast : i_{Q_2} \sqsubseteq \Psi$. It is straight-forward to verify that $\I_I + (\Phi + \Psi)$ is the required bisimulation. The first inclusion follows from

$$
(\I_I + (\Phi + \Psi)); (P_2 \boxplus Q_2)
= (\I_I + (\Phi + \Psi)); ([i_{P_2}; t \sqcup i_{Q_2}; \kappa], \I_{T_1 + T_2}); (P_2 \oplus Q_2); F(\kappa)
= [(i_{P_2}; t \sqcup i_{Q_2}; \kappa), (\Phi + \Psi)]; (P_2 \oplus Q_2); F(\kappa)
\sqsubseteq [(i_{P_1}; t \sqcup i_{Q_1}; \kappa), (\Phi + \Psi)]; (P_2 \oplus Q_2); F(\kappa)
= [(i_{P_1}; t \sqcup i_{Q_1}; \kappa), \I_{T_1 + T_2}; (\Phi + \Psi); (P_2 \oplus Q_2); F(\kappa)
= [(i_{P_1}; t \sqcup i_{Q_1}; \kappa), \I_{T_1 + T_2}; (P_1 \oplus Q_1); F((\Phi + \Psi); \kappa)
= [(i_{P_1}; t \sqcup i_{Q_1}; \kappa), \I_{T_1 + T_2}; (P_1 \oplus Q_1); F(\kappa; (\I_I + (\Phi + \Psi)))
= (P_2 \boxplus Q_2); F(\I_I + (\Phi + \Psi)),
$$

and the second inclusion is shown analogously. Finally we have $i_{P_1} \boxplus i_{Q_1} ; (\I_I + (\Phi + \Psi)) = t; (\I_I + (\Phi + \Psi)) = t = i_{P_2} \boxplus i_{Q_2}$. 

In the next lemma we want to investigate the terminal states of $P \boxplus Q$. Obviously, the terminal states of $P$ as well as the terminal states of $Q$ are such states. In addition, the new initial state of $P \boxplus Q$ is also a terminal state if the initial states of $P$ and $Q$ are also terminal.
Lemma 7. Let \((P, i_P) : S_1 \to F(S_1)\) and \((Q, i_Q) : S_2 \to F(S_2)\) be rooted \(F\) co-algebras. Then we have \(t_{P \sqcup Q} = (t_P; i_P \sqcap t_Q; i_Q^\sim); t \sqcup (t_P; t \sqcup t_Q; \kappa; \kappa)\).

Proof. First of all, from the computation

\[
((t_P; i_P \sqcap t_Q; i_Q^\sim); t \sqcup (t_P; t \sqcup t_Q; \kappa; \kappa); (P \sqcup Q)) \\
= ((t_P; i_P \sqcap t_Q; i_Q^\sim); (i_P; t \sqcup i_Q; \kappa); (P \sqcup Q) \sqcup (t_P; t \sqcup t_Q; \kappa); (P \sqcup Q); F(\kappa)) \\
\sqsubseteq ((t_P; i_P; P; F(\kappa) \sqcup (Q; F(\kappa))) \sqcup (t_P; F(\kappa) \sqcup t_Q; F(\kappa))); (P \sqcup Q)) \\
\sqsubseteq ((t_P; P; F(Q; F(\kappa)) \sqcup (t_P; F(\kappa) \sqcup t_Q; F(\kappa))); F(\kappa)) \\
= \downarrow_{F(I+{(S_1+S_2)\sqcup})} t_P; P = \downarrow_{F(S_1)} \text{ and } t_Q; Q = \downarrow_{F(S_2)}
\]

we conclude \((t_P; i_P \sqcap t_Q; i_Q^\sim); t \sqcup (t_P; t \sqcup t_Q; \kappa; \kappa) \sqsubseteq t_{P \sqcup Q}.\) Now consider

\[
v \sqsubseteq t_{P \sqcup Q} \\
\iff v; (P \sqcup Q) \sqsubseteq \downarrow_{F(I+{(S_1+S_2)\sqcup})} \\
\iff v; [(i_P; t \sqcup i_Q; \kappa), \downarrow_{F(S_1+S_2)}]; (P \sqcup Q); F(\kappa) \sqsubseteq \downarrow_{F(I+{(S_1+S_2)\sqcup})} \\
\iff v; [(i_P; t \sqcup i_Q; \kappa), \downarrow_{F(S_1+S_2)}]; (P \sqcup Q) \sqsubseteq \downarrow_{F(I+{(S_1+S_2)\sqcup})}
\]

Lemma 1

\[
v; t^\sim; (i_P; t \sqcup i_Q; \kappa); (P \sqcup Q) \sqsubseteq \downarrow_{F(S_1+S_2)} \\
\text{and } v; (P \sqcup Q); F(\kappa) \sqsubseteq \downarrow_{F(S_1+S_2)} \\
\iff v; t^\sim; (i_P; P; F(\kappa) \sqcup t_Q; F(\kappa)) \sqsubseteq \downarrow_{F(S_1+S_2)} \\
\text{and } v; (P \sqcup Q); F(\kappa) \sqsubseteq \downarrow_{F(S_1+S_2)} \\
\iff v; t^\sim; t_P; F(\kappa) \sqsubseteq \downarrow_{F(S_1+S_2)} \\
\text{and } v; (P \sqcup Q); F(\kappa) \sqsubseteq \downarrow_{F(S_1+S_2)}
\]

If we use the first inclusion, we obtain

\[
v; t^\sim; i_P; P; F(\kappa) \sqsubseteq \downarrow_{F(S_1+S_2)} \iff v; t^\sim; i_P; P \sqsubseteq \downarrow_{F(S_1)} \\
\iff v; t^\sim; i_P \sqsubseteq t_P \\
\iff v; t^\sim \sqsubseteq t_P; i_P^\sim
\]

Lemma 1

2 Analogously, the second inclusion is equivalent to \(v; t^\sim \sqsubseteq t_Q; i_Q^\sim\) so that \(v; t^\sim \sqsubseteq t_P; i_P \sqcap t_Q; i_Q^\sim\) follows. Now we consider the third inclusion, and we get

\[
v; \kappa^\sim; t^\sim; P; F(\kappa) \sqsubseteq \downarrow_{F(S_1+S_2)} \iff v; \kappa^\sim; t^\sim; P \sqsubseteq \downarrow_{F(S_1)} \\
\iff v; \kappa^\sim; t^\sim \sqsubseteq t_P
\]

Lemma 1
The fourth inclusion is equivalent to \( v; \kappa^-; \kappa^- \subseteq t_Q \) by a similar computation. The two properties together imply \( v; \kappa^- = v; \kappa^-; (\iota^-; \iota \sqcup \kappa^-; \kappa) \subseteq t_P; \iota \sqcup t_Q; \kappa \). Together with the previous result we get

\[
v = v; (\iota^-; \iota \sqcup \kappa^-; \kappa) \subseteq (t_P; i_P \sqcap t_Q; i_Q^-); \iota \sqcup (t_P; \iota \sqcup t_Q; \kappa); \kappa.
\]

This finally implies \( t_{P \bowtie Q} \subseteq (t_P; i_P \sqcap t_Q; i_Q^-); \iota \sqcup (t_P; \iota \sqcup t_Q; \kappa); \kappa \).

The next lemma is of particular interest. It shows that the sequential composition with a co-algebra \( U \) distributes from the right over \( \bowtie \).

**Theorem 2.** Let \( (P, i_P): S_1 \to F(S_1), (Q, i_Q): S_2 \to F(S_2) \) and \( (U, i_U): S_3 \to F(S_3) \) be rooted \( F \) co-algebras. Then we have

\[
((P.U) \bowtie (Q.U)), i_{(P.U)\bowtie(Q.U)}) \sim ((P \bowtie Q).U, i_{(P \bowtie Q).U}).
\]

**Proof.** First we define \( \alpha : I + ((S_1 + S_3) + (S_2 + S_3)) \to (I + (S_1 + S_2)) + S_3 \) by

\[
\alpha = [I; \iota, ([I; \kappa; \iota, \kappa], [\kappa; \iota, \kappa])]
\]

\[
= \iota^-; \iota \sqcup \kappa^-; \iota^-; \iota; \kappa; \iota \sqcup \kappa^-; \iota^-; \kappa^-; \kappa
\]

\[
\sqcup \kappa^-; \kappa^-; \iota^-; \kappa; \kappa \sqcup \kappa^-; \kappa^-; \kappa^-; \kappa.
\]

In matrix form we have

\[
\alpha = \begin{pmatrix}
II & (\downarrow I S_1 & \downarrow I S_2)
\downarrow S_1 & (\downarrow S_1 S_1 & \downarrow S_1 S_2)
\downarrow S_2 & (\downarrow S_2 S_1 & \downarrow S_2 S_2)
\end{pmatrix}
\]

For simplicity we will skip the inner brackets of a matrix such as \( \alpha \) in the remainder of this proof. It is easy to compute that \( \alpha^-; \alpha = I_{(I + (S_1 + S_2)) + S_3} \) and that

\[
\alpha; \alpha^- = \begin{pmatrix}
II & \downarrow I S_1 & \downarrow I S_3 & \downarrow I S_1 & \downarrow I S_3
\downarrow S_1 & \downarrow S_1 S_1 & \downarrow S_1 S_3 & \downarrow S_1 S_1 & \downarrow S_1 S_3
\downarrow S_1 & \downarrow S_1 S_3 & \downarrow S_1 S_3 & \downarrow S_1 S_1 & \downarrow S_1 S_3
\end{pmatrix},
\]

i.e., \( \alpha^- \) splits \( \alpha; \alpha^- \). We want to show that \( \Phi = (II + (R_{P.U} + R_{Q.U})): \alpha; R_{(P \bowtie Q).U}^- \) is the required bisimulation from \( (P.U) \bowtie (Q.U) \) to \( (P \bowtie Q).U \). In order to do
so, we will use the abbreviations

\[
v_1 = \begin{pmatrix}
i_p; P; F(\iota; \kappa; \iota) \cup i_Q; F(\kappa; \kappa; \iota) \\
P; F(\iota; \kappa; \iota) \\
Q; F(\kappa; \kappa; \iota) \\
U; F(\kappa)
\end{pmatrix}
\]

\[
v_2 = \begin{pmatrix}
i_p; P; F(\iota; \iota; \iota) \cup i_Q; F(\iota; \iota; \iota) \cup t_Q^\sim; i_U; U; (F(\kappa; \kappa; \kappa) \cup F(\iota; \iota; \iota)) \\
P; F(\iota; \iota; \iota) \\
U; F(\kappa; \kappa; \kappa) \\
Q; F(\iota; \iota; \iota) \\
U; F(\kappa; \kappa; \kappa)
\end{pmatrix}
\]

\[
\Xi = \begin{pmatrix}
\mathbb{I}_I & X^-; t_P & X^-; i_U & X^-; t_Q & X^-; i_U \\
t_Q^-; X & \mathbb{I}_{S_1} \cup t_P^-; t_P & t_Q^-; i_U & t_Q^-; t_Q & t_Q^-; i_U \\
i_U^-; X & i_U^-; t_P & \mathbb{I}_{S_3} & i_U^-; t_Q & \mathbb{I}_{S_3} \\
t_Q^-; X & t_Q^-; t_P & \mathbb{I}_{S_2} \cup t_Q^-; t_Q & t_Q^-; i_U & \mathbb{I}_{S_3}
\end{pmatrix}
\]

\[
X = t_P; i_P^- \cap t_Q; i_Q^-.
\]

Before we continue we want to show the following property:

\[(*) \quad \Xi; \alpha; v_1 = (\mathbb{I}_I + (\Xi_{P,U} + \Xi_{Q,U})); v_2; F(\alpha).
\]

First, we want to compute the matrix representation of \(\Xi_{(P\oplus Q).U}\). We compute

\[
\mathbb{I}_{I+(S_1+S_2)} \cup t_{P\oplus Q}^-; t_P\oplus Q
= \mathbb{I}_{I+(S_1+S_2)} \cup (\iota^-; X^- \cap \kappa^-; (\iota^-; t_P^- \cap \kappa^-; t_Q^-)); (X \cup (t_P; \iota \cup t_Q; \kappa); \kappa)
\]

by Lemma 7

\[
= \begin{pmatrix}
\mathbb{I}_I \cup X^-; X & X^-; (t_P; \iota \cup t_Q; \kappa) \\
(\iota^-; t_P^- \cap \kappa^-; t_Q^-); X & \mathbb{I}_{S_1+S_2} \cup (\iota^-; t_P^- \cap \kappa^-; t_Q^-); (t_P; \iota \cup t_Q; \kappa)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\mathbb{I}_I \cup X^-; X & X^-; t_P & X^-; t_Q \\
t_P^-; X & \mathbb{I}_{S_1} \cup t_P^-; t_P & t_P^-; t_Q \\
t_Q^-; X & t_Q^-; t_P & \mathbb{I}_{S_2} \cup t_Q^-; t_Q
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\mathbb{I}_I \cup X^-; t_P & X^-; t_Q \\
t_P^-; X & \mathbb{I}_{S_1} \cup t_P^-; t_P & t_P^-; t_Q \\
t_Q^-; X & t_Q^-; t_P & \mathbb{I}_{S_2} \cup t_Q^-; t_Q
\end{pmatrix}
\]

where the last equation follows from the fact that \(X^-; X \subseteq \mathbb{I}_I = \mathbb{I}_I\). Furthermore, we have

\[
t_{P\oplus Q}^-; i_U = (\iota^-; X^-; i_U \cup \kappa^-; (\iota^-; t_P^- \cup \kappa^-; t_Q^-); i_U)
\]

\[
= \begin{pmatrix}
X^-; i_U \\
(\iota^-; t_P^- \cup \kappa^-; t_Q^-); i_U
\end{pmatrix}
\]
A similar computation shows

\[
\begin{pmatrix}
\alpha; \\
\alpha; \\
\alpha;
\end{pmatrix}
\]

so that we obtain

\[
\Xi_{(P \Box Q);U} = \begin{pmatrix}
I_{I+(s_1+s_2)} \cup t_{P \Box Q}; t_{P \Box Q}; i_U \\
\iota_U; t_{P \Box Q} \\
\iota_U \cup t_{P \Box Q}; t_{P \Box Q}; t_P; i_U \\
\end{pmatrix}
\]

It is easy to see that we obtain \( \Xi \) from \( \Xi_{(P \Box Q);U} \) by duplicating the last row and column in the third row and column, respectively. Using the relation \( \alpha \) this translates to \( \alpha; \Xi_{(P \Box Q);U}; \alpha^- = \Xi \). Notice that this property implies \( \alpha; \Xi_{(P \Box Q);U} = \Xi; \alpha \) and \( \Xi_{(P \Box Q);U} = \alpha^-; \Xi; \alpha \). Furthermore, we have

\[
X^-; i_P; P = (t_P; i_P \cap t_Q; i_Q); i_P; P
\]

\[
\subseteq t_P; i_P; i_P; P
\]

\[
\subseteq t_P; P \quad \text{\( i_P \) univalent}
\]

A similar computation shows \( X^-; i_Q; Q = \downarrow_{i_P; F(S_1)} \) so that \( X^-; (i_P; P; F(\iota; \kappa; \iota) \cup i_Q; Q; F(\iota; \kappa; \iota)) = \downarrow_{i_P; F(S_1)} \) follows. This property and the definition of \( t_P \) and \( t_Q \) is used in the third equality of the computation starting with the left-hand side of (*)

\[
\Xi; \alpha; v_1
\]

\[
= \Xi; \begin{pmatrix}
I_f F(S_1) \downarrow i_P t_{P \Box Q} \downarrow t_{P \Box Q} \\
\downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \\
\downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \\
\downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \\
\downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \\
\downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \\
\downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \\
\downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \\
\downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \\
\downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \downarrow t_{P \Box Q} \\
\end{pmatrix}
\]

\[
= \Xi; \begin{pmatrix}
(i_P; P; F(\iota; \kappa; \iota) \cup i_Q; Q; F(\kappa; \iota)) \\
(i_P; P; F(\iota; \kappa; \iota)) \\
(i_Q; Q; F(\kappa; \iota)) \\
(i_P; P; F(\iota; \kappa; \iota)) \\
(i_Q; Q; F(\kappa; \iota)) \\
(i_P; P; F(\iota; \kappa; \iota)) \\
(i_Q; Q; F(\kappa; \iota)) \\
(i_P; P; F(\iota; \kappa; \iota)) \\
(i_Q; Q; F(\kappa; \iota)) \\
(i_P; P; F(\iota; \kappa; \iota)) \\
\end{pmatrix}
\]

\[
= \Xi; \begin{pmatrix}
P; F(\iota; \kappa; \iota) \\
P; F(\iota; \kappa; \iota) \\
Q; F(\kappa; \iota) \\
P; F(\iota; \kappa; \iota) \\
Q; F(\kappa; \iota) \\
\end{pmatrix}
\]

\[
= \Xi; \begin{pmatrix}
P; F(\iota; \kappa; \iota) \\
P; F(\iota; \kappa; \iota) \\
Q; F(\kappa; \iota) \\
P; F(\iota; \kappa; \iota) \\
Q; F(\kappa; \iota) \\
\end{pmatrix}
\]
For the right-hand side we first get

\[
(\mathcal{I}_I + (\Xi_{P;U} + \Xi_{Q;U})); v_2
= \left( \begin{array}{cccc}
\mathcal{I}_I & \mathcal{I}_{S_1} & \mathcal{I}_{S_2} & \mathcal{I}_{S_3} \\
\mathcal{I}_{S_1} & t_{\mathcal{I}_{S_1}} & t_{\mathcal{I}_{S_2}} & t_{\mathcal{I}_{S_3}} \\
\mathcal{I}_{S_2} & t_{\mathcal{I}_{S_2}} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} \\
\mathcal{I}_{S_3} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} \\
\end{array} \right); v_2
\]

\[
= \left( \begin{array}{cccc}
\mathcal{I}_I & \mathcal{I}_{S_1} & \mathcal{I}_{S_2} & \mathcal{I}_{S_3} \\
\mathcal{I}_{S_1} & t_{\mathcal{I}_{S_1}} & t_{\mathcal{I}_{S_2}} & t_{\mathcal{I}_{S_3}} \\
\mathcal{I}_{S_2} & t_{\mathcal{I}_{S_2}} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} \\
\mathcal{I}_{S_3} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} \\
\end{array} \right); v_2
= \left( \begin{array}{cccc}
\mathcal{I}_I & \mathcal{I}_{S_1} & \mathcal{I}_{S_2} & \mathcal{I}_{S_3} \\
\mathcal{I}_{S_1} & t_{\mathcal{I}_{S_1}} & t_{\mathcal{I}_{S_2}} & t_{\mathcal{I}_{S_3}} \\
\mathcal{I}_{S_2} & t_{\mathcal{I}_{S_2}} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} \\
\mathcal{I}_{S_3} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} \\
\end{array} \right); v_2
\]

\[
= \left( \begin{array}{cccc}
\mathcal{I}_I & \mathcal{I}_{S_1} & \mathcal{I}_{S_2} & \mathcal{I}_{S_3} \\
\mathcal{I}_{S_1} & t_{\mathcal{I}_{S_1}} & t_{\mathcal{I}_{S_2}} & t_{\mathcal{I}_{S_3}} \\
\mathcal{I}_{S_2} & t_{\mathcal{I}_{S_2}} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} \\
\mathcal{I}_{S_3} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} & t_{\mathcal{I}_{S_3}} \\
\end{array} \right); v_2
\]

From the definition of \( \alpha \) we recognize that \( \alpha; \alpha = \alpha; \alpha \). We obtain similar properties relating all compositions of injections occurring in the vector above with the corresponding one in previous computation. Therefore, we conclude \((\ast)\).

Now, we want to calculate the matrix representation of \((P \boxplus Q).U\) and \((P.U)\boxplus (Q.U)\). First, we obtain

\[
(P \boxplus Q).U
= R_{(P \boxplus Q);U}; \left( [(i_p \mathcal{I}_{i \cup Q}; \kappa), \mathcal{I}_{S_1 + S_2}]; (P \boxplus Q); F(\kappa) \right) U; F(R^{-}_{(P \boxplus Q);U})
= R_{(P \boxplus Q);U}; \left( [(i_p \mathcal{I}_{i \cup Q}; \kappa), \mathcal{I}_{S_1 + S_2}]; (P \boxplus Q); F(\kappa) \right) U; F(R^{-}_{(P \boxplus Q);U})
\]
\[
R_{(P \oplus Q), U} = R_{(P \oplus Q), U} 
\begin{pmatrix}
(i_P; \iota \cup i_Q; \kappa); (P \oplus Q); F(\kappa); i_U & F(R_{(P \oplus Q), U}) \\
(P \oplus Q); F(\kappa; i_U) & F(R_{(P \oplus Q), U})
\end{pmatrix} 
\]

In order to calculate a similar representation for \((P.U) \oplus (Q.U)\) we have

\[
i_P; R_{P; U}; (P \oplus U) \\
= i_P; R_{P; U}^{\ominus}; (P \oplus U)
\]

and

\[
i_Q; R_{Q; U}; (Q \oplus U) = i_Q; Q; F(\iota \cup i_Q; i_U; U; F(\kappa)), \text{ analogously. Further-}
\]

more, we have

\[
[(i_P; \iota \cup i_Q; \kappa), \iota_{(S_1 + s_3) + (S_2 + s_3)}]; (R_{P; U} + R_{P; U}) \\
= \prod_{P; U; S_1 + s_3 + s_3} \prod_{P; U; S_2 + s_3} \prod_{P; U; S_3 + s_3} \prod_{P; U; S_4 + s_3} \\
= (i_P; R_{P; U} \cup i_Q; R_{Q; U}) \\
= (P \cup Q; U; F(\kappa)) \\
= i_P; R_{P; U}; (P \oplus U) \\
= i_Q; R_{Q; U}; (Q \oplus U)
\]

so that we obtain

\[
\begin{pmatrix}
(i_P; R_{P; U} \cup i_Q; R_{Q; U}) \\
(P \oplus U); F(\kappa; \kappa)
\end{pmatrix}
\]

for \((P \oplus U); F(\kappa; \kappa)\).
\[
\begin{align*}
&= \left( i_{P.U}; R_{P.U}; (P \oplus U); F(\iota; \kappa) \sqcup i_{Q.U}; R_{Q.U}; (Q \oplus U); F(\kappa; \kappa) \right) \\
&= \left( i_{P.U}; P; F(\iota; \kappa) \sqcup i_{Q.U}; Q; F(\kappa; \kappa) \right)
\end{align*}
\]

Combining all computation from above we get

\[(P.U) \boxplus (Q.U)\]

\[
= \left( i_{P.U}; (P \oplus U); (Q \oplus U); F(\iota; \kappa) \right)
\]

Now we are ready to prove the first inclusion of a bisimulation as follows

\[
\Phi; ((P \boxplus Q).U)
\]

\[
= (\mathbb{I}_I + (R_{P.U} + R_{Q.U})); \alpha; R^\sim_{(P \boxplus Q).U}; R_{(P \boxplus Q).U}; v_1; F(R^\sim_{(P \boxplus Q).U})
\]

The second inclusion is shown by

\[
\Phi^\sim; ((P.U) \boxplus (Q.U))
\]

\[
= R_{(P \boxplus Q).U}; \alpha^\sim; (\mathbb{I}_I + (\Xi_{P.U} + \Xi_{Q.U})); v_2; F(R^\sim_{(P \boxplus Q).U})
\]
In this section we want to investigate an operation closely related to iteration or recursion. The co-algebra $\triangleright P$ identifies the terminal states $P$ with its initial state, i.e. introduces a loop in the graph of $P$.

**Definition 10.** Let $(P, i_P) : S \to F(S)$. Furthermore, let $R_{\triangleright P} : S_{\triangleright P} \to S$ be the splitting of the relation $\Xi_{\triangleright P} = \Xi_S \cup (t_P \sqcup i_P) \sim (t_P \sqcup i_P)$. Then we define a rooted co-algebra $(\triangleright P, i_{\triangleright P}) : S_{\triangleright P} \to F(S_{\triangleright P})$ by

$\triangleright P := R_{\triangleright P} ; P ; F(R_{\triangleright P} \sim)$,

$i_{\triangleright P} := i_P ; R_{\triangleright P} \sim$.

Similar to the sequential composition we can define recursion for certain terminal states of $P$ only, i.e., with respect to a vector $r \subseteq t_P$. This recursion operation goes back to the initial state only if a terminal state in $r$ is reached. We denote this operation by $\triangleright \triangleright P$. Formally, the definition only requires to replace $t_P$ in the definition of $\Xi$ by $r$. As before in all proofs we only use that terminal states are terminal, i.e., $t_P ; P = \bot_{g(F(S))}$, so that the corresponding properties (or their obvious generalization) remain true.

First, we want to show that the loop operation respects the notion of bisimilarity.

**Lemma 8.** Let $(P_1, i_{P_1}) : S_1 \to F(S_1)$ and $(P_2, i_{P_2}) : S_2 \to F(S_2)$ be rooted co-algebras with $(P_1, i_{P_1}) \sim (P_2, i_{P_2})$. Then we have $(\triangleright P_1, i_{\triangleright P_1}) \sim (\triangleright P_2, i_{\triangleright P_2})$. 

Last but not least we have

\[
(\triangleright P_{\triangleright P}) ; t\sim = (i_{\triangleright P_{\triangleright P}} \sqcup i_{\triangleright P_{\triangleright P}}) \sim (i_{\triangleright P_{\triangleright P}} \sqcup i_{\triangleright P_{\triangleright P}}) \sim ; t\sim
\]

so that $i_{(P, U)\triangleright Q} = t \sqsubseteq (P, U) ; \Phi^{-}$ from Lemma 1 follows. 

\[\square\]
Proof. Suppose $\Phi : S_1 \to S_2$ is a bisimulation from $P_1$ to $P_2$ with $i_{P_1} \subseteq i_{P_2}; \Phi$. We want to show that $\Psi = R_{\bowtie P_1}; \Phi; R_{\bowtie P_2}$ is the required bisimulation from $\bowtie P_1$ to $\bowtie P_2$. The first inclusion follows from

$$\Psi; \bowtie P_2 = R_{\bowtie P_1}; \Phi; R_{\bowtie P_2}; P_2; F(R_{\bowtie P_2})$$

The second inclusion is shown analogously. Finally, the computation

$$i_{\bowtie P_2} = i_{P_2}; R_{\bowtie P_2}$$

$$\subseteq i_{P_1}; \Phi; R_{\bowtie P_2}$$

$$\subseteq i_{P_1}; \Xi_{\bowtie P_1}; \Phi; R_{\bowtie P_2}$$

$$= i_{P_1}; R_{\bowtie P_1}; i_{P_1}; \Phi; R_{\bowtie P_2}$$

$$= i_{\bowtie P_1}; \Psi$$

shows that $\Psi$ is the required bisimulation. \qed

The main theorem of this section shows that $\bowtie P$ actually solves a recursive equation.

**Theorem 3.** Let $(P, i_P) : S \to F(S)$. Then $(\bowtie P, i_{\bowtie P}) \sim (P, i_P, \bowtie P)$.

Proof. We want to show that $\Phi = R_{\bowtie P}; (i \sqcup \kappa); (\bowtie S + R_{\bowtie P}); R_{\bowtie P}$ is the required bisimulation from $\bowtie P$ to $P, \bowtie P$. In order to do so we first investigate the equivalence relation $\Xi = (\bowtie S + R_{\bowtie P}); R_{\bowtie P}; P; \bowtie P; (\bowtie S + R_{\bowtie P})$ on $S + S$ induced by...
$\Xi_{\circ P}$ and $\Xi_{P \circ P}$. We obtain

\[
\Xi = \left( \begin{array}{c}
\Xi_S \uplus \Xi_{\circ P} \\
\Xi_S \uplus \Xi_{P \circ P}
\end{array} \right); \left( \begin{array}{c}
\Xi_S \uplus t_{\circ P} ; t_{\circ P} ; i_{\circ P} ; \Xi_{\circ P} \\
\Xi_S \uplus t_{P \circ P} ; t_{P \circ P} ; i_{P \circ P}
\end{array} \right); \left( \begin{array}{c}
\Xi_S \uplus \Xi_{\circ P} \\
\Xi_S \uplus \Xi_{P \circ P}
\end{array} \right).
\]

Before we show the first inclusion of a bisimulation we compute

\[(i \sqcup \kappa) ; \Xi ; (P \sqcup P)\]

\[
= (\Xi_S \uplus \Xi_S); \left( \begin{array}{c}
\Xi_S \uplus t_{\circ P} ; t_{\circ P} ; i_{\circ P} ; \Xi_{\circ P} \\
\Xi_{\circ P} \uplus t_{\circ P} ; \Xi_{\circ P}
\end{array} \right); \left( \begin{array}{c}
P ; F(i) \\
P ; F(P)
\end{array} \right)
\]

\[
= (\Xi_S \uplus \Xi_S); \left( \begin{array}{c}
\Xi_S \uplus t_{P \circ P} ; t_{P \circ P} ; i_{P \circ P} \setminus \Xi_{P \circ P} \sqcup P ; F(P)
\end{array} \right)
\]

\[
= \Xi_{\circ P} ; P ; F(i) \sqcup \Xi_{\circ P} ; P ; F(P)
\]

\[
\sqsubseteq \Xi_{\circ P} ; P ; F(i) \sqcup F(P)
\]

F monotonic

Now we obtain the first inclusion by

\[\Phi ; (P \circ P)\]

\[
= R_{\circ P} ; (i \sqcup \kappa) ; (\Xi_S + R_{\circ P} ; R_{P \circ P} ; (P \sqcup P) ; F(R_{P \circ P})
\]

\[
= R_{\circ P} ; (i \sqcup \kappa) ; (\Xi_S + R_{\circ P} ; R_{P \circ P} ; (P \sqcup P) ; F(R_{P \circ P})) \quad \text{see above}
\]

\[
= R_{\circ P} ; P ; F((i \sqcup \kappa) ; (\Xi_S + R_{\circ P} ; R_{P \circ P})
\]

\[
= R_{\circ P} ; P ; F(\Xi_{\circ P} ; (i \sqcup \kappa) ; (\Xi_S + R_{\circ P} ; R_{P \circ P})
\]

\[
= \circ P ; F(\Phi).
\]
For the other inclusion we first obtain
\[
\Phi^\sim; \circ P = R_{P; \circ P}; (I_S + R_{\circ P}); (\uplus \cup \kappa)^\sim; R_{\circ P}^\sim; R_{\circ P}; F(R_{\circ P}^\sim) \\
= R_{P; \circ P}; (I_S + R_{\circ P}); (\uplus \cup \kappa)^\sim; \Xi_{\circ P}; P; F(R_{\circ P}^\sim) \\
= R_{P; \circ P}; (I_S + \Xi_{\circ P}; P) \circ \Xi_{\circ P}; P; \circ P; F(R_{\circ P}^\sim) \\
= R_{P; \circ P}; (I_S + \Xi_{\circ P}; P) \circ \Xi_{\circ P}; P; F(R_{\circ P}^\sim) \\
= R_{P; \circ P}; (I_S + \Xi_{\circ P}; P) \circ \Xi_{\circ P}; P; F(R_{\circ P}^\sim) \\
= R_{P; \circ P}; (\uplus \circ \Xi_{\circ P}; P) \circ \Xi_{\circ P}; P; F(R_{\circ P}^\sim) \\
= R_{P; \circ P}; (\uplus \circ \Xi_{\circ P}; P) \circ \Xi_{\circ P}; P; F(R_{\circ P}^\sim) \\
= R_{P; \circ P}; (\uplus \circ \Xi_{\circ P}; P) \circ \Xi_{\circ P}; P; F(R_{\circ P}^\sim) \\
= R_{P; \circ P}; (\uplus \circ \Xi_{\circ P}; P) \circ \Xi_{\circ P}; P; F(R_{\circ P}^\sim) \\
= R_{P; \circ P}; (\uplus \circ \Xi_{\circ P}; P) \circ \Xi_{\circ P}; P; F(R_{\circ P}^\sim). \\
\]

Before we show that the right-hand side \((P; \circ P); F(\Phi^\sim)\) is also equal to the last expression above we first compute
\[
(I_S + R_{\circ P}^\sim); R_{F; \circ P}; \Phi^\sim \\
= (I_S + R_{\circ P}^\sim); R_{F; \circ P}; R_{P; \circ P}; (I_S + R_{\circ P}); (\uplus \cup \kappa)^\sim; R_{\circ P}^\sim \\
= \Xi; (\uplus \cup \kappa)^\sim; R_{\circ P}^\sim \\
= \left(\frac{I_S \cup t_p^\sim; t_p^\sim; i_p; \Xi_{\circ P}}{\Xi_{\circ P}; t_p^\sim; i_p; \Xi_{\circ P}}\right); \left(\frac{I_S}{I_S}\right); R_{\circ P}^\sim \\
= \left(\frac{I_S \cup t_p^\sim; t_p^\sim; i_p; \Xi_{\circ P}}{\Xi_{\circ P}; t_p^\sim; i_p; \Xi_{\circ P}}\right); R_{\circ P}^\sim \\
= \left(\frac{I_S \cup t_p^\sim; t_p^\sim; i_p; \Xi_{\circ P}}{\Xi_{\circ P}; t_p^\sim; i_p; \Xi_{\circ P}}\right); R_{\circ P}^\sim \\
= (I_S \cup t_p^\sim; t_p^\sim; i_p; \Xi_{\circ P}; R_{\circ P}^\sim) \\
= (I_S \cup t_p^\sim; t_p^\sim; i_p; \Xi_{\circ P}; R_{\circ P}^\sim) \\
= (I_S \cup t_p^\sim; t_p^\sim; i_p; \Xi_{\circ P}; R_{\circ P}^\sim) \\
= (I_S \cup t_p^\sim; t_p^\sim; i_p; \Xi_{\circ P}; R_{\circ P}^\sim) \\
= \left(\frac{I_S \cup t_p^\sim; t_p^\sim; i_p; \Xi_{\circ P}}{R_{\circ P}^\sim}\right), \\
\]
where the last equation follows from
\[
(I_S \cup t_p^\sim; t_p^\sim; i_p; \Xi_{\circ P}; R_{\circ P}^\sim) \subseteq (\Xi_{\circ P} \cup \Xi_{\circ P}; \Xi_{\circ P}; R_{\circ P}^\sim) \\
= \Xi_{\circ P}; R_{\circ P}^\sim 
\]
\[ = R_{\subseteq P} \]
\[ \sqsubseteq (I_S \cup t_p; t_p \sqcup t_p; i_p; \Xi_{\subseteq P}) R_{\subseteq P}. \]

Now, we calculate
\[
(P \subseteq P) ; F(\Phi^-) \\
= R_{P \subseteq P}; (P \oplus \circ P) ; F(R_{P \subseteq P}^- ; \Phi^-) \\
= R_{P \subseteq P}; (P \oplus (R_{P \subseteq P} P ; R_{P \subseteq P}^-)) ; F(R_{P \subseteq P}^-; \Phi^-) \\
= R_{P \subseteq P}; (I_S + R_{P \subseteq P}) ; (P \oplus P) ; F((I_S + R_{P \subseteq P}) ; R_{P \subseteq P}^-; \Phi^-) \\
= R_{P \subseteq P}; (I_S + R_{P \subseteq P}) ; P; F((I_S + R_{P \subseteq P}) ; R_{P \subseteq P}^-; \Phi^-) \\
= R_{P \subseteq P}; (I_S + R_{P \subseteq P}) ; P; F(R_{P \subseteq P}^-) \\
= R_{P \subseteq P}; (I_S + R_{P \subseteq P}) ; P; F(R_{P \subseteq P}^-) \\
\]

which shows the other inclusion. Last but not least the two initial state are in
the relation \( \Phi \), which is shown by
\[
i_p; i; R_{P \subseteq P}^- ; R_{P \subseteq P} ; (I_S + R_{P \subseteq P}) ; (i \sqcup \kappa) ; R_{\subseteq P}^- \\
= i_p; i; (I_S \cup t_p; t_p \sqcup t_p; i_p; i_p) R_{P \subseteq P}^- ; (I_S \cup t_p; t_p; i_p) ; R_{P \subseteq P}^- \\
= i_p; (I_S \cup t_p; t_p \sqcup t_p; i_p) ; R_{P \subseteq P}^- \\
= i_p; (I_S \cup t_p; t_p \sqcup t_p; i_p) ; R_{P \subseteq P}^- \\
\sqsubseteq i_p; R_{\subseteq P}^- \\
\sqsubseteq i_{\subseteq P}. \]

This completes the proof. \( \square \)

### 6 Recursive Equations

In this section we want to show how to solve certain recursive equations using
the operator from the previous section. Therefore, we consider a language that
This completes the proof.

Theorem 4. From the example we see that a recursion on $r_t$ and a vector $r_t \subseteq t_P$ to the expression $t(X)$ recursively. The vector $r_t$ describes those terminal states that should loop.

1. If $t(X) = X$, then $P_t = \{M_{f(t)}(I), I_t\}$ and $r_t = \{I\}$.
2. If $t(X) = Q$, then $P_t = Q$ and $r_t = \emptyset$.
3. If $t(X) = t_1(X) \oplus t_2(X)$, then $P_t = P_{t_1} \oplus P_{t_2}$ and $r_t = (r_{t_1} \mapsto \top \cap r_{t_2} \mapsto \top) \lambda (r_{t_1} \mapsto \top \cap r_{t_2} \mapsto \top)$.
4. If $t(X) = Q.t'(X)$, then $P_t = Q.P_{t'}$ and $r_t = r_{t'}$.

As an example consider the expression $t(X) = (a.X) + b$ where $a$ and $b$ are the labeled transition systems with exactly one transition from the initial state to a terminal state labeled $a$ and $b$, respectively. The labeled transition system $P_t$ can be found in the following figure where the states in $r_t$ and indicated by a hollow bullet.

From the example we see that a recursion on $r_t$ leads to the intended result.

**Theorem 4.** We have $\overset{\circ}{\circ} P_t \sim t(\circ_{r_t} P_t)$.

**Proof.** By Theorem 3 it is sufficient to show that $t(\circ_{r_t} P_t) \sim P_t \overset{\circ}{\circ} P_t$. Therefore we show that $t(P) = P_t \overset{\circ}{\circ} P$ by induction on the structure of $t(X)$ for all co-algebras $P$.

- $t(X) = X$ : In this case $t(P) = \emptyset P \sim P$ by Lemma 5(2).
- $t(X) = Q$ : We have $t(P) = Q = Q I P$ by Lemma 5(1).
- $t(X) = t_1(X) + t_2(X)$ : Now, we get
  
  \[
  t(P) = t_1(P) \oplus t_2(P) \\
  = (P_{t_1} \overset{\circ}{\circ} P) \oplus (P_{t_2} \overset{\circ}{\circ} P) \quad \text{induct. hyp.} \\
  = (P_{t_1} \oplus P_{t_2}) \overset{\circ}{\circ} P. \quad \text{Theorem 1}
  \]

- $t(X) = Q.t'(X)$ : again, we have
  
  \[
  t(P) = Q.t'(P) \\
  = Q.(P_{t'} \overset{\circ}{\circ} P) \quad \text{induct. hyp.} \\
  = (Q.P_{t'}) \overset{\circ}{\circ} P \quad \text{Theorem 2} \\
  = P_{t'} \overset{\circ}{\circ} P.
  \]

This completes the proof. \qed
7 Conclusion and Future Work

In this paper we have defined a sequential composition for rooted co-algebras. In addition, we have shown how recursive equations based on sequential composition and summation can be solved explicitly. From process calculi we know that certain equation, so called guarded equations, have a unique fixed point. Future research will investigate whether this remains true in the more general context of rooted co-algebras. This requires to generalize the notion of a guarded expression to relators of some kind, of course.

Another area of study is the relationship between parallel and sequential composition. In particular, the combination of the two categories - the ordered category of processes based on parallel composition, and the category of rooted co-algebras based on sequential composition. This kind of work might lead to completely algebraic treatment of processes and/or co-algebras.

References
