Towards Quantum Relation Algebras

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Abstract. Propositional quantum logic is a non-classical logic system based on the structure of quantum mechanics. It is based on the concept of physical observations or experimental propositions. These propositions correspond to closed linear subspaces of a Hilbert space or, more abstractly, to elements of an orthomodular lattice. In this paper we want to start the investigation of relations that are based on such a logic system as a first step towards an algebraic version of quantum first-order logic using a relation algebraic approach.

1 Introduction

In 1936 G. Birkhoff and J. von Neumann introduced in [1] a non-classical logic system, called quantum logic, that is based on the principles of quantum mechanics. Quantum logic is a multi-valued logic in which propositions correspond only to physical observations and/or experimental propositions. Such a proposition is represented by a self-adjoint projection or, alternatively, by a closed linear subspace. The collection of these subspaces forms an orthomodular lattice. Nowadays, quantum logic refers to the multi-valued logic based on these kind of lattices. The main difference to classical logic is that orthomodular lattices, and, hence, the logic itself, is not distributive. On the other hand, they provide a strong complement, i.e., an orthocomplement that satisfies the De Morgan equalities. These two characteristics also distinguishes quantum logic from most non-classical logics such as intuitionistic logic, fuzzy logic etc. These logics consider weaker forms of complements and/or implications but are distributive. Since its introduction quantum logic has been studied widely. It has also been used to establish a version of set theory [5] in order to provide foundations of mathematics based on quantum principles.

The study of theory of relations started in the work of A. De Morgan, C. Peirce, and E. Schröder. Relation algebras together with their equational definition was initiated by A. Tarski. His algebraic treatment of relations provides a variable-free and equational treatment of classical logic. A perfect example of this approach is [6] in which A. Tarski and S. Givant used relation algebras in

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order to present a variable-free version of set theory. Since then weaker versions of the relational calculus in its various forms have been studied. However, up to our knowledge, quantum relations, i.e., relations based on quantum logic have not been studied yet. In this paper we want to start the investigation of relations that are based on orthomodular lattices as a first step towards an algebraic version of quantum first-order logic using a relation algebraic approach.

2 Orthomodular Lattices

In this section we want to recall some fundamentals on orthomodular lattices. For more details and proofs omitted here, we refer to [3].

**Definition 1.** An operation \( (\cdot) \) on a bounded lattice \( \langle L, \cdot, +, 0, 1 \rangle \), i.e., a lattice with a smallest element \( 0 \) and a greatest element \( 1 \), is called an orthocomplementation iff

1. \( x \cdot x^\perp = 0 \) and \( x + x^\perp = 1 \),
2. \( x^{\perp\perp} = x \),
3. \( x \leq y \) implies \( y^\perp \leq x^\perp \),

for all \( x, y \in L \). A lattice with an orthocomplementation is called an orthocomplemented lattice. An orthomodular lattice is an orthocomplemented lattice such that

\[ x \leq y \implies x + (x^\perp \cdot y) = y \]

for all \( x, y \in L \).

Every modular orthocomplemented lattice is orthomodular but not vice versa. Therefore, orthomodular lattices are, in general, not distributive. A distributive orthocomplemented lattice is a Boolean algebra. Notice that orthomodular lattices can also be defined equationally using the property

\[ x + (x^\perp \cdot (x + y)) = x + y. \]

In a complemented lattice one may define a classical implication by \( x \to y = x^+ + y \). The so-called Sasaki implication \( x \to_1 y = x^+ + (x \cdot y) \) is equal to the classical implication \( x \to y \) in Boolean algebras by the distributivity law. However, if we consider orthomodular lattices, the two implications are not equal.

In the next lemma we have summarized two basic properties of the orthocomplement and the Sasaki implication in an orthomodular lattice.

**Lemma 1.** Let \( L \) be an orthomodular lattice. Then we have for all \( x, y \in L \):

1. \( (x \cdot y)^\perp = x^\perp + y^\perp \) and \( (x + y)^\perp = x^\perp \cdot y^\perp \).
2. \( x \to_1 y = 1 \) if \( x \leq y \).
An important notion in the theory of orthomodular lattices is the notion of “compatible” or “commutable” elements, denoted by $x \cdot y$. This notion was motivated by commuting projections on Hilbert spaces. It is defined by

$$x \cdot y \iff x \cdot (x^+ + y) = x \cdot y.$$  

**Lemma 2.** Let $L$ be an orthomodular lattice. Then we have for all $x, y \in L$:

1. The properties $x \cdot y, y \cdot x, x^+ \cdot y, x^+ \cdot y^-$, and $x \cdot y^-$ are equivalent,
2. $x \leq y$ implies $x \cdot y$,
3. $0 \cdot x$ and $1 \cdot x$,
4. $x \cdot y$ iff $x = (x \cdot y) + (x + y^-),$ 
5. If one of the elements $x, y, z$ is compatible with the other two, then $(x, y, z)$ is a distributive triple, i.e., the equations $x \cdot (y + z) = x \cdot y + x \cdot z$ and $x + y \cdot z = (x + y) \cdot (x + z)$ hold together with the other four equations obtained by cyclic permutation of $x, y,$ and $z$.

Notice that according to the last property of the previous lemma we may have $x \cdot y, x \cdot z$, and, hence, $x \cdot (y + z) = x \cdot y + x \cdot z$, but not $y \cdot z$.

The following decomposition theorem for orthomodular lattices will become of interest when we consider the set of crisp quantum relations.

**Theorem 1.** Let $L$ be an orthomodular lattice. If $z \in L$ is compatible to all elements of $L$, then $L$ is isomorphic to $[0, z) \times [0, z^-)$ where $[0, z]$ denotes the interval $0$ to $z$, i.e., $[0, z] = \{y \in L \mid y \leq z\}$.

The final lemma of this section summarizes properties of compatibility in the complete case.

**Lemma 3.** Let $L$ be a complete orthomodular lattice. If $x_i \cdot y$ for all $i \in I$, then we have:

1. $\left(\sum_{i \in I} x_i \cdot y\right)$ and $\left(\prod_{i \in I} x_i \cdot y\right)$,
2. $\left(\sum_{i \in I} x_i \cdot y\right) = \sum_{i \in I} x_i \cdot y$ and $\left(\prod_{i \in I} x_i \cdot y\right) + y = \prod_{i \in I} x_i \cdot y$.

# 3 Relation Algebras

In this section we want to recall the basic facts of relation algebras. For further details we refer to [4].

**Definition 2.** A structure $(A, \sqcap, \sqcup, \neg, \vee, \supseteq, \subseteq, \cdot, 1)$ is called a relation algebra iff

\begin{enumerate}
\item[(RA 1)] $(A, \sqcap, \sqcup, \neg, \vee, \supseteq, \subseteq, \cdot, 1)$ is a Boolean algebra,
\item[(RA 2)] $Q; (R; S) = (Q; R); S,$
\item[(RA 3)] $Q; 1 = Q,$
\end{enumerate}
(RA 4) \((Q ∪ R); S = Q; S ⊔ R; S\),
(RA 5) \(Q^\sim = Q\),
(RA 6) \((Q ∪ R)^\sim = Q^\sim ⊔ R^\sim\),
(RA 7) \((Q; R)^\sim = R^\sim; Q^\sim\),
(RA 8) \(Q^\sim; R; R^\sim = R\).

If we denote by \(\supseteq\) the order induced by the Boolean part of a relation algebra, then (RA 8) can also be formulated as \(Q^\sim; R \subseteq R\).

For technical reasons we want to split axiom (RA 4) of the previous definition into the two inclusions

\begin{align*}
(RA4a) & \quad Q; S ⊔ R; S \subseteq (Q ∪ R); S, \\
(RA4b) & \quad (Q ∪ R); S \subseteq Q; S ⊔ R; S.
\end{align*}

Notice that (RA 4a) simply states that composition is monotone in the first parameter.

The properties of the next lemma can be found in [4]. It is important to mention that the proof of those properties do not require all axioms of a relation algebra.

Lemma 4. Let \((A; ∩, ∪, \overline{\cdot}); 0; 1; 1, 2, ..., \mathbb{1})\) be a structure satisfying the axioms of a complemented lattice and the Axioms (RA 3), (RA 4a), (RA 5), (RA 6), and (RA 7). Then we have:

1. \(Q = \mathbb{1}; Q\).
2. Composition is monotone in the second parameter, i.e., we have \(Q ⊆ R\) implies \(S; Q \subseteq S; R\),
3. \(1; 2 = 1; 2\).

Notice that [4] actually provides a huge list of properties that do not use Axiom (RA 2), (RA 4b), or (RA 8). Even though they will be important in further study of quantum relations, none of those additional properties is needed in the current paper.

4 Quantum Relations

We want to introduce relations that are based on quantum logic. But first, let us recall that a regular (or classical) relation on a set \(A\) is a subset of the cartesian product \(A × A\), i.e., it is a set of pairs. Alternatively, a relation can be described by its membership function that assigns to each pair either true or false depending whether the pair of elements is in the relation or not. Fuzzy relations and/or \(L\)-fuzzy relations replace the binary truth values by elements of the unit interval \([0, 1]\) of elements of a complete Heyting algebra \(L\). Notice that both structures are distributive but do not provide an orthocomplementation. We will now replace these kind of structures by a complete orthomodular lattice.

Definition 3. A quantum relation \(R\) on a set \(A\) is a function \(R : A × A → L\) where \((L; \cdot, +, (\cdot)^{-}; 0, 1)\) is a complete orthomodular lattice. On the set of all quantum relation \(QRel_L(A)\) on \(A\) based on a fixed complete orthomodular lattice \(L\) we define the following operations and constants:
1. \((R \cap S)(x, y) = R(x, y) \cdot S(x, y)\) and \((R \cup S)(x, y) = R(x, y) + S(x, y)\),
2. \(R^\ast(x, y) = R(y, x)\) and \(\overline{R}(x, y) = (R(x, y))^\perp\),
3. \(\overline{\top}(x, y) = 1\) and \(\overline{\bot}(x, y) = 0\),
4. \((R; S)(x, z) = \sum_{y \in A} R(x, y) \cdot S(y, z)\),
5. \(\mathbb{I}(x, y) = 1\) if \(x = y\) and \(\mathbb{I}(x, y) = 0\) otherwise.

Notice that the operations and constants introduced in the previous definition are a straightforward generalization of the corresponding operations and constants of classical relations. However, the lack of distributivity implies that certain properties of those operations are no longer valid. For example, composition (or relative product) \(\cdot\) of two relations is no longer associative.

**Lemma 5.** (QRA 1) The structure \(\langle \text{QRel}_L(A), \cap, \cup, \overline{();}, \overline{\bot}, \overline{\top} \rangle\) is an orthomodular lattice.

**Proof.** All operations and constants are defined component-wise so that the proof is an easy verification of the required properties, and, therefore, omitted. \(\square\)

Since all lattice operations, including complement, are defined component-wise, we also obtain
\[ Q \downarrow R \iff Q(x, y) \downarrow R(x, y) \] for all \(x, y \in L\).

Notice that the two compatibility relations in the equivalence above are based on different orthomodular lattices.

**Lemma 6.** The operations and constants of \(\text{QRel}_L(A)\) satisfy the following:

(QRA 3) \(Q; \mathbb{I} = Q\),
(QRA 4a) \(\cdot\) is monotone in the first parameter, i.e., \(Q \sqsubseteq R\) implies \(Q; S \sqsubseteq R; S\),
(QRA 5) \(Q^{-\ast} = Q\),
(QRA 6) \((Q \cup R)^\ast = Q^{-\ast} \cup R^{-\ast}\),
(QRA 7) \((Q; R)^\ast = R^{-\ast}; Q^{-\ast}\).

**Proof.** We only show (QRA 4a) as an example. The other properties follow analogously from the definition of the corresponding operations. Consider the computation
\[
(Q; R)^\ast(x, z) = (Q; R)(z, x)
= \sum_{y \in A} Q(z, y) \cdot R(y, x)
= \sum_{y \in A} R^\ast(x, y) \cdot Q^\ast(y, z)
= (R^\ast; Q^\ast)(x, z),
\]
which verifies 4. \(\square\)
As already mentioned we cannot expect that all axioms of relation algebras are satisfied since the underlying lattice of truth values is not distributive. However, we obtain the following restricted versions of the remaining axioms.

**Lemma 7.** Suppose $Q, R, S \in \text{QRel}_L(A)$. Then we have:

1. If $Q(x, y) \downarrow S(y, z)$ and $R(x, y) \downarrow S(y, z)$ for all $x, y, z \in A$, then $(Q \sqcup R); S = Q; S \sqcup R; S$.
2. If $Q(u, x) \downarrow R(x, y), Q(u, x) \downarrow S(y, z)$, and $R(x, y) \downarrow S(y, z)$ for all $u, x, y, z \in A$, then $Q; (R; S) = (Q; R); S$.
3. If $Q(x, u) \downarrow R(u, z)$ for all $u, x, z \in A$, then $Q^\rightarrow; R; S \equiv R$.

**Proof.**

1. The first property follows immediately from

   \[(Q \sqcup R); S)(x, z) =\sum_{y \in A} (Q \sqcup R)(x, y) \cdot S(y, z) =\sum_{y \in A} (Q(x, y) + R(x, y)) \cdot S(y, z) =\sum_{y \in A} Q(x, y) \cdot S(y, z) + R(x, y) \cdot S(y, z)\]

   by Lemma 2(5).

2. From Lemma 3(1) we obtain $Q(u, x) \downarrow (R(x, y) \cdot S(y, z))$ for all $y \in A$. Similarly, $S(y, z) \downarrow (Q(u, x) \cdot R(x, y))$ for all $x \in A$ follows. This implies

   \[(Q; R; S)(u, z) =\sum_{x \in A} Q(u, x) \cdot (R; S)(x, z) =\sum_{x \in A} Q(u, x) \cdot \left(\sum_{y \in A} R(x, y) \cdot S(y, z)\right) =\sum_{y \in A} \sum_{x \in A} Q(u, x) \cdot R(x, y) \cdot S(y, z)\]

   by Lemma 3(2).
3. Consider the following computation

\[(Q^\perp;Q;R)(u, z)\]
\[= \sum_{x \in A} Q^\perp(u, x) \cdot Q;R(x, z)\]
\[= \sum_{x \in A} Q(x, u) \cdot ((Q;R)(x, z))^\perp\]
\[= \sum_{x \in A} Q(x, u) \cdot \left(\prod_{y \in A} (Q(x, y) \cdot R(y, z))^\perp\right) \quad \text{Lemma 5(1)}\]
\[\leq \sum_{x \in A} Q(x, u) \cdot (Q(x, u) \cdot R(u, z))^\perp\]
\[= \sum_{x \in A} Q(x, u) \cdot ((Q(x, u))^\perp + (R(u, z))^\perp) \quad \text{Lemma 5(1)}\]
\[= \sum_{x \in A} Q(x, u) \cdot (R(u, z))^\perp\]
\[\leq \sum_{x \in A} (R(u, z))^\perp\]
\[= (R(u, z))^\perp\]
\[= R(u, z).\]

This completes the proof. \(\square\)

In order to obtain suitable axioms for quantum relation algebras that correspond to the Axioms \((RA\ 2)\), \((RA4b)\), and \((RA\ 8)\) we need to able to formalize the compatibility properties of the previous lemma in the language of relations, i.e., without referring to the elements of the universe \(A\). Notice that the compatibility relation of \(Q_{RL}(A)\) does not immediately provide the required properties because of its component-wise behavior.

We want to focus on so-called ideal relations. A quantum relation \(Q \in Q_{RL}(A)\) is called an ideal if it is a constant function, i.e., if there is a \(z \in L\) so that \(Q(x, y) = z\) for every pair \(x, y \in A\). In the theory of relation algebras, or in a distributive case, this can be expressed by \(\top;Q;\top = Q\). Composition of quantum relations is, in general, not associative so that the previous equation becomes problematic.

**Lemma 8.** Suppose the structure \((A, \cap, \cup, (\cdot)^\perp, \top, \bot)\) satisfies \((QRA\ 1)\), \((QRA\ 3)\), and \((QRA\ 4a)\). Then the following two properties are equivalent:

1. \((\top;Q;\top = \top;Q;\top) = Q,\)
2. \(Q;\top = Q\) and \(\top;Q = Q.\)
Proof. 1.⇒2. The first equation follows immediately from

\[
\top; Q = (\top; Q); I \quad \text{(QRA 3)}
\]
\[
\subseteq (\top; Q); \top = Q \quad \text{(QRA 1) and Lemma 4(2)}
\]
\[
Q = I; Q \quad \text{(1), Lemma 4(1)}
\]
\[
\subseteq \top; Q \quad \text{(QRA 3) and (QRA 4a)}
\]

The second equation is shown analogously.

2.⇒1. This is obvious. \(\Box\)

Due to the previous lemma we will call a relation \(Q\) an ideal if \(Q; \top = Q\) and \(\top; Q = Q\). Now, in order to obtain an abstract version of the compatibility properties of Lemma 7 we are going to use another pair \((\uparrow, \downarrow)\) of operations already studied in the theory of \(L\)-fuzzy relations \([2, 8]\). These operations allow a definition of the notion of crisp relations, i.e., \(L\)-valued relations that assign only 0 or 1 to each pair in \(A \times A\). This property cannot be expressed in the language of relations without those operations \([7]\). The operation \(\downarrow\) is supposed to map a relation \(R\) to the greatest crisp relation \(R^\downarrow\) included in \(R\). Similarly, \(R^\uparrow\) is the smallest crisp relation containing \(R\). Since we deal with complemented lattice it will be sufficient to introduce only one operation. The other can then be defined using the first and complements.

Consider the following operation on a lattice \(L\) defined by

\[
x^\uparrow = \begin{cases} 
0 & \text{if } x = 0, \\
1 & \text{otherwise.}
\end{cases}
\]

Notice that we have to following two properties:

1. \((x \cdot y)^\uparrow = x^\uparrow \cdot y^\uparrow\) for all \(x, y \in L\),
2. \((\sum_{i \in I} x_i)^\uparrow = \sum_{i \in I} x_i^\uparrow\) for all sets \(\{x_i \mid i \in I\} \subseteq L\).

We may now lift this operation to the level of quantum relations by defining

\[
R^\uparrow(x, y) = (R(x, y))^\uparrow.
\]

The properties of the following lemma will turn out to be equivalent to the abstract definition of the arrow operations in the theory of \(L\)-fuzzy relations.

Lemma 9. Suppose \(Q, R, S \in \mathcal{QRel}_L(A)\). Then we have:

(QRA 9) \(\uparrow\) is a closure, i.e., monotone, expanding, and idempotent, and we have \(\bot^\uparrow = \bot\),

(QRA 10) \((R^\uparrow; S^\uparrow)^\uparrow = R^\uparrow \cap S^\uparrow\),

(QRA 11) \((Q \cap R^\uparrow)^\uparrow = Q^\uparrow \cap R^\uparrow\),

(QRA 12) If \(Q \neq \bot\) is an ideal, then \(Q^\uparrow = \top\).
Proof. (QRA 9) follows immediately from the definition.  
(QRA 10) This property follows from

\[(R^\sim; S^\uparrow)(x, z) = ((R^\sim; S^\uparrow)(x, z))^\uparrow\]

\[= \left( \sum_{y \in A} R^\sim(x, y) \cdot S^\uparrow(y, z) \right)^\uparrow\]

\[= \sum_{y \in A} (R^\sim(x, y) \cdot S^\uparrow(y, z))^\uparrow\]  (2) from above

\[= \sum_{y \in A} (R(y, x))^\uparrow \cdot (S(y, z))^\uparrow\]  (1) from above

\[= \sum_{y \in A} R^\uparrow(y, x) \cdot S^\uparrow(y, z)\]

\[= \sum_{y \in A} R^\sim(y, x) \cdot S^\uparrow(y, z)\]

\[= (R^\sim; S^\uparrow)(x, z)\].

(QRA 11) Consider the following computation

\[(Q \sqcap R^\uparrow)^\uparrow(x, y) = ((Q \sqcap R^\uparrow)(x, y))^\uparrow\]

\[= ((Q(x, y) \cdot R^\uparrow(x, y))^\uparrow\]

\[= ((Q(x, y) \cdot (R(x, y))^\uparrow)^\uparrow\]

\[= (Q(x, y))^\uparrow \cdot (R(x, y))^\uparrow\]  (1) from above

\[= Q^\uparrow(x, y) \cdot R^\uparrow(x, y)\]

\[= (Q \sqcap R^\uparrow)(x, y)\].

(QRA 12) From \(Q; \top = Q\) we obtain \(Q(x, z) = (Q; \top)(x, z) = \sum_{y \in A} Q(x, y) \cdot \top(y, z) = (Q; \top)(x, z') = Q(x, z').\) Similarly, we get \(Q(x, z) = Q(x', z)\) using \(\top; Q = Q\). This shows that \(Q\) assigns the same element from \(L\) to every pair in \(A \times A\). Since \(Q \neq \bot\), i.e., \(Q(x, y) \neq 0\), we immediately obtain \(Q^\uparrow = \top\). □

We call a relation \(Q\) crisp iff \(Q^\uparrow = Q\). Now, let us define the dual operation \(\downarrow\) by \(R^\downarrow := \overline{R^\uparrow}\).

Lemma 10. Suppose \(\uparrow\) is an operation on an orthomodular lattice satisfying (QRA 9) and (QRA 11). Then we have:

1. If \(Q\) is crisp, then so is \(\overline{Q}\).
2. \( Q^\uparrow \subseteq R \) iff \( Q \subseteq R^\downarrow \) (Galois correspondence),
3. \( Q \) is crisp iff \( Q^\downarrow = Q \).
4. \((Q \sqcup R^\downarrow)^\downarrow = Q^\uparrow \sqcup R^\downarrow\).

Proof. 1. Consider the following computation

\[
\begin{align*}
\overline{Q}^\uparrow \cap Q &= \overline{Q}^\uparrow \cap Q^\uparrow \\
&= (\overline{Q} \cap Q^\uparrow)^\uparrow \tag{QRA 11} \\
&= (\overline{Q} \cap Q)^\uparrow \tag{QRA 9} \\
&= \perp^\uparrow \tag{QRA 9}
\end{align*}
\]

2. If \( Q^\uparrow \subseteq R \), then (QRA 9) implies \( \overline{R}^\uparrow \subseteq \overline{Q}^\downarrow = \overline{Q}^\uparrow \) where the last equality follows from (1) since \( Q^\uparrow \) is crisp. We obtain \( Q \subseteq Q^\uparrow \subseteq \overline{R}^\uparrow = R^\downarrow \). Conversely, suppose \( Q \subseteq R^\downarrow \). Then we have \( Q^\uparrow \subseteq \overline{R}^\uparrow \). By (1) the right-hand side is equal to \( \overline{R}^\uparrow \) since \( \overline{R}^\uparrow \) is crisp. We obtain \( \overline{R} \subseteq \overline{R}^\uparrow \subseteq \overline{Q}^\uparrow \) from (QRA 9), and, hence, \( Q^\uparrow \subseteq R \).
3. This follows from (2) and the fact that we have \( R^\downarrow = \overline{R}^\uparrow \subseteq \overline{R} = R \) for all relations \( R \) by (QRA 9).
4. This property is an immediate consequence (QRA 11) using the definition of \( \downarrow \) and (1) and (3). \( \square \)

Property (2) of the previous lemma together with (QRA 10), (QRA 11), and (QRA 12) are the axiom of arrow categories [8, 9]. However, notice that arrow categories use scalar relations instead of ideal relations used in (QRA 12). In the distributive case both notions are equivalent.

Lemma 11. Let \( L \) be a complete orthomodular lattice. Then we have:

(QRA 13) Crisp relations are compatible with all relations, i.e., \( Q^\uparrow = Q \) implies \( Q \downarrow \subseteq R \) for all \( R \).

Proof. From the computation

\[
\begin{align*}
(Q \cap (\overline{Q} \sqcup R))(x, y) &= Q(x, y) \cdot ((Q(x, y))^\downarrow + R(x, y)) \\
&= Q(x, y) \cdot R(x, y) \tag{Lemma 2(3)} \\
&= (Q \cap R)(x, y)
\end{align*}
\]

we conclude the assertion. \( \square \)

Notice that the previous property of crisp relations cannot be used as alternative definition for crispness. If we consider the Boolean algebra \( B_4 \) with four elements, then all relations in \( \text{QRel}_{B_4}(\{a_1, a_2\}) \) are compatible with each other, but not all relations are crisp. On the other hand, (QRA 13) and Theorem 1
imply that the orthomodular lattice of quantum relations can be decomposed into a product of orthomodular lattices for each crisp relation. If we consider atomic relations (with respect to crisp relations) we actually obtain structures that are isomorphic to the underlying orthomodular lattice of truth values. As an example consider the relation \( Q \in Q_{\text{Rel}}(\{a_1, a_2\}) \) defined by \( Q(a_1, a_1) = 1 \) and \( Q(x, y) = 0 \) otherwise. The interval \([\bot, Q]\) contains the relations that map \((a_1, a_1)\) to an arbitrary element from \(L\) and every other pair to 0. This interval is obviously isomorphic to \(L\). This construction will be studied in future work.

Now, consider the following operation

\[ Q \ll R := (Q \rightarrow_1 R)^\dagger. \]

**Lemma 12.** Let be \( Q, R \in Q_{\text{Rel}}(A) \). Then

\[
(Q \ll R)(x, y) = \begin{cases} 
1 & \text{if } Q(x, y) \leq R(x, y), \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** This follows immediately from Lemma 5(2) and the definition of the arrow operation. \(\square\)

If \( Q \) is an ideal, then \( Q \ll R \) computes a so-called \( \alpha \)-cut of \( R \). In the theory of \( L \)-fuzzy relations an \( \alpha \)-cut of a relation \( R \) is a crisp relation that assigns 1 to a pair \((x, y)\) if \( R(x, y) \geq \alpha \) and 0 otherwise. Notice that in arrow categories and related structures [8] scalars and residuals with respect to composition is used. Alternatively, this could also be done using ideals and implication operations, i.e., the \( \alpha \)-cut operation obtained here is the same as in arrow categories.

Now, define

\[ Q \equiv R := (Q \ll R) \cap (R \ll Q). \]

Obviously, we have \((Q \equiv R)(x, y) = 1\) if \( Q(x, y) = R(x, y) \) and 0 otherwise for quantum relations \( Q, R \in Q_{\text{Rel}}(A) \).

Let us denote by \( Q_J \) the relation \((J \equiv Q):\top \cap J\). Based on this notation we are now able to define a “row-wise” compatibility relation by

\[ Q \& R : \iff Q_J \downarrow R \text{ for all ideals } J. \]

We obtain the following result.

**Lemma 13.** Let \( Q, R \in Q_{\text{Rel}}(A) \) be quantum relations. Then \( Q \& R \) is equivalent to \( Q(x, y) \downarrow R(x, z) \) for all \( x, y, z \in A \).
Proof. If \( J \) is the ideal induced by \( a \in L \), i.e., \( J(x, y) = a \) for all \( x, y \in A \), then we have
\[
Q_J(x, y) = (J \equiv Q; \top)(x, y) \cdot J(x, y)
\]
\[
= \left( \sum_{x \in A} (J \equiv Q)(x, z) \cdot \top(z, y) \right) \cdot a
\]
\[
= \left( \sum_{x \in A} (J \equiv Q)(x, z) \right) \cdot a
\]
\[
= \begin{cases} 1 \cdot a & \text{if } \exists z : Q(x, z) = a \\ 0 \cdot a & \text{otherwise} \end{cases}
\]
\[
= \begin{cases} a & \text{if } \exists z : Q(x, z) = a \\ 0 & \text{otherwise} \end{cases}
\]
Definition of \( J \equiv Q \)

"\( \Rightarrow \)". Choose the ideal \( J \) induced by \( Q(x, y) \). Then \( Q_J(x, z) = Q(x, y) \) by the computation above. Furthermore from \( Q_J \downarrow R \) and its component-wise definition we obtain \( Q_J(x, z) \downarrow R(x, z) \), and, hence, \( Q(x, y) \downarrow R(x, z) \).

"\( \Leftarrow \)". Suppose \( J \) is the ideal induced by \( a \) and \( x, z \in A \). If there is no element \( y \in A \) so that \( Q(x, y) = a \), then \( Q_J(x, u) = 0 \) for all \( u \in A \). In particular, we have \( Q_J(x, z) = 0 \), and, hence, \( Q_J(x, z) \downarrow R(x, z) \). If there is a \( y \in A \) so that \( Q(x, y) = a \), then \( Q_J(x, u) = a \) for all \( u \in A \). From \( Q(x, y) \downarrow R(x, z) \) we conclude \( Q_J(x, z) \downarrow R(x, z) \). Together we have just shown that \( Q_J \downarrow R \). \( \square \)

Based on this operation we can reformulate two of the properties of Lemma 7 in the language of relations:

\( \text{(QRA 4b)} \) \( Q^\perp \# S \) and \( R^\perp \# S \) implies \( (Q \sqcup R); S = Q; S \sqcup R; S \).

\( \text{(QRA 8)} \) \( Q^\perp \# R \) implies \( Q^\perp; R \sqsubseteq R \).

We will use \( \text{(QRA 4)} \) to refer to \( \text{(QRA 4a)} \) and \( \text{(QRA 4b)} \). Let us denote by \( \text{QRA}^- \) the set of axioms \( \text{(QRA 1)}, \text{(QRA 3)}-(\text{QRA 13}). \)

Lemma 14. Suppose \( \langle A; \cap, \sqcup, \sqcap, \sqsupseteq, \downarrow, \top, \bot, =, (\cdot)^+, (\cdot)^\perp \rangle \) is a structure that satisfies \( \text{QRA}^- \). Then \( Q \) is crisp implies \( Q \# R \) for all \( R \).

Proof. If \( \bot = \top \), then the assertion is trivial. Therefore, assume for the rest of this proof that \( \bot \neq \top \). By \( \text{(QRA 13)} \) we have \( Q \downarrow J \), and, hence, \( J \rightarrow_1 Q = \overline{J} \sqcup Q \) and \( Q \rightarrow_1 J = \overline{Q} \sqcup J \). We obtain
\[
J \ll Q = (\overline{J} \sqcup Q)^\perp
\]
\[
= J^\perp \sqcup Q \quad \text{Lemma 10(4) since } Q \text{ is crisp}
\]
\[
= \overline{J}^\perp \sqcup Q
\]
\[
= \begin{cases} \overline{J}^\perp \sqcup Q & \text{if } J \neq \bot, \\ \bot \sqcup Q & \text{if } J = \bot, \end{cases}
\]
\[
= \begin{cases} Q & \text{if } J \neq \bot, \\ \top & \text{if } J = \bot. \end{cases}
\]
\( \text{(QRA 9)} \) and \( \text{(QRA 12)} \)
Analogously, using the fact that also $Q$ is crisp, we get

$$Q \ll J = \begin{cases} Q & \text{if } J \neq \bot, \\ \top & \text{if } J = \bot. \end{cases}$$

Together we obtain

$$J \equiv Q = \begin{cases} Q & \text{if } J = \bot, \\ Q & \text{if } J = \top, \\ \bot & \text{if } J \neq \bot \text{ and } J \neq \top. \end{cases}$$

In all three cases we obtain a crisp relation. Since crisp relations are closed under composition and meet by \textbf{(QRA 10)} and \textbf{(QRA 11)} the relation $Q_J$ is also crisp. We conclude $Q_J \downarrow R$, and, hence, $Q \uplus R$, follows from \textbf{(QRA 13)}.

For the remaining axiom let us denote by $Q_J^J$ the relation $\top; ((J \equiv Q); \top) \cap J$. Notice that composition of crisp quantum relations is associative so that the brackets in the definition of $Q_J^J$ are redundant in the concrete case. In addition, we will later show that composition of crisp relations is also associative in the abstract case. However, at this point, seen as axiom, we cannot drop the brackets. Based on the above notation we are now able to define an “overall” compatibility relation by

$$Q \uplus R \iff Q_J^J \downarrow R \text{ for all ideals } J.$$

We obtain the following result.

**Lemma 15.** Let $Q, R \in Q\text{Rel}_L(A)$ be quantum relations. Then $Q \uplus R$ is equivalent to $Q(u, x) \downarrow R(y, z)$ for all $u, x, y, z \in A$.

**Proof.** If $J$ is the ideal induced by $a \in L$, i.e., $J(x, y) = a$ for all $x, y \in A$, then we have

$$Q_J^J(x, y) = (\top; ((J \equiv Q); \top))(x, y) \cdot J(x, y)$$

$$= \left( \sum_{u \in A} \top(x, u) \cdot ((J \equiv Q); \top)(u, y) \right) \cdot a$$

$$= \left( \sum_{u, v \in A} (J \equiv Q)(u, v) \cdot \top(v, y) \right) \cdot a$$

$$= \left( \sum_{u, v \in A} (J \equiv Q)(u, v) \right) \cdot a$$

$$= \begin{cases} 1 \cdot a & \text{if } \exists u, v : Q(u, v) = a \\ 0 \cdot a & \text{otherwise} \end{cases} \quad \text{Definition of } J \equiv Q$$

$$= \begin{cases} a & \text{if } \exists u, v : Q(u, v) = a \\ 0 & \text{otherwise.} \end{cases}$$
Choose the ideal \( J \) induced by \( Q(u, x) \). Then \( Q^J(y, z) = Q(u, x) \) by the computation above. Furthermore from \( Q^J \downarrow R \) and its component-wise definition we obtain \( Q^J(y, z) \downarrow R(y, z) \), and, hence, \( Q(u, x) \downarrow R(y, z) \).

Suppose \( J \) is the ideal induced by \( a \) and \( y, z \in A \). If there is no elements \( u, x \in A \) so that \( Q(u, x) = a \), then \( Q^J(r, s) = 0 \) for all \( r, s \in A \). In particular, we have \( Q^J(y, z) = 0 \), and, hence, \( Q^J(x, z) \downarrow R(x, z) \). If there are \( u, x \in A \) so that \( Q(u, x) = a \), then \( Q^J(r, s) = a \) for all \( r, s \in A \). From \( Q(u, x) \downarrow R(y, z) \) we conclude \( Q^J(y, z) \downarrow R(y, z) \). Together we have just shown that \( Q^J \downarrow R \).

Now, we are ready to present our last axiom:

\[ \text{(QRA 2)} \quad Q^\prec \not\in R, \quad Q^\not\preceq S, \quad \text{and} \quad R^\prec \not\in S \text{ implies } Q ; (R ; S) = (Q ; R) ; S. \]

We will use \( \text{QRA} \) to denote \( \text{QRA}^- \) plus Axiom \( \text{(QRA 2)} \).

**Lemma 16.** Suppose \( \langle A, \cap, \cup, R, \Pi, ;, (\cdot)^\prec, I, (\cdot)^\top \rangle \) is a structure that satisfies \( \text{QRA} \). Then \( Q \) is crisp implies \( Q^\not\in R \) for all \( R \).

**Proof.** This property is shown similar to Lemma 14.

Finally, we can summarize the properties of crisp relations in the following theorem.

**Theorem 2.** Suppose \( \langle A, \cap, \cup, R, \Pi, ;, (\cdot)^\prec, I, (\cdot)^\top \rangle \) is a structure that satisfies \( \text{QRA} \). Then the set of crisp relations is a relation algebra.

**Proof.** The crisp relation are closed under all operations and satisfy the assumption of the Axioms \( \text{(QRA 2)}, \text{(QRA 4b)}, \text{and (QRA 8)}. \)

Maddux has already considered relation algebras in which composition only satisfies weaker forms of associativity \[4\]. An operation is called semi-associative if \( (Q; R) ; \Pi = Q ; (R ; \Pi) \) for all relations \( Q \). This property is of particular interest if we consider the atom structure of a relation algebra. Here we only want to establish this property within our theory.

**Theorem 3.** Suppose \( \langle A, \cap, \cup, R, \Pi, ;, (\cdot)^\prec, I, (\cdot)^\top \rangle \) is a structure that satisfies \( \text{QRA} \). Then composition is semi-associative.

**Proof.** If \( \bot = \Pi \), then the assertion is trivial. Now, assume \( \bot \neq \Pi \). Since \( \Pi \) is crisp by \( \text{(QRA 12)} \) we get \( Q^- \not\in \Pi, \quad Q^\not\preceq \Pi, \quad \text{and} \quad Q^\sim \not\in \Pi \) from Lemma 14 and Lemma 16. We compute

\[
(Q; \Pi) ; \Pi = Q ; (\Pi ; \Pi) \quad \text{(QRA 2)}
\]

\[
= Q ; \Pi. \quad \text{Lemma 4(3)}
\]

This completes the proof.
5 Conclusion and Future Work

We consider this paper as a first step towards an algebraic treatment of first-order quantum logic based on the theory of relations. As a result we obtained a set of axioms that seems adequate for quantum relation algebras. This set of axioms is based on the axioms of relation algebras as well as the set of axioms used for arrow operation within the theory of $L$-fuzzy relations. We did not define a quantum relation algebra as a structure satisfying (QRA 1)-(QRA 13) because we believe that more study, and more experience in using those structures in particular, is necessary before providing such a definition.

Besides the obvious study of the basic properties of quantum relations, the current paper indicated several possible projects for future work. One possible study could further investigate the decomposition using atomic crisp relations in the abstract setting. The relationship between the structure obtained by this decomposition and the set of ideal or scalar relations is here of particular interest since all of them give access to the underlying orthomodular lattice of truth values. Another project could investigate a variable-free version of quantum set theory [5] similar to [6]. Last but not least, the relationship between a theory of quantum relation and programs on quantum computers is another interesting area.

References