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Abstract. We show that the class of Boolean contact algebras has the joint embedding property and the amalgamation property, and that the class of connected Boolean contact algebras has the joint embedding property but not the amalgamation property.

1 Introduction

Boolean contact algebras (BCAs) arise in spatial–temporal reasoning, and are hybrid algebraic–relational structures. Their history goes back to the 1920's, augmenting Leśniewski's mereology –whose models may be regarded as Boolean algebras with the least element removed – with a binary predicate of "being in contact". A contact relation on a Boolean algebra B is, loosely speaking, a symmetric and reflexive relation $\mathcal{C}$ on the nonzero elements of B with additional compatibility properties. Standard models are Boolean algebras of regular closed sets of some topological space $\mathcal{X}$ where two such sets are in contact if their intersection is not empty.

From a mereological "part of" relation P – which is the Boolean order in a BCA – and a contact relation $\mathcal{C}$, several other binary relations can be defined as indicated in Table 1. The relations $DC, EC, PO, TPP, TPP^\smile, NTPP, NTPP^\smile$ along with the identity relation $1'$ have achieved some prominence in constraint spatial reasoning where they are called the *RCC8 relations*. The conditions are given by a table of relational constraints based on composition among these relations. Details and background can be found in the overview article by Renz and Nebel [12].

Recently, Bodirsky and Wölfl [3] have shown that a class $\mathbf{K}$ of finite models of RCC8 theory has both the joint embedding property and the amalgamation

* The ordering of authors is alphabetical and equal authorship is applied.

Table 1. Relations defined from $\mathcal{C}$ and $\leq$

(1.1)	$P := \leq$	part of
(1.2)	$PP := P \cap -1'.$	proper part of
(1.3)	$O := P^{\smile} \ ; \ P$	overlap
(1.4)	$PO := O \cap -(P \cup P^{\smile})$	partial overlap
(1.5)	$EC := C \cap -O$	external contact
(1.6)	$TPP := PP \cap (EC \ ; \ EC)$	tangential proper part
(1.7)	$NTPP := PP \cap -TPP$	non–tangential proper part
(1.8)	$DC := -\mathcal{C}$	disconnected

property. Thus, by Fraïssé's method, there is a countable homogeneous structure into which all members of **K** can be embedded. It follows from their investigations that for any k, the network consistency problem for RCC8 restricted to networks of treewidth at most k can be solved in polynomial time. Since each RCC8 networks can be embedded into a BCA it is of interest to consider analogous problems for BCAs. In the present paper we start such investigation and we show that the class of BCAs has the amalgamation property, and that the class of connected BCAs does not. Our tools are free constructions of Boolean algebras and two forms of graph products which have been studied since the 1960s.

2 Basic definitions and notation

If $\langle U, R\rangle$ and $\langle V, S\rangle$ are binary relational structures, and $f : U \to V$ is a mapping, then f is called a *homomorphism* if aRb implies $f(a)Sf(b)$. We call f a *strong homomorphism*, if f is onto and for each $s, t \in V$ with sSt there are $a, b \in U$ such that aRb and $f(a) = s, \ f(b) = t$. A *graph* is a reflexive and symmetric binary relation.

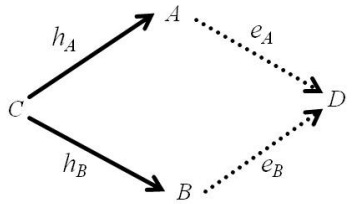
Fix a countable first order language $\mathcal{L}$. A class **K** of $\mathcal{L}$ – structures has the

1. *Hereditary property* (HP) if **K** is closed under substructures.
2. *Joint embedding property* (JEP) if for any $A, B \in \mathbf{K}$, there is some $C \in \mathbf{K}$ and embeddings $e_A : A \to C, \ e_B : B \to C$. Such C is called a *free product of A and B* (with respect to **K**) if for every $M \in \mathbf{K}$ and homomorphisms $f_A : A \to M, \ f_B : B \to M$ there is a unique homomorphism $f : C \to M$ such that $f \circ e_A = f_A$ and $f \circ e_B = f_B$.
3. *Amalgamation property* (AP) if for all $A, B, C \in \mathbf{K}$ such that C is (isomorphic to) a common substructure of A and B, say, with embeddings

$h_B : C \hookrightarrow B$ and $h_A : C \hookrightarrow A$, there are some $D \in \mathbf{K}$ and embeddings $e_A : A \hookrightarrow D$ and $e_B : B \hookrightarrow D$ such that $e_A \circ h_A = e_B \circ h_B$. D is called an *amalgamated product of A and B over C*.

4. *Disjoint amalgamation property* (DAP) – also called *strong amalgamation property* – if it has the AP and the amalgamated product may be chosen in such a way that $e_A[A] \cap e_B[B] = e_A \circ h_A[C]$. If we assume that C is a common substructure of A and B, then an amalgamated product D of A and B over C containing A and B as substructures may be chosen in such a way that $A \cap B = C$.

Fig. 1. Amalgamation property



Throughout, $\langle B, +, \cdot, -, 0, 1\rangle$ will denote a Boolean algebra. With some abuse of language, we shall usually identify algebras with their base set. We denote by B^+ the set of all nonzero elements of B. If A is a (Boolean) subalgebra of B, we write $A \leq B$. If $M \subseteq B$ we denote by $[M]^B_{\text{filt}}$ the filter of B generated by M.

It is well known that the class of all Boolean algebras has the free amalgamation property:

Theorem 1. *[9, Theorem 11.19.] If A, B, C are Boolean algebras and $h_A : C \hookrightarrow A$ and $h_B : C \hookrightarrow B$ are embeddings, then there is (up to isomorphism) a unique amalgamated free product of A and B over C.*

In the sequel we will use a characterization of the amalgamated free product of Boolean algebras from [9]:

Lemma 1. *Suppose that $A, B \leq D$, and that C is a subalgebra of both A and B. Then, D is the amalgamated free product of A and B over C, if $A \cup B$ generates D, and for all $a \in A, b \in B$ with $a \cdot b = 0$ there is some $c \in C$ such that $a \leq c$ and $b \leq -c$.*

We shall denote the (Boolean) amalgamated product of A and B over C by $A \oplus_C B$. If C is the two element Boolean algebra, we just write $A \oplus B$ and call

it the *free product of A and B*. The existence of $A \oplus B$ shows that the class of all Boolean algebras has the JEP as well.

The following lemma describes the ultrafilters of $A \oplus_C B$ in terms of $\mathrm{Ult}(A)$ and $\mathrm{Ult}(B)$ see [9], Ch. 4, Exercise 9:

Lemma 2. *Suppose that D is the amalgamated free product of A, B over C. Then, each ultrafilter of D is of the form $[F \cup G]^D_{\mathrm{filt}}$ where $F \in \mathrm{Ult}(A)$, $G \in \mathrm{Ult}(B)$, and $F \cap C = G \cap C$. Furthermore, if $H' \in \mathrm{Ult}(D)$ such that $H' \cap A = F$ and $H' \cap B = G$, then $H = H'$.*

Proof. Since ultrafilters are maximal filters it is enough to show that $H := [F \cup G]^D_{\mathrm{filt}}$ is an ultrafilter in D when $F \in \mathrm{Ult}(A)$, $G \in \mathrm{Ult}(B)$, and $F \cap C = G \cap C$. We first show that H is proper: If there are $a \in F, b \in G$ such that $a \cdot b = 0$, with Lemma 1 choose some $c \in C$ with $a \leq c$ and $b \leq -c$. Then, $c \in F \cap C$ and $-c \in G \cap C$, and thus, by the hypothesis $F \cap C = G \cap C$, we have $c, -c \in F \cap G \cap C$, contradicting that F and G are proper filters.

Assume that there is some $a \in D$ such that $a, -a \notin H$. Since D is generated by $A \cup B$, there are $a_0, \ldots, a_n \in A$, $b_0, \ldots, b_n \in B$ such that $a = a_0 \cdot b_0 + \ldots + a_n \cdot b_n$; in particular, there is some $i \leq n$ with $a_0 \cdot b_0 \notin H$ and $-(a_0 \cdot b_0) \notin H$. It follows that $a_0 \notin F$ or $b_0 \notin G$, and $-a_0 \notin F$ and $-b_0 \notin G$. This contradicts that $F \in \mathrm{Ult}(A)$ and $G \in \mathrm{Ult}(B)$. Finally, suppose that $H' \in \mathrm{Ult}(D)$ such that $H' \cap A = F$ and $H' \cap B = G$, and assume that $H \neq H'$. Then, there are $x \in H, y \in H'$ such that $x \cdot y = 0$. Choose some $c \in C$ such that $x \leq c$, $y \leq -c$ and proceed as in the first part of the proof. □

If $H \in \mathrm{Ult}(D)$ corresponds to $\langle F, G \rangle$ we will indicate this by $H \sim \langle F, G \rangle$.

For unexplained definitions and properties in model theory we refer the reader to [8], and in Boolean algebras to [9].

3 Boolean contact algebras

A contact relation $\mathcal{C}$ on a Boolean algebra B is defined by the following five properties:

C_0. $x \mathcal{C} y$ implies $x, y \neq 0$.
C_1. $x \neq 0$ implies $x \mathcal{C} x$.
C_2. $x \mathcal{C} y$ implies $y \mathcal{C} x$. (symmetry)
C_3. $x \mathcal{C} y$ and $y \leq z$ implies $x \mathcal{C} z$. (monotonicity)
C_4. $x \mathcal{C} (y + z)$ implies ($x \mathcal{C} y$ or $x \mathcal{C} z$). (distributivity)

A pair $\langle B, \mathcal{C}_B \rangle$ is called a *Boolean contact algebra* (BCA). This is the definition of contact algebras given in [2] in the Handbook of Spatial Logics, and by Dimov and Vakarelov [4] in their seminal papers on proximity structures.

Other properties have been considered in the literature: Contact algebras which satisfy

C_5. $\mathcal{C}(x) = \mathcal{C}(y)$ implies $x = y$. (extensionality)
C_6. If $(\forall z)(x\mathcal{C}z$ or $y\mathcal{C} - z)$ then $x\mathcal{C}y$. (interpolation)

are called *connection algebras* in [15] in the context of proximity structures. Finally, in [14] a Boolean connection algebra is a BCA $\langle B, \mathcal{C} \rangle$ which satisfies C_5, and, in addition, the following condition:

C_7. If $x \notin \{0, 1\}$, then $x\mathcal{C} - x$. (connectivity)

BCAs which satisfy C_5 and C_7 are sometimes called *RCC algebras* after the Region Connection Calculus of [11], and BCAs which satisfy C_7 have been called *generalized RCC algebras* in [10].

The smallest contact relation is given by $x\mathcal{C}_{\min}y \iff x \cdot y \neq 0$. Usually, $\mathcal{C}_{\min}$ is called the *overlap relation*, denoted by $\mathcal{O}$; clearly, $\mathcal{O}$ satisfies C_5. The largest contact relation is $\mathcal{C}_{\max} = B^+ \times B^+$; it satisfies C_7. The four element Boolean algebra has exactly the contact relations $\mathcal{C}_{\min}$ and $\mathcal{C}_{\max}$. The only extensional contact relation on a finite Boolean algebra is the overlap relation $\mathcal{O}$.

Note that a BCA homomorphism is injective: If a Boolean homomorphism $f : \mathfrak{B} \to \mathfrak{B}_0$ is not injective, then there is some $x \in B^+$ such that $f(x) = 0$. By C_2, $x\mathcal{C}x$, but $f(x)(-\mathcal{C}_0)f(x)$ by C_0.

There is a well developed duality theory for BCAs. Generalizing results of [7] in the finite case and of [5] in the area of discrete proximity algebras, in [6] the following duality between contact relations and graphs on its space of ultrafilters is exhibited:

Theorem 2. *Let B be a Boolean algebra. There is a bijective order preserving correspondence between the contact relations on B and the reflexive and symmetric relations on* $\mathrm{Ult}(B)$ *which are closed in the product topology of* $\mathrm{Ult}(B)$.

Proof. We just give the construction; details can be found in [6]. If R is a reflexive and symmetric closed relation on $\mathrm{Ult}(B)$, set $\mathcal{C}_R = \bigcup\{F \times G : \langle F, G \rangle \in R\}$. Conversely, if $\mathcal{C}$ is a contact relation on B, let $R_{\mathcal{C}} = \{\langle F, G \rangle \in \mathrm{Ult}(B)^2 : F \times G \subseteq \mathcal{C}\}$. □

$R_{\mathcal{C}}$ is called the *dual of* $\mathcal{C}$, and $\mathcal{C}_R$ is called the *dual of* R. Our proofs will sometimes be based on this duality, as graphs on ultrafilters are often easier to handle than contact relations. As an application we present a result from [6]:

Theorem 3. *Let $\langle B, \mathcal{C}\rangle$ be a BCA.*

1. *$\mathcal{C}$ satisfies C_6 if and only if $R_{\mathcal{C}}$ is an equivalence relation.*
2. *If B is finite, then $\mathcal{C}$ satisfies C_7 if and only if $R_{\mathcal{C}}$ is connected.*

Our final aim in this section is to describe the dual of the situation that $\langle A, \mathcal{C}_A\rangle$ is a substructure of $\langle B, \mathcal{C}_B\rangle$. To this end, we first quote Proposition 3.2 of [4]:

Lemma 3. *If $\langle B, \mathcal{C}\rangle$ is a BCA, and F', G' are filters of B such that $F' \times G' \subseteq \mathcal{C}$, then there are ultrafilters F, G of B such that $F' \subseteq F,\ G' \subseteq G$, and $F \times G \subseteq \mathcal{C}$.*

Theorem 4. *Let $\langle A, \mathcal{C}_A\rangle$ and $\langle B, \mathcal{C}_B\rangle$ be BCAs with dual relations R_A and R_B, and suppose that $A \leq B$. Then, $\mathcal{C}_A = \mathcal{C}_B \restriction A$ if and only if*

$$R_A = \{\langle F \cap A, G \cap A\rangle : F, G \in \mathrm{Ult}(B), \langle F, G\rangle \in R_B\}. \tag{3.1}$$

Proof. "$\Rightarrow$": For the "$\subseteq$" direction, suppose that $\langle F', G'\rangle \in R_A$. We need to find $F, G \in \mathrm{Ult}(B)$ such that $F' = F \cap A$, $G' = G \cap A$, and $\langle F, G\rangle \in R_B$.

By definition, $\langle F', G'\rangle \in R_A$ implies that $F' \times G' \subseteq \mathcal{C}_A$, and $\mathcal{C}_A \subseteq \mathcal{C}_B \restriction A$ implies that $F' \times G' \subseteq \mathcal{C}_B$. Let F_0 be the filter of B generated by F', and G_0 the filter of B generated by G'. Since F' is a filter of A, it is closed under $\cdot$, and therefore, $F_0 = \{b \in B : (\exists a)[a \in F' \text{ and } a \leq b]\}$. Similarly, $G_0 = \{b \in B : (\exists a)[a \in G' \text{ and } a \leq b]\}$. Since $F' \times G' \subseteq \mathcal{C}_B$ and by C_3, it follows that $F_0 \times G_0 \subseteq \mathcal{C}_B$. Now, by Lemma 3, there are ultrafilters F, G of B such that $F_0 \subseteq F,\ G_0 \subseteq G$, and $F \times G \subseteq \mathcal{C}_B$, i.e. $\langle F, G\rangle \in R_B$. Since $F' \subseteq F, G' \subseteq G$, and F', G' are ultrafilters of A, we have $F' = F \cap A$ and $G' = G \cap A$.

For "$\supseteq$", let $F, G \in \mathrm{Ult}(B)$ and $\langle F, G\rangle \in R_B$, i.e. $F \times G \subseteq \mathcal{C}_B$. Since $\mathcal{C}_B \restriction A \subseteq \mathcal{C}_A$ we have $(F \cap A) \times (G \cap A) \subseteq \mathcal{C}_A$ which shows $\langle F \cap A, G \cap A\rangle \in R_A$.

"$\Leftarrow$": Let $a, b \in A$ and $a\mathcal{C}_A b$; we show that $a\mathcal{C}_B b$. Since $a\mathcal{C}_A b$, by Lemma 3 there are $F', G' \in \mathrm{Ult}(A)$ such that $a \in F', b \in G'$ and $\langle F', G'\rangle \in R_A$. By the hypothesis, there are $F, G \in \mathrm{Ult}(B)$ such that $F' = F \cap A, G' = G \cap A$ and $F \times G \subseteq \mathcal{C}_B$. Hence, $a\mathcal{C}_B b$.

Conversely, let $a, b \in A$ and $a\mathcal{C}_B b$. Then, by Lemma 3 there are ultrafilters F, G of B with $a \in F, b \in G$, and $F \times G \subseteq \mathcal{C}_B$. By the hypothesis, $\langle F \cap A, G \cap A\rangle \in R_A$, and thus, $(F \cap A) \times (G \cap A) \subseteq \mathcal{C}_A$. It follows from $a, b \in A$ that $a\mathcal{C}_A b$. □

Corollary 1. *Let $\langle A, \mathcal{C}_A\rangle$ be a BCA, and $A \leq B$. Suppose that R_A is the dual of $\mathcal{C}_A$, and define a relation R_B on $\mathrm{Ult}(B)$ by*

$$\langle F, G\rangle \in R_B \iff \langle F \cap A, G \cap A\rangle \in R_A. \tag{3.2}$$

Then, the dual $\mathcal{C}_B$ of R_B is the largest contact relation $\mathcal{C}$ on B such that $\langle A, \mathcal{C}_A\rangle$ is a substructure of $\langle B, \mathcal{C}\rangle$.

Proof. Since R_B satisfies (3.1), $\langle A, \mathcal{C}_A\rangle$ is a substructure of $\langle B, \mathcal{C}_B\rangle$. Next, suppose that $\langle A, \mathcal{C}_A\rangle$ is a substructure of $\langle B, \mathcal{C}\rangle$, and that R is the dual of $\mathcal{C}$. Suppose that $\langle F, G\rangle \in R$; then, by (3.1) $\langle F \cap A, G \cap A\rangle \in R_A$, and therefore, $\langle F \cap A, G \cap A\rangle \in R_B$ by the "$\Leftarrow$" direction of (3.2). Thus, $R \subseteq R_B$, and it follows from Theorem 2 that $\mathcal{C} \subseteq \mathcal{C}_B$. □

The result of Theorem 4 comes as no surprise: If A is a subalgebra of B, then by duality for Boolean algebras the mapping $p : \mathrm{Ult}(B) \twoheadrightarrow \mathrm{Ult}(A)$ defined by $H \mapsto H \cap A$ is onto. If, furthermore, $\langle A, \mathcal{C}_A\rangle$ and $\langle B, \mathcal{C}_B\rangle$ are BCAs with dual relations R_A and R_B, then Theorem 4 shows that $\mathcal{C}_B$ is an extension of $\mathcal{C}_A$ if and only if $R_A = \{\langle p(F), p(G)\rangle : \langle F, G\rangle \in R_B\}$; in other words, R_A is the image of R_B under the action of p on $\mathrm{Ult}(B) \times \mathrm{Ult}(B)$.

With some abuse of language we call R_B an extension of R_A if $A \leq B$, and (3.1) holds.

Since the Stone space of a finite Boolean algebra B is discrete, and each ultrafilter is determined by an atom of B, the construction can be somewhat simplified in this case: For a finite BCA $\langle B, \mathcal{C}\rangle$ we let $\Gamma(\langle B, \mathcal{C}\rangle)$ be the structure $\langle U, E\rangle$ where $U = At(B)$, and $\langle a, b\rangle$ is an edge in E if and only if $a\mathcal{C}b$. Conversely, if E is a graph on U we let $\Delta(\langle U, E\rangle)$ be the structure $\langle B, \mathcal{C}\rangle$ where B is the powerset algebra of U and $\mathcal{C}$ is the relation on B defined by $a\mathcal{C}b$ if and only if there are $x, y \in U$ with $x \in a$, $y \in b$ and xEy. Theorem 2 and Theorem 4 can now be reformulated as follows:

Theorem 5. *Let $\mathfrak{B} = \langle B, C\rangle$ be a finite BCA, and $\mathfrak{U} = \langle U, E\rangle$ be a finite graph, i.e. $|U| < \omega$.*

1. *$\Gamma(\mathfrak{B})$ is a graph, and $\Delta(\mathfrak{U})$ is a BCA.*
2. *$\Delta(\Gamma(\mathfrak{B})) \cong \mathfrak{B}$, and $\Gamma(\Delta(\mathfrak{U})) \cong \mathfrak{U}$.*
3. *If $\mathfrak{A} = \langle A, \mathcal{C}_A\rangle$ and $\mathfrak{B} = \langle B, \mathcal{C}_B\rangle$ are finite BCAs and $e : \mathfrak{A} \hookrightarrow \mathfrak{B}$ is a BCA embedding, then $h : \Gamma(\mathfrak{B}) \twoheadrightarrow \Gamma(\mathfrak{A})$ is a strong graph homomorphism, where for $b \in At(B)$, $h(b)$ is defined as follows: If $b \in At(B)$, then the smallest element of $[\{b\}]^B_{\mathrm{filt}} \cap e[A]$ is an atom of $e[A]$, and we set $h(b) = a$. Since e is an embedding, h is well defined.*

Example 1. Let B be the Boolean algebra generated by its atoms $At(B) = \{b_0, b_1, b_2, b_3\}$, and set $R = \{\langle b_0, b_1\rangle, \langle b_1, b_2\rangle, \langle b_2, b_3\rangle\}$. If $x, y \in B \setminus \{0, 1\}$, $x = b_{i_0} + \ldots + b_{i_k}$, $y = b_{j_0} + \ldots + b_{j_n}$ let

$$x\mathcal{C}_B y \iff (\exists i_r, j_s)[i_r = j_s \text{ or } b_{i_r} R b_{j_s} \text{ or } b_{j_s} R b_{i_r}].$$

Then, $\mathcal{C}_B$ is a contact relation on B, and $\Delta(\mathfrak{B})$ is the graph $R \cup R^{\smile} \cup 1'$ on $\{a_0, \ldots, a_3\}$. Suppose that A is the Boolean algebra generated by its atoms $AT(A) = \{d_0, d_1, d_2\}$, $S = \{\langle d_0, d_1\rangle, \langle d_1, d_2\rangle\}$, and for $x = d_{i_0} + \ldots + d_{i_k}$, $y = d_{j_0} + \ldots + d_{j_n}$ let

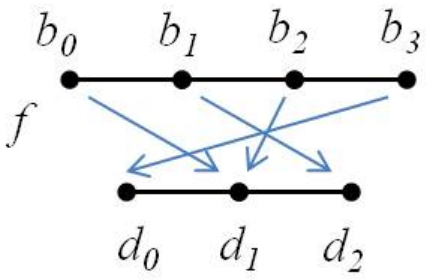
$$x\mathcal{C}_A y \iff (\exists i_r, j_s)[i_r = j_s \text{ or } d_{i_r} S d_{j_s} \text{ or } d_{j_s} R d_{i_r}].$$

Then, $\mathcal{C}_A$ is a contact relation on A, and $\Delta(\mathfrak{A})$ is the graph $S \cup S^{\smile} \cup 1'$ on $\{d_0, d_1, d_2\}$. Now, suppose that $f : At(B) \twoheadrightarrow At(A)$ is defined by

$$f(b_0) = f(b_2) = d_1,\ f(b_1) = d_2,\ f(b_3) = d_0,$$

see Figure 2. Then, f is a strong homomorphism and its dual embedding e of $\mathfrak{A}$

Fig. 2. A strong homomorphism



into $\mathfrak{B}$ is determined by $e(d_0) = b_3, e(d_1) = b_0 + b_2$, and $e(d_2) = b_1$. □

4 Extension properties of Boolean contact algebras

Let $\mathbf{K}$ be the class of all BCAs, and $\mathbf{K}_{\mathcal{C}_6}$, respectively, $\mathbf{K}_{\mathcal{C}_7}$, be the collection of all BCAs which satisfy C_6, respectively, C_7. Since the axioms C_0 – C_4 and C_7 are universal, both $\mathbf{K}$ and $\mathbf{K}_{\mathcal{C}_7}$ possess HP.

Theorem 6. $\mathbf{K}$, $\mathbf{K}_{\mathcal{C}_6}$, *and* $\mathbf{K}_{\mathcal{C}_7}$ *have the JEP.*

Proof. We shall exhibit two different ways to embed two BCAs into a third one, both arising from familiar constructions in graph theory. Let $\mathfrak{A} := \langle A, \mathcal{C}_A \rangle$ and $\mathfrak{B} := \langle B, \mathcal{C}_B \rangle$ be BCAs with dual relations R_A, respectively, R_B, and let $D := A \oplus B$ be the (Boolean) free product of A and B. $h : D \hookrightarrow 2^{\mathrm{Ult}(D)}$ is the Stone embedding, i.e. $h(a) = \{F \in \mathrm{Ult}(D) : a \in F\}$. The collection $\{h(a) : a \in D\}$ is a clopen basis for the Stone topology on $\mathrm{Ult}(D)$. Suppose w.l.o.g. that $A, B \leq D$, and $H, H' \in \mathrm{Ult}(D)$ with $H \sim \langle F, G \rangle,\ H' \sim \langle F', G' \rangle$, i.e. $H \cap A = F,\ H \cap B = G$ and $H' \cap A = F',\ H' \cap B = G'$.

Construction 1

Our first construction is based on the *direct product of graphs* [13]: Set

(4.1) $\quad HR^0_D H'$ if and only if $FR_A F'$ and $G = G'$ or $F = F'$ and $GR_B G'$.

Clearly, R_D^0 is reflexive and symmetric, since R_A and R_B have these properties. Next, we show that R_D^0 is closed in the product topology of $\mathrm{Ult}(D) \times \mathrm{Ult}(D)$: Suppose that $\langle H, H' \rangle \in \mathrm{cl}(R_D^0)$, and assume that $\langle H, H' \rangle \notin R_D^0$. There are three cases:

1. $H \cap A \neq H' \cap A$ and $H \cap B \neq H' \cap B$: Then, there are $a \in A,\ b \in B$ such that $a \in H \cap A,\ -a \in H' \cap A$, and $b \in H \cap B,\ -b \in H' \cap B$. It follows that $a \cdot b \in H$ and $-a \cdot -b \in H'$. Since $\langle H, H' \rangle \in \mathrm{cl}(R_D^0)$, $(h(a \cdot b) \times h(-a \cdot -b)) \cap R_D^0 \neq \emptyset$, and therefore there are $P, P' \in \mathrm{Ult}(D)$ such that $a \cdot b \in P,\ -a \cdot -b \in P'$, and $\langle P, P' \rangle \in R_D^0$. Let w.l.o.g. $P \cap A = P' \cap A$. Now, $a \cdot b \in P$ implies $a \in P \cap A$, and $-a \cdot -b \in P'$ implies $-a \in P' \cap A$. This, however, contradicts $P \cap A = P' \cap A$.
2. $H \cap A = H' \cap A$ and $\langle H \cap B, H' \cap B \rangle \notin R_B$: Since $\langle H \cap B, H' \cap B \rangle \notin R_B$ there are $b \in H \cap B$ and $b' \in H' \cap B$ such that $\langle b, b' \rangle \notin \mathcal{C}_B$. It follows from $\langle H, H' \rangle \in \mathrm{cl}(R_D^0)$ that $(h(b) \times h(b')) \cap R_D^0 \neq \emptyset$, i.e. there are $P, P' \in \mathrm{Ult}(D)$ such that $b \in P,\ b' \in P'$ and $\langle P, P' \rangle \in R_D^0$. Since $P \cap B$ and $P' \cap B$ are ultrafilters of B and $\langle b, b' \rangle \notin \mathcal{C}_B$, it follows that $P \cap B \neq P' \cap B$. By definition of R_D^0, we have $\langle P \cap B, P' \cap B \rangle \in R_B$. But then, by definition of R_B, it follows that $(P \cap B) \times (P' \cap B) \subseteq \mathcal{C}_B$, which in turn implies that $b \mathcal{C}_B b'$, a contradiction.
3. $H \cap B = H' \cap B$ and $\langle H \cap A, H' \cap A \rangle \notin R_A$: This is analogous to the previous case.

To prove that R_D^0 extends both R_A and R_B we use Theorem 4 and show that

$$R_A = \{\langle H \cap A, H' \cap A \rangle : H, H' \in \mathrm{Ult}(D) \text{ and } \langle H, H' \rangle \in R_D^0\}, \tag{4.2}$$

$$R_B = \{\langle H \cap B, H' \cap B \rangle : H, H' \in \mathrm{Ult}(D) \text{ and } \langle H, H' \rangle \in R_D^0\}. \tag{4.3}$$

Without loss of generality we prove only (4.2):

"$\subseteq$": Let $F, F' \in \mathrm{Ult}(A)$ with $\langle F, F' \rangle \in R_A$. Choose any $G \in \mathrm{Ult}(B)$. If $H \sim \langle F, G \rangle,\ H' \sim \langle F', G \rangle$, then $\langle H, H' \rangle \in R_D^0$ and $F = H \cap A,\ F' = H' \cap A$.

"$\supseteq$": Let $\langle H, H' \rangle \in R_D^0$, $H \sim \langle F, G \rangle,\ H' \sim \langle F', G' \rangle$, and w.l.o.g. $G = G'$. Then, $F = H \cap A,\ F' = H' \cap A$, and $\langle F, F' \rangle \in R_A$ by definition of R_D^0.

Up to now we have shown that $\mathbf{K}$ has the JEP. To prove that $\mathbf{K}_{\mathcal{C}_7}$ has the JEP, suppose that both $\mathcal{C}_A$ and $\mathcal{C}_B$ satisfy C_7, and let $x \in D,\ x \neq 0, 1$. Then, there are $a_0, \ldots, a_n \in A,\ a_i \neq 0$, and $b_0, \ldots, b_n \in B,\ b_j \neq 0$, such that $x = a_0 \cdot b_0 + \ldots + a_n \cdot b_n$ and

$$a_i \cdot b_i \not\leq a_0 \cdot b_0 + \ldots + a_{i-1} \cdot b_{i-1} + a_{i+1} \cdot b_{i+1} + \ldots + a_n \cdot b_n,$$

in particular, $a_i \cdot b_i \neq 0$ for all $i \leq n$. By distributivity,

$$-x = \sum \left\{ \prod \{-a_i : i \in I\} \cdot \prod \{-b_i : i \in \{0, \ldots, n\} \setminus I\} : I \subseteq \{0, \ldots, n\} \right\}.$$

Since $-x \neq 0$, there is some $I \subseteq \{0, \ldots, n\}$ such that

$$\prod\{-a_i : i \in I\} \cdot \prod\{-b_i : i \in \{0, \ldots, n\} \setminus I\} \neq 0.$$

We consider three cases:

1. Suppose that $I = \{0, \ldots, n\}$. Then, $\prod\{-a_i : i \leq n\} = -(a_0 + \ldots + a_n) \neq 0$, and thus, $a_0 + \ldots + a_n \neq 1$. Therefore, $(a_0 + \ldots + a_n)\mathcal{C}_A - (a_0 + \ldots + a_n)$, and so there are ultrafilters F, F' of A such that

   $$a_0 + \ldots + a_n \in F,\ -(a_0 + \ldots + a_n) \in F', \langle F, F' \rangle \in R_A.$$

   Since F is an ultrafilter, suppose w.l.o.g. that $a_0 \in F$, and choose some $G \in \mathrm{Ult}(B)$ with $b_0 \in G$; such G exists, since $b_0 \neq 0$. Setting $H \sim \langle F, G \rangle$ and $H' \sim \langle F', G \rangle$, we obtain $\langle H, H' \rangle \in R_D^0$, and $a_0 \cdot b_0 \in H$. Since $a_0 \cdot b_0 \leq x$ we obtain $x \in H$. Furthermore, $-(a_0 + \ldots + a_n) \in F'$ and $-(a_0 + \ldots + a_n) \leq -x$ imply that $-x \in H'$; hence, $x\mathcal{C}_D^0 - x$.

2. If $I = \emptyset$, then $b_0 + \ldots + b_n \neq 1$ and we proceed analogously.

3. Let $I \neq \emptyset, \{0, \ldots, n\}$, and choose $I = \{i_0, \ldots, i_k\}, J = \{i_{k+1}, \ldots, i_n\}$ such that $I \cup J = \{0, \ldots, n\}, I \cap J = \emptyset$, and I is maximal with the property

   $$-a_{i_0} \cdot \ldots \cdot -a_{i_k} \cdot -b_{i_{k+1}} \cdot \ldots \cdot -b_{i_n} \neq 0. \tag{4.4}$$

   Possibly permuting the indices, we may suppose that $I = \{0, \ldots, k\}$. By our previous considerations we have $0 \leq k < n$, and the maximality of k implies that for every j with $k < j \leq n$,

   $$-a_0 \cdot \ldots \cdot -a_k \cdot -a_j \cdot \prod\{-b_i : k < i \leq n, i \neq j\} = 0.$$

   The independence of A and B now implies that

   $$-a_0 \cdot \ldots \cdot -a_k \cdot -a_j = 0,$$

   see e.g. [9], Lemma 11.7. Hence, for every $k < j \leq n$ we have in particular

   $$-a_0 \cdot \ldots \cdot -a_k \cdot a_j \neq 0. \tag{4.5}$$

   It follows from (4.4) that $-b_{k+1} \cdot \ldots \cdot -b_n \neq 0$, and from $b_j \neq 0$ we obtain $-b_{k+1} \cdot \ldots \cdot -b_n \neq 1$.

   Since $\mathcal{C}_B$ satisfies C_7, $(-b_{k+1} \cdot \ldots \cdot -b_n)\mathcal{C}_B(b_{k+1} + \ldots + b_n)$, and so there are $G, G' \in \mathrm{Ult}(B)$ such that $-b_{k+1} \cdot \ldots \cdot -b_n \in G$, $b_{k+1} + \ldots + b_n \in G'$ and $\langle G, G' \rangle \in R_B$. Since G' is an ultrafilter, let w.l.o.g. $b_n \in G'$. Choose some ultrafilter F of A with $-a_0 \cdot \ldots \cdot -a_k \cdot a_n \in F$, and let $H \sim \langle F, G \rangle$ and $H' \sim \langle F, G' \rangle$. Then,

   $$\langle H, H' \rangle \in R_D^0, \quad -a_0 \cdot \ldots \cdot -a_k \cdot a_n \cdot -b_{k+1} \cdot \ldots \cdot -b_n \in H, \quad a_n \cdot b_n \in H'.$$

   It follows from $a_n \cdot b_n \leq x$ and $-a_0 \cdot \ldots \cdot -a_k \cdot a_n \cdot -b_{k+1} \cdot \ldots \cdot -b_n \leq -x$ that $x \in H$ and $-x \in H'$, hence, $x\mathcal{C}_D^0 - x$.

Concerning C_6, observe that the fact that R_A and R_B are transitive does not imply that R_D is transitive: If $\langle F_0, F_1\rangle \in R_A, F_0 \neq F_1$, $\langle G_1, G_2\rangle \in R_B, G_1 \neq G_2$, then $\langle\langle F_0, G_1\rangle, \langle F_1, G_1\rangle\rangle \in R_D$, $\langle\langle F_1, G_1\rangle, \langle F_1, G_2\rangle\rangle \in R_D$, on the other hand, $\langle\langle F_0, G_1\rangle, \langle F_1, G_2\rangle\rangle \notin R_D$.

Construction 2

The second construction is related to the *Kronecker product of graphs* [16]: Set

$$\langle H, H'\rangle \in R_D^1 \Longleftrightarrow \langle H \cap A, H' \cap A\rangle \in R_A \text{ and } \langle H \cap B, H' \cap B\rangle \in R_B. \tag{4.6}$$

Since both R_A and R_B are reflexive and symmetric, so is R_D^1. To show that R_D^1 is closed, let $\langle H, H'\rangle \in \mathrm{cl}(R_D^1)$. Assume that $\langle H, H'\rangle \notin R_D^1$; then, $\langle H \cap A, H' \cap A\rangle \notin R_A$ or $\langle H \cap B, H' \cap B\rangle \notin R_B$. Let w.l.o.g. $\langle H \cap A, H' \cap A\rangle \notin R_A$; then, there are $a \in H \cap A$ and $a' \in H' \cap A$ such that $a(-\mathcal{C}_A)a'$. Since $\langle H, H'\rangle \in h(a) \times h(a')$ and $\langle H, H'\rangle \in \mathrm{cl}(R_D^1)$, it follows that $(h(a) \times h(a')) \cap R_D^1 \neq \emptyset$, and therefore there are $P, P' \in \mathrm{Ult}(D)$ such that $a \in P$, $a' \in P'$, and $\langle P, P'\rangle \in R_D^1$. But then, $\langle P \cap A, P' \cap A\rangle \in R_A$, i.e. $(P \cap A) \times (P' \cap A) \subseteq \mathcal{C}_A$. Hence, $a\mathcal{C}_A a'$, contradicting our assumption.

It is straightforward to show as above that $\mathcal{C}_D^1$ extends both $\mathcal{C}_A$ and $\mathcal{C}_B$. If both R_A and R_B are transitive, then, clearly, so is R_D^1 and thus, $\mathcal{C}_D^1$ satisfies C_6 in this case by Theorem 3. Finally, $\mathcal{C}_D^0 \subseteq \mathcal{C}_D^1$ shows that $\mathcal{C}_D^1$ satisfies C_7, if $\mathcal{C}_A$ and $\mathcal{C}_B$ satisfy C_7. □

Theorem 7. $\mathbf{K}$ *and* $\mathbf{K}_{\mathcal{C}_6}$ *have the disjoint AP.*

Proof. Suppose that $\langle A, \mathcal{C}_A\rangle$, $\langle B, \mathcal{C}_B\rangle$, $\langle C, \mathcal{C}_C\rangle \in \mathbf{K}$ with $\langle C, \mathcal{C}_C\rangle$ a substructure of both $\langle A, \mathcal{C}_A\rangle$ and $\langle B, \mathcal{C}_B\rangle$, and let D be the Boolean amalgamated free product of A and B over C; then, $A \cap B = C$ [9].

We first note that the first construction of Theorem 6 need not work: Suppose that $\langle F, F'\rangle \in R_A$ such that $F \cap C \neq F' \cap C$; then, there is no $G \in \mathrm{Ult}(B)$ for which $F \cap C = G \cap C$ and $F' \cap C = G \cap C$. Thus, we use the second construction and define for $H, H' \in \mathrm{Ult}(D)$

$$\langle H, H'\rangle \in R \Longleftrightarrow \langle H \cap A, H' \cap A\rangle \in R_A \text{ and } \langle H \cap B, H' \cap B\rangle \in R_B. \tag{4.7}$$

As in proof of Theorem 6 one shows that R is reflexive, symmetric, and closed. To show that R extends both R_A and R_B we use Theorem 4, in other words, we show that

$$R_A = \{\langle H \cap A, H' \cap A\rangle : H, H' \in \mathrm{Ult}(D) \text{ and } \langle H, H'\rangle \in R\}, \tag{4.8}$$

$$R_B = \{\langle H \cap B, H' \cap B\rangle : H, H' \in \mathrm{Ult}(D) \text{ and } \langle H, H'\rangle \in R\}. \tag{4.9}$$

Without loss of generality we show only (4.8):

"$\subseteq$": Let $F, F' \in \mathrm{Ult}(A)$ with $\langle F, F'\rangle \in R_A$. We need to find $H, H' \in \mathrm{Ult}(D)$ such that $H \cap A = F$, $H' \cap A = F'$, and $\langle H, H'\rangle \in R$, i.e. additionally $\langle H \cap B, H' \cap$

$B\rangle \in R_B$. Let $P := F \cap C$ and $P' := F' \cap C$. Since $\langle C, \mathcal{C}_C\rangle$ is a substructure of $\langle A, \mathcal{C}_A\rangle$, we have $\langle P, P'\rangle \in \mathcal{C}_C$ by Theorem 4. Since $\langle C, \mathcal{C}_C\rangle$ is a substructure of $\langle B, \mathcal{C}_B\rangle$ as well, there are $G, G' \in \mathrm{Ult}(B)$ such that $P = G \cap C, P' = G' \cap C$, and $\langle G, G'\rangle \in R_B$.

Suppose that $H := [F \cup G]^D_{\mathrm{filt}}$ and $H' := [F' \cup G']^D_{\mathrm{filt}}$. Since $F \cap C = G \cap C$ and $F' \cap C = G' \cap C$, both H and H' are ultrafilters of D by Lemma 2, and the choice of G and G' shows that H and H' fulfill the desired condition.

"$\supseteq$": Let $\langle H, H'\rangle \in R$. By definition of R, $\langle H \cap A, H' \cap A\rangle \in R_A$.

Clearly, if both R_A and R_B are transitive, then so is R, and thus, if $\mathcal{C}_A$ and $\mathcal{C}_B$ satisfy C_6 so does $\mathcal{C}_R$ by Theorem 3. □

Theorem 8. $\mathbf{K}_{\mathcal{C}_7}$ *does not have AP.*

Proof. Our strategy is as follows: We will define three connected graphs, $\mathfrak{U} = \langle U, E\rangle, \mathfrak{U}' = \langle U', E'\rangle$, $\mathfrak{V} = \langle V, R\rangle$, and strong homomorphisms $f : \mathfrak{U} \twoheadrightarrow \mathfrak{V}$, $g : \mathfrak{U}' \twoheadrightarrow \mathfrak{V}$. Then we shall show that there is no connected graph $\mathfrak{W} = \langle W, L\rangle$ for which there exist strong homomorphisms $f' : \mathfrak{W} \twoheadrightarrow \mathfrak{U}$, $g' : \mathfrak{W} \twoheadrightarrow \mathfrak{U}'$ which satisfy $f \circ f' = g \circ g'$. The result then follows from the duality theorem 5 and Theorem 3(2).

Consider the graph $\langle U, E\rangle$ where $U = \{0, 1, 2, 3\}$ and E is the smallest graph on U containing the chain $0E1E2E3$; observe that E is connected. Suppose that $\langle U', E'\rangle$ is a copy of $\langle U, E\rangle$, where $U' = \{0', 1', 2', 3'\}$, and $xE'y$ if and only if xEy. Let $V = \{a, b, c\}$, and R be smallest graph on V containing the chain $aRbRc$. Let $f : U \twoheadrightarrow V$ and $g : U' \twoheadrightarrow V$ be defined by

$$\begin{array}{lll} f(0) = f(2) = b, & f(1) = c, & f(3) = a, \\ g(0') = g(2') = b, & g(1') = a, & g(3') = c. \end{array}$$

1. *f is a graph homomorphism*:

$$\begin{aligned} f(\langle 0, 1\rangle) &= \langle b, c\rangle \in R, \\ f(\langle 1, 2\rangle) &= \langle c, b\rangle \in R, \\ f(\langle 2, 3\rangle) &= \langle b, a\rangle \in R. \end{aligned}$$

   The rest follows from symmetry of E and R.
2. *f is a strong homomorphism, i.e. each edge of R is the image of an edge of E*:

$$\begin{aligned} \langle a, b\rangle &= \langle f(3), f(2)\rangle \text{ and } \langle 3, 2\rangle \in E, \\ \langle b, c\rangle &= \langle f(0), f(1)\rangle \text{ and } \langle 0, 1\rangle \in E. \end{aligned}$$

   The rest follows from symmetry of E and R.

3. *g is a graph homomorphism*:

$$g(\langle 0', 1'\rangle) = \langle b, a\rangle \in R,$$
$$f(\langle 1', 2'\rangle) = \langle a, b\rangle \in R,$$
$$f(\langle 2', 3'\rangle) = \langle b, c\rangle \in R.$$

   The rest follows from symmetry of E' and R.
4. *g is a strong homomorphism, i.e. each edge of R is the image of an edge of E*:

$$\langle a, b\rangle = \langle g(1'), g(2')\rangle \text{ and } \langle 1', 2'\rangle \in E',$$
$$\langle b, c\rangle = \langle g(2'), g(3')\rangle \text{ and } \langle 2', 3'\rangle \in E'.$$

   The rest follows from symmetry of E and R.

Thus, the duals of the graphs and mappings satisfy the conditions of Theorem 5, i.e. $\Delta(\mathfrak{V})$ is isomorphic to a substructure of $\Delta(\mathfrak{U})$ and $\Delta(\mathfrak{U}')$.

Suppose that there are a graph $\langle W, L\rangle$ and strong homomorphisms $f' : \langle W, L\rangle \twoheadrightarrow \langle U, E\rangle$, $g' : \langle W, L\rangle \twoheadrightarrow \langle U', E'\rangle$ such that $f \circ f' = g \circ g'$. We shall prove that $\langle W, L\rangle$ is not connected. Our first step is to prove that $f'^{-1}(0) \cap g'^{-1}(0') \neq \emptyset$ or $f'^{-1}(2) \cap g'^{-1}(2') \neq \emptyset$: Assume otherwise. If $f'(x) = 0$, then $f(f'(x)) = b$, and $g^{-1}(b) = \{0', 2'\}$ together with the hypothesis imply that $g'(x) = 2'$. Similarly, if $g'(x) = 2'$, then $g(g'(x)) = b$ and $f^{-1}(b) = \{0, 2\}$ imply that $f'(x) = 0$. Therefore, we have $f'^{-1}(0) = g'^{-1}(2')$, and, similarly

$$f'^{-1}(2) = g'^{-1}(0'),\ f'^{-1}(1) = g'^{-1}(3'),\ f'^{-1}(3) = g'^{-1}(1'), \tag{4.10}$$

the last three cases being analogous. Let $x, y \in W$ such that $f'(x) = 1$, $f'(y) = 2$, and xLy. These exist, since $1E2$, and f' is a strong homomorphism. Then, $g'(x) = 3'$, $g'(y) = 0'$ by (4.10). Since xLy and g' preserves L, we must have $3'E'0'$, contradicting the definition of E'.

Thus, we have $f'^{-1}(0) \cap g'^{-1}(0') \neq \emptyset$ or $f'^{-1}(2) \cap g'^{-1}(2') \neq \emptyset$. As the cases are analogous, we only consider the former. Our aim is to show that if $x \in f'^{-1}(0) \cap g'^{-1}(0')$ and $y \notin f'^{-1}(0) \cap g'^{-1}(0')$, then $x(-L)y$. Thus, let $f'(x) = 0,\ g'(x) = 0'$, and $f'(y) \neq 0$ or $g'(y) \neq 0'$, and assume that xLy. There are three cases:

Case 1: $f'(y) = 0,\ g'(y) \neq 0'$: By the hypothesis, $f(f'(y)) = f(0) = b = g(g'(y))$. Since $g^{-1}(b) = \{0', 2'\}$, and $g'(y) \neq 0'$, we must have $g'(y) = 2'$. Thus, if xLy, then $0'E'2'$ which is not the case. □

Case 2: $f'(y) \neq 0,\ g'(y) = 0'$: By the hypothesis, $g(g'(y)) = g(0') = b = f(f'(y))$. Since $f^{-1}(b) = \{0, 2\}$, and $f'(y) \neq 0$, we must have $f'(y) = 2$. Thus, if xLy, then $0E2$ which is not the case. □

Case 3: $f'(y) \neq 0,\ g'(y) \neq 0'$: If xLy, then $\langle f'(x), f'(y)\rangle \in E$ and $E(0) = \{1\}$ imply that $f'(y) = 1$; similarly, $g'(y) = 1'$. Thus, $f(f'(y)) = f(1) = c$ and $g(g'(y)) = a$, a contradiction. □

Hence, $f'^{-1}(0) \cap g'^{-1}(0')$ and its complement in W are disconnected. The other case is analogous, and it follows now from duality that $\Delta(\mathfrak{U})$ and $\Delta(\mathfrak{U}')$ cannot be amalgamated over $\Delta(\mathfrak{V})$. □

5 Outlook

If **K** is the class of all finite BCAs, then since **K** has HP, JEP, and AP, there is a countable universal homogeneous BCA by the Fraïssé construction. In future work we will investigate this algebra and the relation algebra generated by its contact relation.

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