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Abstract. Contact relations on an algebra have been studied since the early part of the previous century, and have recently become a powerful tool in several areas of artificial intelligence, in particular, qualitative spatial reasoning and ontology building. In this paper we investigate the structure of the set of all contact relations on a Boolean algebra.

1 Introduction

Contact relations arise historically in two different contexts: Proximity relations were introduced by Efremovič to express the fact that two objects are – in some sense – close to each other [1]. The other source of contact relations is pointless geometry (or topology), which goes back to the works of [2], [3], [4], [5] and others. The main difference to traditional geometry is the way in which the building blocks are defined: Instead of taking points as the basic entity and defining other geometrical objects from these, the pointless approach starts from certain collections of points, for example, plane regions or solids, and defines points from these. One reason behind this approach is the fact that points are (unobservable) abstract objects, while regions or solids occur naturally in physical reality, as we sometimes painfully observe.

A standard example of a contact relation is the following: Consider the set of all closed disks in the plane, and say that two such disks are in contact if they have a nonempty intersection. More generally we say that two regular closed sets are in contact if they have a nonempty intersection. This relation is, indeed, considered to be the *standard contact* between regular closed sets of a topological space. Motivated by certain problems arising in qualitative spatial reasoning, Boolean algebras equipped with a contact relation have been intensively studied in the artificial intelligence community, and we invite the reader to consult [6] or [7] for some background reading.

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2 Notation and basic definitions

We assume that the reader has a working knowledge of lattice theory, Boolean algebras, and topology. Our standard references for these are, respectively, [8], [9], and [10].

For any set U, we denote by $\mathrm{Rel}(U)$ the set of all binary relations on U, and by $1'$ the identity relation on U. If $x \in U$, then $\mathrm{dom}_R(x) = \{y : yRx\}$, and, if $M \subseteq U$, we let $\mathrm{dom}_R(M) = \bigcup_{x \in M} \mathrm{dom}_R(x)$. Similarly, we define $\mathrm{ran}_R(x)$ and $\mathrm{ran}_R(M)$. If R is understood, we will usually drop the subscript; furthermore, we will usually write $R(x)$ for $\mathrm{ran}_R x$.

Two distinct elements $x, z \in U$ are called *R–connected*, if there are $y_0, \ldots, y_k \in U$ such that $x = y_0, z = y_k$, and $y_0 R y_1 R \ldots R y_k$. If x and z are R–connected, we write $x \xrightarrow{R} z$. A subset W of U is called *R–connected*, if any two different elements of W are connected. A maximally R–connected subset of U is called a *component of R*. A *clique of R* is a nonempty subset M of U with $M \times M \subseteq R$.

Throughout, $\langle B, +, \cdot, ^*, 0, 1\rangle$ will denote a Boolean algebra (BA), and $\mathbf{2}$ is the two element BA. If A is a subalgebra of B, we will write $A \leq B$. For $M \subseteq B$, $[M]$ is the subalgebra of B generated by M, and $M^+ = M \setminus \{0\}$, $M^- = M \setminus \{1\}$. If I, J are ideals of B, then $I \vee J$ denotes the ideal generated by $I \cup J$, i.e. $I \vee J = \{a : (\exists b, c)[a \in I, b \in J \text{ and } a = b + c]\}$.

$At(B)$ is the set of atoms of B, and $\mathrm{Ult}(B)$ its set of ultrafilters. We assume that $\mathrm{Ult}(B)$ is equipped with the Stone topology $\tau_{\mathrm{Ult}(B)}$ via the mapping $h : B \to 2^{\mathrm{Ult}(B)}$ with $h(x) = \{U \in \mathrm{Ult}(B) : x \in U\}$; the product topology on $\mathrm{Ult}(B)^2$ is denoted by $\tau_{\mathrm{Ult}(B)^2}$. Note that $\tau_{\mathrm{Ult}(B)^2}$ is the Stone space of the free product $B_0 \oplus B_1$, where $B_0, B_1 \cong B$, see e.g. Section *11.1.* of [9].

Recall the following result for topological spaces X_0, X_1,

Lemma 1. *[10, Proposition 2.3.1.] If S_i is a basis for X_i, $i \leq 1$, then $\{\langle W_0, W_1\rangle : W_0 \in S_0,\ W_1 \in S_1\}$ is a basis for the product topology on $X_0 \times X_1$.*

In particular, the sets of the form $h(a) \times h(b)$ where $a, b \in B$ are a basis for the product topology on $\mathrm{Ult}(B)^2$. Furthermore, note that for $M \subseteq \mathrm{Ult}(B)$, $F \in \mathrm{cl}(M)$ if and only if $F \subseteq \bigcup M$.

We denote by $\mathrm{Rel}^{rs}(\mathrm{Ult}(B))$ the collection of all reflexive and symmetric relations on $\mathrm{Ult}(B)$, and by $\mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$ the collection of all reflexive and symmetric relations on $\mathrm{Ult}(B)$ that are closed in $\tau_{\mathrm{Ult}(B)^2}$. Note that $1' \in \mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$, and that $\mathrm{int}(1') \neq \emptyset$ if and only if B has an atom.

B is called a *finite–cofinite* algebra (FC–algebra), if every element $\neq 0, 1$ is a finite sum of atoms or the complement of such an element. If B is an FC–algebra, and $|B| = \kappa$, then B is isomorphic to the BA $FC(\kappa)$ which is generated by the finite subsets of κ. If $\gamma \in \kappa$, we let F_γ be the ultrafilter of $FC(\kappa)$ generated by $\{\gamma\}$, and F_κ be the ultrafilter of cofinite sets. If $M \subseteq \mathrm{Ult}(B)$, $x \in B$, we say that M *admits* x, if $x \in \bigcap M$, i.e. if $M \subseteq h(x)$.

3 Boolean contact algebras

Suppose that $C \in \mathrm{Rel}(B)$, and consider the following properties: For all $x, y, z \in B$

C_0. $0(-C)x$
C_1. $x \neq 0 \Rightarrow xCx$
C_2. $xCy \Rightarrow yCx$
C_3. xCy and $y \leq z \Rightarrow xCz$. (The compatibility axiom)
C_4. $xC(y+z) \Rightarrow (xCy$ or $xCz)$ (The sum axiom)
C_5. $C(x) = C(y) \Rightarrow x = y$. (The extensionality axiom)
C_6. $(xCz$ or $yCz^*) \Rightarrow xCy$ (The interpolation axiom)
C_7. $(x \neq 0 \wedge x \neq 1) \Rightarrow xCx^*$ (The connection axiom)

C is called a *contact relation* (CR), and the structure $\langle B, C\rangle$ is called a *Boolean contact algebra* (BCA), if C satisfies $C_0 - C_4$. C is called an *extensional contact relation* (ECR) if it satisfies $C_0 - C_5$. If C satisfies C_7, we call it *connected*. The collection of contact relations on B will be denoted by $\mathscr{C}_B$.

As mentioned in the introduction, a standard example for a BCA, indeed, the original motivation for studying contact relations, is the collection of regular closed sets of the Euclidean plane with *standard contact* defined by $aCb \iff a \cap b \neq \emptyset$; an in–depth investigation of BCAs in relation to topological properties can be found in [11].

Another important example of a contact relation on B is the *overlap relation* O on B defined by $xOy \iff x \cdot y \neq 0$.

Lemma 2. *C is an extensional contact relation if and only if for all $x, y \notin \mathbf{2}$ with $x \cdot y = 0$, there is some $z \in B^+$ such that $z \leq y$ and $x(-C)z$.*

Proof. "$\Rightarrow$": We have shown in [12] that for an extensional contact relation and all $z \neq 0$, $z = \sum\{t : t(-C)z^*\}$. Suppose that $x, y \notin \mathbf{2}$, and that $x \cdot y = 0$. Assume that xCz for all $0 \lneq z \leq y$; then $x(-C)z$ implies that $z \cdot y = 0$, i.e. $z \leq y^*$. Since $x^* = \sum\{t : t(-C)x\}$, it follows that $x^* \leq y^*$, i.e. $y \leq x$. This contradicts the hypothesis that $y \neq 0$ and $x \cdot y = 0$.

"$\Leftarrow$": This is obvious. □

The following concepts have their origin in proximity theory [1], which has a close connection to the theory of contact relations, see e.g. [13]. A *clan* is a subset Γ of B which satisfies

Γ_1. If $x, y \in \Gamma$ then xCy.
Γ_2. If $x + y \in \Gamma$ then $x \in \Gamma$ or $y \in \Gamma$.
Γ_3. If $x \in \Gamma$ and $x \leq y$, then $y \in \Gamma$.

In the sequel, we will use upper case Greek letters Γ, Δ etc to denote clans. When C is understood, the set of clans of $\langle B, C\rangle$ will be denoted by $\mathrm{Clan}(B)$; clearly, each clan is contained in a maximal clan, and we will denote the set of maximal clans by $\mathrm{MaxClan}(B)$. A *cluster* is a clan Γ for which $\{x\} \times \Gamma \subseteq C$ implies $x \in \Gamma$ for all $x \in B$.

For later use we note the following:

Lemma 3. *[12] Suppose that C is a contact relation on B. Then,*

1. *aCb if and only if there is a clan containing a and b if and only if there are ultrafilters F, G of B such that $a \in F$, $b \in G$ and $F \times G \subseteq C$.*
2. *If $\Gamma \in Clan(B)$, then $B \setminus \Gamma$ is an ideal of B.*

4 Contact relations and ultrafilters

The connection between (ultra-) filters on B and contact relations was established in [14], and, more generally, in [11]. Our aim in this Section is to establish the following representation theorem[1] :

Theorem 1. *Suppose that B is a Boolean algebra. Then, there is a bijective order preserving correspondence between the contact relations on B and the reflexive and symmetric relations on* $\mathrm{Ult}(B)$ *that are closed in the product topology of* $\mathrm{Ult}(B)^2$.

Proof. Let $q : \mathrm{Rel}^{rsc}(\mathrm{Ult}(B)) \to \mathrm{Rel}(B)$ be defined by $q(R) := \bigcup\{F \times G : \langle F, G\rangle \in R\}$; then, clearly, q preserves $\subseteq$. We first show that $q(R) \in \mathscr{C}_B$; this was shown mutatis mutandi in [14] for proximity structures, and for completeness, we repeat the proof. Since no ultrafilter of B contains 0, $q(R)$ satisfies C_0. The reflexivity of R implies C_1, and the symmetry of R implies C_2. Since ultrafilters are closed under $\leq$, $q(R)$ satisfies C_3. For C_4, let $a\ q(R)\ (b+c)$; then, there are $F, G \in \mathrm{Ult}(B)$ such that $a \in F,\ b+c \in G$, and $\langle F, G\rangle \in R$. Since G is an ultrafilter, $b \in G$ or $c \in G$, and it follows that aCb or aCc.

To show that q is injective, suppose that $R, R' \in \mathrm{Rel}^{rsc}(\mathrm{Ult}(B)), q(R) = q(R')$, and assume that $\langle F, G\rangle \in R' \setminus R$. Since R is closed, there are $a, b \in B$ such that $a \in F, b \in G$, and $(h(a) \times h(b)) \cap R = \emptyset$. Now, since $q(R) = q(R')$ it follows that $F \times G \subseteq \bigcup\{F' \times G' : \langle F', G'\rangle \in R\}$, and thus, there are $F', G' \in \mathrm{Ult}(B)$ such that $a \in F',\ b \in G'$, and $\langle F', G'\rangle \in R$. This contradicts $(h(x) \times h(y)) \cap R = \emptyset$.

For surjectivity, let $C \in \mathscr{C}_B$, and set $p(C) := \{\langle F, G\rangle : F \times G \subseteq C\}$. We first show that $p(C) \in \mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$: It is straightforward to show that symmetry of C implies symmetry of $p(C)$, and C_1 implies that $p(C)$ is symmetric [14].

Next, suppose that $\langle F, G\rangle \in \mathrm{cl}(p(C))$, and assume that $\langle F, G\rangle \notin p(C)$. Then, $F \times G \not\subseteq C$, and thus, there are $a \in F,\ b \in G$ such that $a(-C)b$. Now, $h(a) \times h(b)$ is an open neighbourhood of $\langle F, G\rangle$, and $\langle F, G\rangle \in \mathrm{cl}(p(C))$ implies that there is some $\langle F', G\rangle \in p(C)$ such that $\langle F', G'\rangle \in h(a) \times h(b)$. But then, $F' \times G' \subseteq C$ and $\langle a, b\rangle \in F' \times G'$ implies aCb, a contradiction.

All that remains to show is $C = q(p(C))$: By Lemma 3 and the definitions of the mappings,

$$\begin{aligned} aCb &\Longleftrightarrow (\exists F, G)[a \in F, b \in G \text{ and } F \times G \subseteq C] \\ &\Longleftrightarrow (\exists F, G)[a \in F, b \in G \text{ and } \langle F, G\rangle \in p(C)] \Longleftrightarrow \langle a, b\rangle \in q(p(C)). \end{aligned}$$

This completes the proof. □

[1] One of the referees has kindly pointed out that a more general result has independently been shown in [15].

Finally, we turn to the connection between clans and closed sets of ultrafilters; if $M \subseteq \text{Ult}(B)$, we let $\Gamma_M = \bigcup M$; conversely, if $\Gamma \in \text{Clan}(B)$, we set $\text{uf}(\Gamma) = \{F \in \text{Ult}(B) : F \subseteq \Gamma\}$. We will also write R_C instead of $q^{-1}(C)$.

Theorem 2. *1. $\bigcup \text{uf}(\Gamma) = \Gamma$ for each clan Γ.*
2. If $\Gamma \in Clan(B)$, then $\text{uf}(\Gamma)$ is a closed clique in R_C.
3. If M is a clique in R_C, then Γ_M is a clan, and $\text{uf}(\Gamma) = \text{cl}(M)$.
4. A maximal clique M of R_C is closed.

Proof. 1. Suppose that $\Gamma \in \text{Clan}(B)$. Then,

$$\begin{aligned} x \in \bigcup \text{uf}(\Gamma) &\iff (\exists F \in \text{Ult}(B))[F \in \text{uf}(\Gamma) \text{ and } x \in F], \\ &\iff (\exists F \in \text{Ult}(B))[F \subseteq \Gamma \text{ and } x \in F], \\ &\iff x \in \Gamma, \end{aligned}$$

since Γ is a union of ultrafilters.

2. It was shown in [11] that $\Gamma \in \text{Clan}(B)$ is a clique; for completeness, we give a proof:

$$\begin{aligned} \Gamma \in \text{Clan}(B) &\Rightarrow (\forall F, G \in \text{Ult}(B))[F, G \subseteq \Gamma \Rightarrow F \times G \subseteq C], \\ &\Rightarrow (\forall F, G \in \text{Ult}(B))[F, G \in \text{uf}(\Gamma) \Rightarrow F \times G \subseteq C], \\ &\Rightarrow (\forall F, G \in \text{Ult}(B))[F, G \in \text{uf}(\Gamma) \Rightarrow \langle F, G \rangle \in R_C]. \end{aligned}$$

All that remains to be shown is that $\text{uf}(\Gamma)$ is closed:

$$F \in \text{cl}(\text{uf}(\Gamma)) \iff F \subseteq \bigcup U_\Gamma \iff F \subseteq \Gamma \iff F \in \text{uf}(\Gamma).$$

3. Since Γ_M is a union of ultrafilters, it clearly satisfies Γ_2 and Γ_3. For Γ_1, consider

$$\begin{aligned} x, y \in \Gamma_M &\Rightarrow (\exists F, G \in \text{Ult}(B))[F, G \in M \text{ and } x \in F, y \in G], \\ &\Rightarrow (\exists F, G \in \text{Ult}(B))[\langle F, G \rangle \in R_C \text{ and } x \in F, y \in G], \\ &\Rightarrow xCy. \end{aligned}$$

For the rest, note that

$$F \in \text{uf}(\Gamma) \iff F \subseteq \Gamma_M \iff F \subseteq \bigcup M \iff F \in \text{cl}(M).$$

4. Let M be a maximal clique of R_C; then $\Gamma_M \in \text{Clan}(B)$. By 2. above, $\text{uf}(\Gamma)$ is a closed clique that contains M. Maximality of M now implies that $M = \text{uf}(\Gamma)$, and thus, M is closed. □

5 The lattice of contact relations

In this section we will show that $\mathscr{C}_B$ is a lattice under the inclusion ordering. We will do this in two steps: First, we show that $\text{Rel}^{rsc}(\text{Ult}(B))$ is a lattice and then, with the help of Theorem 1, we show how to carry it over to $\mathscr{C}_B$.

It is well known that the collection $\mathscr{T}$ of closed sets of a T_1 space X is a complete and atomic dual Heyting algebra under the operations

$$\text{(5.1)} \qquad \bigvee A = \mathrm{cl}(\bigcup A), \quad \bigwedge A = \bigcap A, \quad a \xrightarrow{\mathrm{d}} b = \mathrm{cl}(b \cap -a), \quad 0 = \emptyset, \quad 1 = X,$$

where $A \subseteq \mathscr{T}$, and $a, b \in \mathscr{T}$. Since X is a T_1 space, the atoms of $\mathscr{T}$ are the singletons.

Theorem 3. *The collection* $\mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$ *of closed reflexive and symmetric relations on* $\mathrm{Ult}(B)$ *is a complete and atomic sublattice of the lattice of closed sets of* $\mathrm{Ult}(B)^2$ *with smallest element* $1'$*, largest element is* $\mathrm{Ult}(B)^2$*, and a dual Heyting algebra where* $R \overset{d}{\Rightarrow} S := \mathrm{cl}(R \setminus S) \cup 1'$*.Its atoms have the form* $1' \cup \{\langle F, G\rangle, \langle G, F\rangle\}$*, where F and G are distinct ultrafilters of B.*

Proof. Since $1'$ is the smallest reflexive and symmetric relation on $\mathrm{Ult}(B)$, and closed since $\tau_{\mathrm{Ult}(B)}$ is compact and Hausdorff, it is the smallest element of $\mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$, and, clearly, $\mathrm{Ult}(B)^2$ is the largest element of $\mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$. Since $\tau_{\mathrm{Ult}(B)^2}$ is a T_1 space, singletons are closed, and therefore, atoms have the form $1' \cup \{\langle F, G\rangle, \langle G, F\rangle\}$ for $F, G \in \mathrm{Ult}(B)$, $F \neq G$.

By the remarks preceding the Theorem, all that is left to show is that the operations $\bigvee$ and $\bigwedge$ do not destroy reflexivity or symmetry, and that $R \overset{d}{\Rightarrow} S \in \mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$. Let $\mathscr{R} = \{R_i : i \in I\} \subseteq \mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$. Since the intersection of reflexive symmetric relations is a reflexive and symmetric relation, and the intersection of closed sets is closed, we have $\bigwedge \mathscr{R} = \bigcap \mathscr{R} \in \mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$.

Set $R = \bigcup \mathscr{R}$, and observe that R is reflexive and symmetric. Let $\langle F, G\rangle \in \mathrm{cl}(R)$, and $h(x) \times h(y)$ be a basic neighbourhood of $\langle F, G\rangle$; then $(h(x) \times h(y)) \cap R \neq \emptyset$. Since R is symmetric, $(h(y) \times h(x)) \cap R \neq \emptyset$, and, since every basic neighbourhood of $\langle G, F\rangle$ is of the form $h(y) \times h(x)$ for an open neighbourhood $h(x) \times h(y)$ of $\langle F, G\rangle$, we conclude that $\langle G, F\rangle \in \mathrm{cl}(R)$. It follows that $\bigvee \mathscr{R} \in \mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$.

Finally, let $R, S \in \mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$, and $\langle F, G\rangle \in \mathrm{cl}(R \setminus S)$. Then, $R \setminus S$ is a symmetric relation, and we have shown in the preceding paragraph that the closure of a symmetric relation is symmetric. Now, by (5.1), $\mathrm{cl}(R \setminus S)$ is the smallest closed set T of $\tau_{\mathrm{Ult}(B)^2}$ with $R \subseteq S \cup T$, and, since $1'$ is closed, $R \overset{d}{\Rightarrow} S$ is the smallest element T of $\mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$ with $R \subseteq S \cup T$. ◻

Corollary 1. *$\mathscr{C}_B$ is a complete and atomic dual Heyting algebra with smallest element O, largest element $B^+ \times B^+$ and the operations*

$$\sum\{C_i : i \in I\} = q\left(\bigvee_i q^{-1}(C_i)\right),$$

$$\prod\{C_i : i \in I\} = q\left(\bigwedge_i q^{-1}(C_i)\right),$$

$$C \xrightarrow{\mathrm{d}} C' = q(q^{-1}(C) \overset{d}{\Rightarrow} q^{-1}(C')).$$

Furthermore, if $\{C_\alpha : \alpha \in I\}$ *is a descending chain of contact relations, then* $\bigwedge_{\alpha\in I} C_\alpha = \bigcap_{\alpha\in I} C_\alpha$.

Proof. First, recall that $aOb \iff a \cdot b \neq 0$; then, $O = \bigcup\{F \times F : F \in \mathrm{Ult}(B)\}$, and it follows that $q(1') = O$. Clearly, $q(\mathrm{Ult}(B) \times \mathrm{Ult}(B)) = B^+ \times B^+$, and the atoms of $\mathscr{C}_B$ are the relations of the form $O \cup (F \times G) \cup (G \times F) = q(1' \cup \{\langle F,G\rangle, \langle G,F\rangle\})$, where $F, G \in \mathrm{Ult}(B)$ and $F \neq G$.

Since $q : \mathrm{Rel}^{rsc}(\mathrm{Ult}(B)) \to \mathscr{C}_B$ is bijective and order preserving by Theorem 1 and $\mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$ is a complete and atomic dual Heyting algebra, so is $\mathscr{C}_B$ with the indicated operations.

In proving the final claim, the only not completely trivial case is C_4: Let $a\left(\bigcap_{\alpha\in I} C_\alpha\right)(s+t)$, and assume that $a\left(-\bigcap_{\alpha\in I} C_\alpha\right)s$ and $a\left(-\bigcap_{\alpha\in I} C_\alpha\right)t$. Then. there are $\alpha, \beta \in I$ such that $\alpha \leq \beta$ and $a(-C_\alpha)s,\ a(-C_\beta)t$. From $C_\beta \subseteq C_\alpha$ we obtain $a(-C_\beta)s$ and $a(-C_\beta)t$, contradicting $aC_\beta(s+t)$. ◻

The explicit definition of the operations in $\mathscr{C}_B$ is somewhat involved, except for the supremum: Suppose that $\mathscr{R} = \{R_i : i \in I\} \subseteq \mathrm{Rel}^{rsc}(\mathrm{Ult}(B))$; then,

$$
\begin{aligned}
\langle a,b\rangle \in q\left(\bigvee \mathscr{R}\right) &\iff \langle a,b\rangle \in q(\mathrm{cl}(\bigcup_{i\in I} R_i)),\\
&\iff (\exists \langle F,G\rangle \in \mathrm{cl}(\bigcup_{i\in I} R_i))[\langle a,b\rangle \in F \times G],\\
&\iff (\exists \langle F_0,G_0\rangle \in \bigcup_{i\in I} R_i)[\langle a,b\rangle \in F_0 \times G_0],\ \text{since } \mathrm{cl}(\bigcup \mathscr{R}) \text{ is closed},\\
&\iff (\exists i \in I)[\langle a,b\rangle \in F_0 \times G_0 \text{ and } \langle F_0,G_0\rangle \in R_i]\\
&\iff (\exists i \in I)[\langle a,b\rangle \in q(R_i)],\\
&\iff \langle a,b\rangle \in \bigcup_{i\in I} q(R_i),
\end{aligned}
$$

so that supremum in $\mathscr{C}_B$ is just the union. Regarding the meet, it can be shown that

$$\prod\{C_i : i \in I\} = \{\langle a,b\rangle \in \bigcap\{C_i : i \in I\} : (\forall s,t)[b = s+t \Rightarrow x\left(\bigcap_{i\in I} C_i\right) s \text{ or } a\left(\bigcap_{i\in I} C_i\right) t]\};$$

we omit the somewhat tedious calculations. Note that the meet operation in $\mathscr{C}_B$ is usually not set intersection. For a simple example, let B be the BA with atoms a,b,c,d, and let $C_0 = O \cup (F_a \times F_b) \cup (F_b \times F_a)$, and $C_1 = O \cup (F_c \times F_d) \cup (F_c \times F_d)$. Then, $(a+c)(C_0 \cap C_1)(b+d)$, but $C_0 \cap C_1$ does not satisfy C_4.

Since the Stone topology of a finite BA is discrete, we note

Corollary 2. *If B is finite, then $\mathscr{C}$ is isomorphic to* $\mathrm{Rel}^{rs}(\mathrm{Ult}(B))$.

Since the ultrafilters of a finite BA are determined by $At(B)$, the contact relations on B are uniquely determined by the reflexive and symmetric relations on $At(B)$. Thus, the adjacency relations of [16] determine the contact relations on finite BAs and vice versa.

In the sequel we shall usually write R_C (or just R, if C is understood) instead of $p(C)$ to indicate that $p(C) \in \mathrm{Rel}(\mathrm{Ult}(B))$. Furthermore, we let $\hat{R} = R \setminus 1'$.

Now that we have established the overall algebraic structure of $\mathscr{C}$, we consider collections of contact relations on B that satisfy additional axioms; for $5 \leq i \leq 7$, set $\mathscr{C}_i = \{C \in \mathscr{C} : C \models C_i\}$.

If $B \neq \mathbf{2}$, then for the bounds of $\mathscr{C}$ we observe

$$O \in \mathscr{C}_5 \cap \mathscr{C}_6, \quad O \notin \mathscr{C}_7, \quad B^+ \times B^+ \in \mathscr{C}_7 \cap \mathscr{C}_6, \quad B^+ \times B^+ \notin \mathscr{C}_5.$$

Theorem 1 implies that $\mathscr{C}_6$ has the following interesting characterization:

Theorem 4. *$\mathscr{C}_6$ is isomorphic to the lattice of closed equivalence relations on* $\mathrm{Ult}(B)$.

Proof. We first show that $C \models C_6$ if and only if R_C is transitive. The "only if" part was shown in [14], so suppose that $C \models C_6$. Let $\langle F, G\rangle, \langle G, H\rangle \in R_C$, and assume that $\langle F, H\rangle \notin R_C$. Then, $F \times H \not\subseteq C$, and thus, there are $x, y \in B^+$ such that $x \in F, y \in H$, and $x(-C)y$. By C_6 there is some $t \in B$ such that $x(-C)t$ and $t^*(-C)y$. Since $\langle F, G\rangle \in R_C$, we cannot have $t \in G$, and thus, $t^* \in G$. But $y \in H$ and $\langle G, H\rangle \in R_C$ imply that $t^* C y$, a contradiction.

By Theorem 1, there is an isotone one–one correspondence between $\mathscr{C}_6$ and the collection of closed equivalence relations on $\mathrm{Ult}(B)$. Thus, all that remains is to show that the latter is a lattice. It is well known that all equivalence relations on a set form a complete lattice under set inclusion, where the meet is just set intersection, and the join of a family of equivalence relations is the transitive closure of its union. Since an arbitrary intersection of closed sets is closed, and each family of closed equivalence relations has an upper bound, namely, the universal relation on $\mathrm{Ult}(B)$, the collection of all closed equivalence relations on $\mathrm{Ult}B$ is also a complete lattice. □

The following property of clans has been investigated in the theory of proximity spaces and their topological representation, see e.g. [11]:

Γ_5. Every maximal clan is a cluster.

It is known that C_6 implies Γ_5, and it was unclear whether the converse holds as well. In the following example we will exhibit a contact relation on $FC(\omega)$, that satisfies Γ_5, but which satisfies neither C_6 nor C_5.

Example 1. Suppose that $B = FC(\omega)$; for $n \in \omega$, let F_n be the ultrafilter generated by $\{n\}$; furthermore, let U be the ultrafilter of cofinite sets. Now, define C by

$$C = O \cup \bigcup \{F_n \times F_m : n \equiv m \mod 2\}. \tag{5.2}$$

In other words,

$$xCy \iff x = y \text{ or } (\exists n, m)[n \in x,\ m \in y,\ n \equiv m \bmod 2]. \tag{5.3}$$

Since each cofinite set contains both odd and even numbers, we have xCy for each cofinite set x and each $y \in B^+$; incidentally, this shows that $C \not\models C_5$. There are exactly two maximal clans in C, namely,

1. $\Gamma_0 = \bigcup\{F_n : n \equiv 0 \mod 2\} \cup U$,
2. $\Gamma_1 = \bigcup\{F_n : n \equiv 1 \mod 2\} \cup U$.

Let $x \in B$, and $\{x\} \times \Gamma_0 \subseteq C$. If x is cofinite, then $x \in \Gamma_0$ by 1. above. If x is finite and contains an even number, say, n, then $x \in F_n \subseteq \Gamma_0$. If x is finite and contains only odd numbers, then $x \notin F_n$ for any even n, and also, $x \notin U$. Therefore, $\{x\} \times \Gamma_0 \not\subseteq C$. Thus, Γ_0 is a cluster, and similarly, Γ_1 is a cluster.

Next, let $x = \{n\}$, where n is even, and set $y = \{n+1\}$; then, $x(-C)y$. Suppose that $z \in B^+$ such that $x(-C)z$; then, in particular, z is finite, i.e. z^* is cofinite, and hence, z^*Cy. This shows that $C \notin \mathscr{C}_6$. □

Turning to C_5, we make the following observation:

Theorem 5. *1. $\mathscr{C}_5$ is an ideal of $\mathscr{C}$.*

2. *Let $F, G \in \operatorname{Ult}(B)$, $F \neq G$, and $C = O \cup (F \times G) \cup (G \times F)$. Then, $C \in \mathscr{C}_5$ if and only if neither F nor G are principal.*
3. *B is isomorphic to a finite–cofinite algebra if and only if $\mathscr{C}_5 = \{O\}$.*
4. *B is atomless if and only if $\mathscr{C}_5$ contains all atoms of $\mathscr{C}$.*

Proof. 1. Clearly $\downarrow \mathscr{C}_5 = \mathscr{C}_5$. Let $C, C' \in \mathscr{C}_5$, and assume that $C \cup C' \notin \mathscr{C}_5$. Then, there exists some $x \in B, x \neq 1$, such that $x(C \cup C')y$ for all $y \in B^+$. Since $C \in \mathscr{C}_5$, there is some $y \neq 0$ such that $x(-C)y$; then, $x \cdot y = 0$ and $xC'y$. Since $C' \in \mathscr{C}_5$, by Lemma 2 there is some $0 \lneq z \leq y$ such that $x(-C')z$. But then, xCz, implying xCy, a contradiction. Hence, $C \cup C' \in \mathscr{C}_5$.

2. "⇒": Suppose that $C \in \mathscr{C}_5$, and assume that w.l.o.g. F is generated by the atom x. Then, $x^* \cdot y \neq 0$ for all $y \notin \{0, x\}$ which implies that x^*Cy for all such y. Since $F \neq G$, we cannot have $x \in G$, hence, $x^* \in G$ and $G \times F \subseteq C$ imply that also x^*Cx.

"⇐": Suppose that F, G are non–principal, and assume that $C \not\models C_5$. Then, there is some $x \neq 1$ such that, in particular, xCy for all $y \neq 0, y \leq x^*$. Let w.l.o.g. $x \in F$; then, $B^+ \cap \downarrow x^* \subseteq G$, which implies that G is generated by x; otherwise, there are nonzero disjoint $y, z \leq x^*$, whose sum is x^*, which cannot be, since $y, z \in G$.

3. The "only if" direction was shown in [17]. Conversely, if $\mathscr{C}_5 = \{O\}$, then, whenever F, G are distinct ultrafilters of B, then $O \cup (F \times G) \cup (G \times F) \notin \mathscr{C}_5$. By 1., this implies that one of F, G must be principal. Hence, B has at most one non–principal ultrafilter, and therefore, B is a finite–cofinite algebra.

4. This follows immediately from the fact that B is atomless if and only if it contains no principal ultrafilters. □

$\mathscr{C}_5$ is generally not generated by the atoms of $\mathscr{C}$: Suppose that $|B| = \kappa \geq \omega$ and that B is atomless. Let $x \in B,\ x \neq 0,1$; then, $|\{y : y \leq x\}| = \kappa$ or $|\{y : y \leq x^*\}| = \kappa$. Suppose w.l.o.g. the latter; then, $h(x^*)$ contains a proper closed subset M of cardinality 2^κ. Let $R = h(x) \times M \cup M \times h(x) \cup 1'$; then, R is a closed graph on $\mathrm{Ult}(B)$, and $C_R \models C_5$.

Finally, turning to $\mathscr{C}_7$, we first note that $\mathscr{C}_7 = \uparrow \mathscr{C}_7$; however, $\mathscr{C}_7$ is, in general, not a filter. To see this, consider the BA with atoms a,b,c, and let F_x be the ultrafilter generated by x $\in \{a,b,c\}$. Then, for $\{x,y\} \subseteq \{a,b,c\},\ x \neq y$, the contact relations $O \cup (F_x \times F_y) \cup (F_y \times F_x)$ satisfy C_7, but their meet does not.

However, the situation is brighter when we consider descending chains in $\mathscr{C}_7$:

Lemma 4. *If $\{C_\alpha : \alpha \in I\}$ is a descending chain in $\mathscr{C}_7$, then $\bigcap\{C_\alpha : \alpha \in I\} \in \mathscr{C}_7$.*

Proof. By Theorem 1, it suffices to show that $\bigcap\{C_\alpha : \alpha \in I\} \models C_7$. If $\langle x, cx\rangle \notin \bigcap\{C_\alpha : \alpha \in I\}$, then $x(-C_\alpha)x^*$ for some $\alpha \in I$. This contradicts $C_\alpha \in \mathscr{C}_7$. ◻

Thus, by Zorn's Lemma,

Corollary 3. *For each $C \in \mathscr{C}_7$ there is a minimal $C' \in \mathscr{C}_7$ such that $C' \subseteq C$.*

It was shown in [14] that $C \in \mathscr{C}_7$ if $\langle \mathrm{Ult}(B), R_C\rangle$ is a connected graph, and that the converse is not generally true. It is instructive to recall the example given in [14]:

Example 2. Let $B = FC(\omega)$, and define R on $\mathrm{Ult}(B)$ by

$$R = 1' \cup \{\langle F_n, F_m\rangle : |n-m| = 2\} = 1' \cup \{\langle F_n, F_{n+2}\rangle : n \in \omega\} \cup \{\langle F_{n+2}, F_n\rangle : n \in \omega\}.$$

Clearly, if $|n-m| \neq 2$, then $\langle F_n, F_m\rangle \notin \mathrm{cl}(R)$. Let $x = \{n\}$, and $y = \omega \setminus \{n+2, n-2\}$. Then, $x \in F_n,\ y \in F_\omega$, and thus, $\{F_n\} \times h(y)$ is an open neighbourhood of $\langle F_n, F_\omega\rangle$. Since $\{n+2, n-2\} \cap y = \emptyset$, $\{F_n\} \times h(y) \times R = \emptyset$, and it follows that $\langle F_n, F_\omega\rangle \notin \mathrm{cl}(R)$; similarly, $\langle F_\omega, F_n\rangle \notin \mathrm{cl}(R)$; hence, R is closed. Let $x \in B, x \neq \emptyset, \omega$. If x is finite, let $m = \max(x)$. Then, $m \in x$ and $m+2 \in x^*$, and therefore, $\langle x, x^*\rangle \in F_m \times F_{m+2}$, i.e. xC_Rx^*. Hence, C_R is a connected contact relation on B. However, R is not a connected graph, since, for example, there is no path from F_n to F_{n+1}. Indeed, the connected components of R are $\{F_{2n} : n \in \omega\}$ and $\{F_{2n+1} : n \in \omega\}$, each of which is a chain of type ω, and $\{F_\omega\}$. ◻

If B is finite, the condition is also sufficient:

Theorem 6. *If B is finite, then $C \in \mathscr{C}_7$ implies that R_C is a connected graph.*

Proof. Suppose that M is a connected component of R_C and $M \subsetneq \mathrm{Ult}(B)$. Then, there is no path between any $F_s \in M$ and any $F_t \in \mathrm{Ult}(B) \setminus M$. Let $x = \sum\{s \in At(B) : F_s \in M\}$ and $y = \sum\{t \in At(B) : F_t \notin M\}$; then, $x^* = y$. If xCy, there are $s,t \in At(B)$ such that $s \leq x, t \leq y$ and sCt, i.e. $\langle F_s, F_t\rangle \in R_C$. This contradicts the fact that F_s and F_t are in different components. ◻

Since the minimally connected graphs are trees (and vice versa), we obtain

Corollary 4. *If B is finite, then $C \in \mathscr{C}_7$ is minimal if and only if R_C is a tree and* $\mathrm{dom}(R_C \setminus 1') = \mathrm{Ult}(B)$.

Furthermore, since the only connected equivalence relation on $\mathrm{Ult}(B)$ is the universal relation, we have

Lemma 5. *If B is finite, then $\mathscr{C}_6 \cap \mathscr{C}_7 = B^+ \times B^+$.*

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