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Abstract. Processes can be seen as relations extended in time. In this paper we want to investigate this observation by deriving an ordered category of processes. We model processes as co-algebras of a relator on Dedekind category up to bisimilarity. On those equivalence classes we define a lower semi-lattice structure and a monotone composition operation.

1 Introduction

With this paper we want to start a comprehensive study of processes seen as relations extended in time. At a given time step a process may perform an input/output action. This corresponds to a functional or, more general, a relational behavior. After performing the action the process may switch to another internal state, i.e. may become another process. This observation justifies that process can be modelled in non-well-founded set theories using fixed points of the power set functor [2]. A categorical investigation using this approach led to notion of interaction categories [1]. Interaction categories lack an allegorical structure, and, therefore, just cover some aspects of relations extended in times. The order structure, and, hence, the allegorical structure that is usually available for relation is ignored. Another approach introduced time-extended allegories [10]. This kind of structure provides all relational and, in addition, time related operations. Unfortunately, time-extended allegories do not have obvious models. In this paper we model processes as co-algebras of a relator on Dedekind category up to bisimilarity. We show that there is a natural ordering on the corresponding equivalence classes. Furthermore, we are going to define a notion of composition of processes based on parallel composition. We show that the structure we obtain is an ordered category.

2 Dedekind Categories

Throughout this paper, we use the following notation. To indicate that a morphism R of a category $\mathcal{R}$ has source A and target B we write $R : A \rightarrow B$. The

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collection of all morphisms $R : A \rightarrow B$ is denoted by $\mathcal{R}[A, B]$ and the composition of a morphism $R : A \rightarrow B$ followed by a morphism $S : B \rightarrow C$ by $R; S$. Last but not least, the identity morphism on A is denoted by $\mathbb{I}_A$.

In this section we recall some fundamentals on Dedekind categories [6, 7]. This kind of category is called locally complete division allegories in [4].

Definition 1. *A Dedekind category $\mathcal{R}$ is a category satisfying the following:*

1. *For all objects A and B the collection $\mathcal{R}[A, B]$ of morphisms/relations is a complete distributive lattice. Meet, join, the induced ordering, the least and the greatest element are denoted by $\sqcap, \sqcup, \sqsubseteq, \sqbot_{AB}$ and $\sqtop_{AB}$, respectively.*
2. *There is a monotone operation $\smile$ (called converse) mapping a relation $Q : A \rightarrow B$ to a relation $Q^\smile : B \rightarrow A$ such that $(Q; R)^\smile = R^\smile; Q^\smile$ and $(Q^\smile)^\smile = Q$ for all relations $Q : A \rightarrow B$ and $R : B \rightarrow C$.*
3. *For all relations $Q : A \rightarrow B, R : B \rightarrow C$ and $S : A \rightarrow C$ the modular law $Q; R \sqcap S \sqsubseteq Q; (R \sqcap Q^\smile; S)$ holds¹.*
4. *For all relations $R : B \rightarrow C$ and $S : A \rightarrow C$ there is a relation $S/R : A \rightarrow B$ (called the left residual of S and R) such that for all $Q : A \rightarrow B$ the following holds: $Q; R \sqsubseteq S \iff Q \sqsubseteq S/R$.*

We will also need the weaker notion of an ordered category in this paper. An ordered category $\mathcal{C}$ is a category so that every collection $\mathcal{C}[A, B]$ is an ordered class and composition is a monotone operation. $\mathcal{C}$ is called an ordered category with converse iff it is an ordered category with an converse operation satisfying 2. of the previous definition. Notice that a Dedekind category is an ordered category with converse.

Throughout this paper we will use several basic properties of Dedekind categories such as $\mathbb{I}_A^\smile = \mathbb{I}_A$, the monotonicity of composition in both parameters or the distributivity of $;$ over $\sqcup$ without mentioning. For details we refer to any of [4, 8–10].

For relations $Q : A \rightarrow B$ and $R : A \rightarrow C$ one may define a right residual $Q \backslash R : B \rightarrow C$ by $Q \backslash R := (R^\smile / Q^\smile)^\smile$. This construction is characterized by $X \sqsubseteq Q \backslash R$ iff $Q; X \sqsubseteq R$.

As a description of recursive data types as well as labelled transition systems a special class of functors is of interest [3].

Definition 2. *Let F be a functor between ordered categories with converse. We call F a relator iff F is monotonic and preserves converse, i.e. $R \sqsubseteq S$ implies $F(R) \sqsubseteq F(S)$ and $F(R^\smile) = F(R)^\smile$.*

Notice, that the definition in [3] is slightly different. The allegories they consider are tabular (see below) so that the preservation of converse is equivalent to monotonicity. The notion of a relator was introduced by Y. Kawahara in [5].

Recall that a natural transformation $\eta : F \rightarrow G$ between two functors is a family of morphisms so that $F(f); \eta = \eta; G(f)$ for all suitable f and g . In the

¹ By convention the precedence of the operations decreases in the following order $\smile, ;, \sqcap$.

context of Dedekind categories and relators F and G one is often interested in lax natural transformations satisfying the weaker property $F(Q); \eta \sqsubseteq \eta; F(Q)$.

An important class of relations is given by mappings.

Definition 3. Let $Q : A \rightarrow B$ be a relation. Then we call

1. Q univalent iff $Q^\smile; Q \sqsubseteq \mathbb{I}_B$,
2. Q total iff $\mathbb{I}_A \sqsubseteq Q; Q^\smile$,
3. Q a map iff Q is univalent and total,
4. Q injective iff $Q^\smile$ is univalent,
5. Q surjective iff $Q^\smile$ is total.

Notice, that if Q is a bijective mapping, i.e. a mapping that is injective and surjective, we have $Q^\smile; Q = \mathbb{I}_B$ and $Q; Q^\smile = \mathbb{I}_A$.

A relator preserves all notions of the last definition. In particular, its restriction to the class of mappings is a functor between the corresponding subcategories.

In the next lemma we collect some fundamental facts used in this paper. A proof may be found in [4, 8, 9].

Lemma 1. Let $Q, T : A \rightarrow B$, $R : B \rightarrow C$, $S : A \rightarrow D$, $V : B \rightarrow A$, and $U, Z : A \rightarrow C$. Then we have

1. $Q; R \sqcap S; \pi_{DC} = (Q \sqcap S; \pi_{DB}); R$ and $(Q; R \sqcap U); \pi_{CD} = (Q \sqcap U; R^\smile); \pi_{BD}$,
2. If U is univalent, then $U; (Q \sqcap T) = U; Q \sqcap U; T$,
3. If U is univalent, then $(V \sqcap R; U^\smile); U = V; U \sqcap R$,
4. If $X \sqsubseteq \mathbb{I}_A$, then $X; X = X$ and $X; (Q \sqcap T) = Q \sqcap X; T$,
5. If U is univalent, then $U \sqcap (U \sqcap Z); \pi_{CA} = U \sqcap Z$.

Another important concept are splittings. They generalize several well-known constructions on sets.

Definition 4. Let $\mathcal{R}$ be a Dedekind category, and $Q : A \rightarrow A$ be partial equivalence relation, i.e. a symmetric idempotent relation such that $Q^\smile = Q$ and $Q; Q = Q$. An object B together with a relation $R : B \rightarrow A$ is called a splitting of Q (or R splits Q) iff $R; R^\smile = \mathbb{I}_B$ and $R^\smile; R = Q$. $\mathcal{R}$ has splittings iff all symmetric idempotent relations in $\mathcal{R}$ split.

A splitting is unique up to isomorphism. If Q is a partial identity the object B of the splitting corresponds to the subset given by Q . Analogously, if Q is an equivalence relation B corresponds to the set of equivalence classes.

The notion dual to a splitting of a difunctional relation is the notion of a tabulation [4].

Definition 5. Let $\mathcal{R}$ be a Dedekind category. A pair of maps $f : C \rightarrow A$ and $g : C \rightarrow B$ tabulates a relation $Q : A \rightarrow B$ iff $f^\smile; g = Q$ and $f; f^\smile \sqcap g; g^\smile = \mathbb{I}_C$. $\mathcal{R}$ is called tabular iff every relation is tabular, i.e. there is a pair a mappings tabulating the relation.

Notice, that tabulations are strongly related to the representability of the Dedekind category [4].

A tabulation of the greatest relation $\mathbb{T}_{AB}$ is called a relational product of A and B . We use the notation $A \times B$ for the product object and $\pi : A \times B \rightarrow A$ and $\rho : A \times B \rightarrow B$ for the projections. A relational product constitutes a product in the subcategory of mappings, and is, therefore, an abstract counterpart of a cartesian product of sets. We use the notation $\langle Q, R \rangle := Q; \pi^\smile \sqcap R; \rho^\smile$ for relations $Q : C \rightarrow A$ and $R : C \rightarrow B$. Notice that if Q and R are mappings, this construction computes the unique product map induced by Q and R , i.e. the unique map $h : C \rightarrow A \times B$ with $h; \pi = Q$ and $h; \rho = R$. In addition, we use $S \times T := \langle \pi; S, \rho; T \rangle : A \times B \rightarrow C \times D$ for relations $S : A \times C$ and $T : B \times D$. Related to these two constructions we have in tabular Dedekind categories:

1. $\langle Q, R \rangle; \pi = Q$ if R is total, and $\langle Q, R \rangle; \rho = R$ if Q is total,
2. $\langle Q, R \rangle; (S \times T) = \langle Q; S, R; T \rangle$ for all $Q : C \rightarrow A$, $R : C \rightarrow B$, $S : A \rightarrow D$ and $T : B \rightarrow E$;
3. $(U \times V); (S \times T) = U; S \times V; T$ for all $U : C \rightarrow A$, $V : F \rightarrow B$, $S : A \rightarrow D$ and $T : B \rightarrow E$.

In the remainder of the paper we will use the properties above of relational products without mentioning.

Relational products are associative, i.e. the relation $\text{ass} : (A \times B) \times C \rightarrow A \times (B \times C)$ defined by $\text{ass} = \langle \pi; \pi, \langle \pi; \rho, \rho \rangle \rangle$ is a bijective function. Its converse (or inverse) is the relation $\text{ass}^\smile = \langle \langle \pi, \rho; \pi \rangle, \rho; \rho \rangle$. In addition, the family of the relations ass is a natural transformation between the relators $(\cdot \times \cdot) \times \cdot \times (\cdot \times \cdot)$ and $\cdot \times (\cdot \times \cdot)$, i.e. we have $\text{ass}; (Q \times (R \times S)) = ((Q \times R) \times S); \text{ass}$ for all suitable relations Q, R and S .

For further details on relational products we refer to any of [8–10].

An object 1 is called a unit iff $\mathbb{I}_1 = \mathbb{T}_{11}$ and $\mathbb{T}_{A1}$ is total for all objects A . Notice that a unit is a terminal object in the subcategory of mappings. Therefore it is unique up to isomorphism and a neutral element for $\times$. In particular, the projection $\pi : A \times 1 \rightarrow A$ is an isomorphism.

In the remainder of this paper we will use the partial identity

$$i := \mathbb{I}_{(L_1 \times L_2) \times (L_2 \times L_3)} \sqcap \pi; \rho; \pi^\smile; \rho^\smile$$

and the univalent relation (partial function) $\text{com} : (L_1 \times L_2) \times (L_2 \times L_3) \rightarrow L_1 \times L_3$ defined by $\text{com} := i; (\pi \times \rho)$.

In the following lemma we have summarized some properties of the relations introduced so far.

- Lemma 2.**
1. $\text{ass}; (\mathbb{I}_{L_0 \times L_1} \times \text{com}); \text{com} = (\text{com} \times \mathbb{I}_{L_2 \times L_3}); \text{com},$
 2. $\langle \mathbb{I}_{L_0 \times L_1}, \langle \rho, \rho \rangle \rangle; \text{com} = \mathbb{I}_{L_0 \times L_1},$
 3. $\langle \langle \pi, \pi \rangle, \mathbb{I}_{L_0 \times L_1} \rangle; \text{com} = \mathbb{I}_{L_0 \times L_1}.$

Because of lack of space we omit the proof of this lemma.

3 Bisimulation

In this section we want to recall some basic properties of bisimulations in the relation algebraic framework [11]. The approach taken in that paper is quite general and covers a huge class of different bisimulations. In current we choose the behavior operation from [11] to be the identity.

Definition 6. Let $\mathcal{R}$ be a Dedekind category, $F : \mathcal{R} \rightarrow \mathcal{R}$ be an endorelator, and $P_1 : S_1 \rightarrow F(S_1)$ and $P_2 : S_2 \rightarrow F(S_2)$ be F coalgebras. A relation $\Phi : S_1 \rightarrow S_2$ is called a bisimulation (from P_1 and P_2) iff

$$\Phi^\smile; P_1 \sqsubseteq P_2; F(\Phi^\smile) \text{ and } \Phi; P_2 \sqsubseteq P_1; F(\Phi).$$

Using residuals the two inclusions can be rewritten as $\Phi \sqsubseteq \tau(\Phi)$ where

$$\tau(\Phi) := P_1^\smile \setminus (F(\Phi); P_2^\smile) \sqcap (P_1; F(\Phi)) / P_2.$$

The next lemma shows the basic properties of bisimulations in the relation algebraic context.

Lemma 3. Let $\mathcal{R}$ be a Dedekind category, $F : \mathcal{R} \rightarrow \mathcal{R}$ be an endorelator, $P_1 : S_1 \rightarrow F(S_1)$, $P_2 : S_2 \rightarrow F(S_2)$ and $P_3 : S_3 \rightarrow F(S_3)$ be F coalgebras, and $\Phi : S_1 \rightarrow S_2, \Psi : S_2 \rightarrow S_3$ and $\Phi_i : S_1 \rightarrow S_2$ ($i \in I$) be bisimulations. Then the following relations are bisimulations

1. $\perp_{AB}$,
2. $\mathbb{I}_A$,
3. $\Phi^\smile$,
4. $\Phi; \Psi$,
5. $\bigsqcup_{i \in I} \Phi_i$.

The previous lemma shows that there is a greatest bisimulation (between two processes). Furthermore, τ is monotonic and, therefore, has a greatest fixed point Θ . Θ is also the greatest post fixed point, i.e. it satisfies $\Theta \sqsubseteq \tau(\Theta)$, and hence is the greatest bisimulation. Furthermore the existence of Θ is independent of the two processes. In order to call two processes bisimilar we have to require that Θ is total and surjective, i.e. that Θ relates the processes and all of their derivatives. In particular, we use the following definition.

Definition 7. Let $\mathcal{R}$ be a Dedekind category, $F : \mathcal{R} \rightarrow \mathcal{R}$ be an endorelator, and $P_1 : S_1 \rightarrow F(S_1)$ and $P_2 : S_2 \rightarrow F(S_2)$ be F coalgebras. Then

1. P_1 is called a future instance of P_2 , or P_1 is eventually bisimilar to P_2 , denoted by $P_1 \preceq P_2$, iff there is a total bisimulation from P_1 to P_2 ;
2. P_1 and P_2 are called bisimilar, denoted by $P_1 \approx P_2$, iff there is a total and surjective bisimulation between P_1 and P_2 .

Notice, that P_1 and P_2 are bisimilar iff the greatest fixed point Θ of τ is total and surjective. Furthermore, Θ is difunctional, i.e. $\Theta; \Theta^\smile; \Theta = \Theta$.

Lemma 4. Let $\mathcal{R}$ be a Dedekind category, and $F : \mathcal{R} \rightarrow \mathcal{R}$ be an endorelator. Then the relation $\preceq$ is a pre-ordering on the class of processes and its induced equivalence relation is $\approx$ i.e. $P_1 \preceq P_2$ and $P_2 \preceq P_1$ implies $P_1 \approx P_2$ for all F coalgebras $P_1 : S_1 \rightarrow F(S_1)$ and $P_2 : S_2 \rightarrow F(S_2)$.

Proof. The reflexivity and transitivity of $\preceq$ follows immediately from Lemma 3 2.&4. since the identity is total and the composition of total relations is total. Suppose $P_1 \preceq P_2$ and $P_2 \preceq P_1$, i.e. there is a total bisimulation $\Phi : S_1 \rightarrow S_2$ and a total bisimulation $\Psi : S_2 \rightarrow S_1$. By Lemma 3 3.&5. the relation $\Phi \sqcup \Psi^\sim$ is a bisimulation. This relation is total (since Φ is) and surjective (since Ψ is total). $\square$

Due to the last lemma the class of equivalence classes $[P]$ (with respect to $\approx$) of F coalgebras is ordered by $[P_1] \sqsubseteq [P_2]$ iff $P_1 \preceq P_2$. We denote by $(\mathcal{L}(F), \sqsubseteq)$ the ordered class of those equivalence classes. If the underlying Dedekind category has splittings, then each equivalence class has a canonical representative (see [11]). Intuitively, the canonical representative is given by the maximal identified graph. Instead of using equivalence classes we may use the canonical representatives. The corresponding order structure is also denoted by $(\mathcal{L}(F), \sqsubseteq)$.

Lemma 5. *Let $\mathcal{R}$ be a Dedekind category with splittings, and $F : \mathcal{R} \rightarrow \mathcal{R}$ be an endorelator. Then each pair of F coalgebras $P_1 : S_1 \rightarrow F(S_1)$ and $P_2 : S_2 \rightarrow F(S_2)$ has a greatest lower bound with respect to $\preceq$.*

Proof. Let Θ be the greatest bisimulation from P_1 to P_2 . Then Θ is difunctional, and, hence, $\Theta; \Theta^\sim$ an equivalence relation. Suppose $R : S \rightarrow S_1$ splits Θ , i.e. we have $R; R^\sim = \mathbb{I}_S$ and $R^\sim; R = \Theta; \Theta^\sim$. Then R is total and the relation $P := R; P_1; F(R^\sim) : S \rightarrow F(S)$ is a F coalgebra. We compute

$$\begin{aligned}
R; P_1 &= R; P_1; F(\mathbb{I}_{S_1}) \\
&\sqsubseteq R; P_1; F(\Theta; \Theta^\sim) && \Theta; \Theta^\sim \text{ reflexive} \\
&= R; P_1; F(R^\sim; R) && R \text{ splits } \Theta; \Theta^\sim \\
&= R; P_1; F(R^\sim); F(R) \\
&= P; F(R), \\
R^\sim; P &= R^\sim; R; P_1; F(R^\sim) \\
&= \Theta; \Theta^\sim; P_1; F(R^\sim) && R \text{ splits } \Theta; \Theta^\sim \\
&\sqsubseteq F(\Theta); P_2; F(\Theta^\sim); F(R^\sim) && \Theta^\sim \text{ is a bisimulation} \\
&\sqsubseteq P_1; F(\Theta); F(\Theta^\sim); F(R^\sim) && \Theta \text{ is a bisimulation} \\
&= P_1; F(\Theta; \Theta^\sim; R^\sim) \\
&= P_1; F(R^\sim; R; R^\sim) && R \text{ splits } \Theta; \Theta^\sim \\
&= P_1; F(R^\sim). && R \text{ is a splitting}
\end{aligned}$$

and conclude that R is a bisimulation from P to P_1 . The relation $R; \Theta$ is a bisimulation (from P to P_2) by Lemma 3. Furthermore, this relation is total since

$$\mathbb{I}_S = R; R^\sim; R; R^\sim = R; \Theta; \Theta^\sim; R^\sim = R; \Theta; (R; \Theta)^\sim.$$

Suppose $P' : S' \rightarrow F(S')$ is a F coalgebra with $P' \preceq P_1$ and $P' \preceq P_2$, i.e. there are total bisimulations $\Phi_1 : S' \rightarrow S_1$ and $\Phi_2 : S' \rightarrow S_2$. By Lemma 3 the relation

$\Phi_1^\sim; \Phi_2$ is bisimulation from P_1 to P_2 , and, thus, we have $\Phi_1^\sim; \Phi_2 \sqsubseteq \Theta$. This implies $\Phi_2 \sqsubseteq \Phi_1; \Phi_1^\sim; \Phi_2 \sqsubseteq \Phi_1; \Theta$ since Φ_1 is total. Furthermore, the relation $\Phi_1; R^\sim$ is a bisimulation from P' to P . It remains to show that this relation is total. We compute

$$\begin{aligned}
\mathbb{I}_{S'} &\sqsubseteq \Phi_2; \Phi_2^\sim && \Phi_2 \text{ is total} \\
&\sqsubseteq \Phi_1; \Theta; (\Phi_1; \Theta)^\sim && \text{see above} \\
&= \Phi_1; \Theta; \Theta^\sim; \Phi_1^\sim \\
&= \Phi_1; R^\sim; R; \Phi_1^\sim && R \text{ splits } \Theta; \Theta^\sim \\
&= \Phi_1; R^\sim; (\Phi_1; R^\sim)^\sim.
\end{aligned}$$

This completes the proof. $\square$

From the previous lemma we immediately obtain the following corollary.

Corollary 1. *Let $\mathcal{R}$ be a Dedekind category with splittings, and $F : \mathcal{R} \rightarrow \mathcal{R}$ be an endorelator. Then $(\mathcal{L}(F), \sqsubseteq)$ is a lower semi-lattice.*

We are interested in operation defined on the ordered classes $\mathcal{L}(F)$ of F coalgebras. As an intermediate step the following category turns out to be useful.

Definition 8. *Let $\mathcal{R}$ be a Dedekind category, and $F : \mathcal{R} \rightarrow \mathcal{R}$ be an endorelator. The category $\text{BSIM}(F)$ has F coalgebras as objects and bisimulations as morphisms.*

From Lemma 3 we conclude that $\text{BSIM}(F)$ is an ordered category with converse. Notice that a relator $F : \text{BSIM}(G) \rightarrow \text{BSIM}(H)$ respects the pre-order $\preceq$ and the equivalence relation $\approx$. Therefore, it induces a monotone function $F^L : \mathcal{L}(G) \rightarrow \mathcal{L}(H)$. A generalization to multiple parameters is obvious.

The next lemma provides a method to lift a relator between the underlying Dedekind categories to a relator between coalgebras.

Lemma 6. *Let $\mathcal{R}_1, \mathcal{R}_2$ be a Dedekind categories, $G : \mathcal{R}_1 \rightarrow \mathcal{R}_1$ and $H : \mathcal{R}_2 \rightarrow \mathcal{R}_2$ be endorelators, $F : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ be a relator, and $\eta : F \circ G \rightarrow H \circ F$ a lax natural transformation. Then F_η defined by*

- $F_\eta(P) := F(P); \eta$,
- $F_\eta(\Phi) := F(\Phi)$,

is a relator from $\text{BSIM}(G)$ to $\text{BSIM}(H)$.

Proof. First of all, if $P : S \rightarrow G(S)$, then we have $F(P); \eta : F(S) \rightarrow H(F(S))$, i.e. $F_\eta(P)$ is a H coalgebra. It remains to verify that if Φ is a bisimulation between G coalgebras P_1 and P_2 , then $F(\Phi)$ is a bisimulation between $F(P_1); \eta$ and

$F(P_2); \eta$. All other properties follow from the fact the operations on morphisms in $\mathcal{R}_2$ and $\text{BSIM}(H)$ are the same. Consider the following computation:

$$\begin{aligned}
F(\Phi); F(P_2); \eta &= F(\Phi; P_2); \eta \\
&\sqsubseteq F(P_1; G(\Phi)); \eta && \Phi \text{ bisimulation and } F \text{ relator} \\
&= F(P_1); F(G(\Phi)); \eta \\
&\sqsubseteq F(P_1); \eta; H(F(\Phi)). && \eta \text{ lax nat. trans.}
\end{aligned}$$

The other inclusion follows analogously. $\square$

Again, a generalization of the previous lemma to multiple parameters is straight-forward.

If we combine the previous lemma with the observation above, we get the following corollary.

Corollary 2. *Let $\mathcal{R}_1, \mathcal{R}_2$ be a Dedekind categories, $G : \mathcal{R}_1 \rightarrow \mathcal{R}_1$ and $H : \mathcal{R}_2 \rightarrow \mathcal{R}_2$ be endorelators, $F : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ be a relator, and $\eta : F \circ G \rightarrow H \circ F$ a lax natural transformation. Then $F_\eta^L : \mathcal{L}(G) \rightarrow \mathcal{L}(H)$ is a monotone function.*

4 The Category $\text{SProc}(F)$

In order to define a category of processes as morphisms we fix a binary relator $F : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$. The first parameter of the relator is considered to be the object of actions (or labels) performed by a process. Fixing an object L we obtain an endorelator $F(L, \cdot)$. We call the elements of $\mathcal{L}(F(L, \cdot))$ processes of kind L . Recall that processes are either equivalence classes of coalgebras or, alternatively, the canonical representative of such an equivalence class. An object in $\text{SProc}(F)$ is an object L from $\mathcal{R}$. A morphism in $\text{SProc}(F)$ between two objects L_1 and L_2 is a process of kind $L_1 \times L_2$. Recall that every class of morphisms in $\text{SProc}(F)$ is lower semi-lattice.

The standard example of the a structure $\text{SProc}(F)$ is given by an arbitrary Dedekind category with splitting and relational products and the binary relator $F(L, S) := L \times S$.

Composition in $\text{SProc}(F)$ is (synchronous) parallel composition $+$ hiding of internal channels. Therefore, we study both concepts separately and start with parallel composition.

Suppose $\alpha_{L_1, L_2, S_1, S_2}$ is a family a relations from $F(L_1, S_1) \times F(L_2, S_2) \rightarrow F(L_1 \times L_2, S_1 \times S_2)$. In the following we will omit the index of α and define $P_1 \parallel P_2 := (P_1 \times P_2); \alpha$. Throughout the paper we will require several properties of α . If we denote by $\mathbf{1}_L : 1 \rightarrow F(L \times L, 1)$ the relation

$$\mathbf{1}_L := \top_{1F(L \times L, 1)}; F(\mathbb{I}_{L \times L} \sqcap \pi; \rho^\smile, \mathbb{I}_1),$$

then we state the properties on α as follows:

($\alpha 1$) is a lax natural transformation between the relators $F(L_1, \cdot) \times F(L_2, \cdot)$ and $F(L_1 \times L_2, \cdot \times \cdot)$, i.e. $(F(\mathbb{I}_{L_1}, Q) \times F(\mathbb{I}_{L_2}, R)); \alpha \sqsubseteq \alpha; F(\mathbb{I}_{L_1 \times L_2}, Q \times R)$,

- ($\alpha 2$) is a natural transformation between $F(\cdot, S_1) \times F(\cdot, S_2)$ and $F(\cdot \times \cdot, S_1 \times S_2)$,
i.e. $(F(U, \mathbb{I}_{S_1}) \times F(V, \mathbb{I}_{S_2})); \alpha = \alpha; F(U \times V, \mathbb{I}_{S_1 \times S_2})$,
($\alpha 3$) $\text{ass}; (\mathbb{I} \times \alpha); \alpha = (\alpha \times \mathbb{I}); \alpha; F(\text{ass}, \text{ass})$,
($\alpha 4a$) $(\mathbb{I}_{(L_1 \times L_2) \times S} \times \mathbf{1}_{L_2}); \alpha; F(i, \pi) = \pi; F(\langle \mathbb{I}_{L_1 \times L_2}, \langle \rho, \rho \rangle \rangle; i, \mathbb{I}_S)$ for all objects
 L_1, L_2 and S ,
($\alpha 4b$) $(\mathbf{1}_{L_1} \times \mathbb{I}_{(L_1 \times L_2) \times S}); \alpha; F(i, \rho) = \rho; F(\langle \langle \pi, \pi \rangle, \mathbb{I}_{L_1 \times L_2} \rangle; i, \mathbb{I}_S)$ for all objects
 L_1, L_2 and S .

($\alpha 1$) ensures that parallel composition is a monotone operation between the lower semi-lattices $\mathcal{L}(F(L_1, \cdot)) \times \mathcal{L}(F(L_2, \cdot))$ and $\mathcal{L}(F(L_1 \times L_2, \cdot))$ since it is defined as the lifting of the product relator using α .

In terms of the actions (or the labels) we will require the stronger property ($\alpha 2$), i.e. that α is a natural transformation. This property as well as ($\alpha 3$) will ensure that composition is associative.

The last property ($\alpha 4$) is used to prove that the process $\mathbf{1}_L$ appearing on the left side of the equations is a left and right identity for composition.

In the standard example α is given by $\langle \pi \times \pi, \rho \times \rho \rangle : (L_1 \times S_1) \times (L_2 \times S_2) \rightarrow (L_1 \times L_2) \times (S_1 \times S_2)$. The next lemma shows that this α satisfies all of the properties above.

Lemma 7. *Let $\mathcal{R}$ be a Dedekind category with relational products, and denote by $F(L, S) := L \times S$ the product relator. Then the family of relations $\alpha := \langle \pi \times \pi, \rho \times \rho \rangle$ satisfies the properties ($\alpha 1$) – ($\alpha 4$).*

Proof. First of all, we have the following property (*)

$$\begin{aligned}
& \langle \langle \pi; U^\sim \rangle \times \langle \pi; Q^\sim \rangle, (\rho; V^\sim) \times (\rho; R^\sim) \rangle \\
&= (\pi; \pi; U^\sim; \pi^\sim \sqcap \rho; \pi; Q^\sim; \rho^\sim); \pi^\sim \sqcap (\pi; \rho; V^\sim; \pi^\sim \sqcap \rho; \rho; R^\sim; \rho^\sim); \rho^\sim \\
&= \pi; \pi; U^\sim; \pi^\sim; \pi^\sim \sqcap \rho; \pi; Q^\sim; \rho^\sim; \pi^\sim \sqcap \pi; \rho; V^\sim; \pi^\sim; \rho^\sim \sqcap \rho; \rho; R^\sim; \rho^\sim; \rho^\sim \\
&\quad \text{by Lemma 1(2)} \\
&= \pi; (\pi; U^\sim; \pi^\sim; \pi^\sim \sqcap \rho; V^\sim; \pi^\sim; \rho^\sim) \sqcap \rho; (\pi; Q^\sim; \rho^\sim; \pi^\sim \sqcap \rho; R^\sim; \rho^\sim; \rho^\sim) \\
&\quad \text{by Lemma 1(2)} \\
&= \pi; ((\pi; U) \times (\pi; V))^\sim \sqcap \rho; ((\rho; Q) \times (\rho; R))^\sim \\
&= \langle (\pi; U) \times (\pi; V), (\rho; Q) \times (\rho; R) \rangle^\sim
\end{aligned}$$

for all suitable Q, R, U and V .

($\alpha 1$) and ($\alpha 2$) We compute

$$\begin{aligned}
& (F(U, Q) \times F(V, R)); \alpha \\
&= ((U \times Q) \times (V \times R)); \langle \pi \times \pi, \rho \times \rho \rangle \\
&= ((U \times Q) \times (V \times R)); \langle \pi \times \pi, \rho \times \rho \rangle^\sim \quad \text{by (*)} \\
&= (\langle \pi \times \pi, \rho \times \rho \rangle; ((U^\sim \times Q^\sim) \times (V^\sim \times R^\sim)))^\sim
\end{aligned}$$

$$\begin{aligned}
&= \langle (\pi \times \pi); (U^\sim \times Q^\sim), (\rho \times \rho); (V^\sim \times R^\sim) \rangle^\sim \\
&= \langle (\pi; U^\sim) \times (\pi; Q^\sim), (\rho; V^\sim) \times (\rho; R^\sim) \rangle^\sim \\
&= \langle (\pi; U) \times (\pi; V), (\rho; Q) \times (\rho; R) \rangle \quad \text{by } (*) \\
&= \langle (\pi \times \pi); (U \times V), (\rho \times \rho); (Q \times R) \rangle \\
&= \langle \pi \times \pi, \rho \times \rho \rangle; ((U \times V) \times (Q \times R)) \\
&= \alpha; F(U \times V, Q \times R)
\end{aligned}$$

verifying $(\alpha 1)$ and $(\alpha 2)$.

$(\alpha 3)$ Consider the following computation

$$\begin{aligned}
(\mathbb{I} \times \alpha); \alpha &= (\mathbb{I} \times \alpha); \langle \pi \times \pi, \rho \times \rho \rangle^\sim \quad \text{by } (*) \\
&= \langle \pi \times \pi, \rho \times \rho \rangle; (\mathbb{I} \times \alpha^\sim)^\sim \\
&= \langle \pi \times \pi, (\rho \times \rho); \alpha^\sim \rangle^\sim \\
&= ((\pi; \pi; \pi^\sim \sqcap \rho; \pi; \rho^\sim); \pi^\sim \sqcap (\pi; \rho; \pi^\sim \sqcap \rho; \rho^\sim); \alpha; \rho^\sim)^\sim \\
&= (\pi; \pi; \pi^\sim; \pi^\sim \sqcap \rho; \pi; \rho^\sim; \pi^\sim \sqcap \pi; \pi^\sim; \alpha; \rho^\sim \sqcap \rho; \rho^\sim; \alpha; \rho^\sim)^\sim \\
&\quad \text{by Lemma 1(3)} \\
&= (\pi; (\pi; \pi^\sim; \pi^\sim \sqcap \rho; \pi^\sim; \alpha; \rho^\sim) \sqcap \rho; (\pi; \rho^\sim; \pi^\sim \sqcap \rho; \rho^\sim; \alpha; \rho^\sim))^\sim \\
&\quad \text{by Lemma 1(3)} \\
&= (\pi; (\pi^\sim \times (\pi^\sim; \alpha)) \sqcap \rho; (\rho^\sim \times (\rho^\sim; \alpha))^\sim)^\sim \\
&= \langle \pi \times (\alpha^\sim; \pi), \rho \times (\alpha^\sim; \rho) \rangle \\
&= \langle \pi \times (\langle \pi \times \pi, \rho \times \rho \rangle; \pi), \rho \times (\langle \pi \times \pi, \rho \times \rho \rangle; \rho) \rangle \quad \text{by } (*) \\
&= \langle \pi \times (\pi \times \pi), \rho \times (\rho \times \rho) \rangle.
\end{aligned}$$

Analogously, we obtain $(\alpha \times \mathbb{I}); \alpha = \langle ((\pi \times \pi) \times \pi), ((\rho \times \rho) \times \rho) \rangle$, and conclude

$$\begin{aligned}
&\text{ass}; (\mathbb{I} \times \alpha); \alpha \\
&= \text{ass}; \langle \pi \times (\pi \times \pi), \rho \times (\rho \times \rho) \rangle \quad \text{see above} \\
&= \langle \text{ass}; (\pi \times (\pi \times \pi)), \text{ass}; (\rho \times (\rho \times \rho)) \rangle \quad \text{Lemma 1(2)} \\
&= \langle ((\pi \times \pi) \times \pi); \text{ass}, ((\rho \times \rho) \times \rho); \text{ass} \rangle \quad \text{ass nat. trans.} \\
&= \langle ((\pi \times \pi) \times \pi), ((\rho \times \rho) \times \rho); (\text{ass} \times \text{ass}) \rangle \\
&= (\alpha \times \mathbb{I}); \alpha; (\text{ass} \times \text{ass}) \quad \text{see above} \\
&= (\alpha \times \mathbb{I}); \alpha; F(\text{ass}, \text{ass}).
\end{aligned}$$

$(\alpha 4)$ First we obtain the following equation

$$\begin{aligned}
\langle (\pi \times \pi); i, \pi; \rho \rangle &= (\pi \times \pi); i; \pi^\sim \sqcap \pi; \rho; \rho^\sim \\
&= (\pi; \pi; \pi^\sim \sqcap \rho; \pi; \rho^\sim); i; \pi^\sim \sqcap \pi; \rho; \rho^\sim
\end{aligned}$$

$$\begin{aligned}
&= \pi; \pi; \pi^\sim; i; \pi^\sim \sqcap \rho; \pi; \rho^\sim; i; \pi^\sim \sqcap \pi; \rho; \rho^\sim && \text{Lemma 1(2)} \\
&= \pi; (\pi; \pi^\sim; i; \pi^\sim \sqcap \rho; \rho^\sim) \sqcap \rho; \pi; \rho^\sim; i; \pi^\sim && \text{Lemma 1(2)} \\
&= \pi; ((\pi^\sim; i) \times \mathbb{I}) \sqcap \rho; \pi; \rho^\sim; i; \pi^\sim \\
&= \langle (i; \pi) \times \mathbb{I}, \pi; i; \rho; \pi^\sim \rangle^\sim.
\end{aligned}$$

Then we conclude

$$\begin{aligned}
&(\mathbb{I}_{(L_1 \times L_2) \times S} \times \mathbf{1}_{L_2}); \alpha; F(i, \pi) \\
&= (\mathbb{I}_{(L_1 \times L_2) \times S} \times \mathbf{1}_{L_2}); \langle \pi \times \pi, \rho \times \rho \rangle; (i \times \pi) \\
&= (\mathbb{I}_{(L_1 \times L_2) \times S} \times \mathbf{1}_{L_2}); \langle (\pi \times \pi); i, (\rho \times \rho); \pi \rangle \\
&= (\mathbb{I}_{(L_1 \times L_2) \times S} \times \mathbf{1}_{L_2}); \langle (\pi \times \pi); i, \pi; \rho \rangle \\
&= (\mathbb{I}_{(L_1 \times L_2) \times S} \times \mathbf{1}_{L_2}); \langle (i; \pi) \times \mathbb{I}_S, \pi; i; \rho; \pi^\sim \rangle^\sim && \text{see above} \\
&= (\langle (i; \pi) \times \mathbb{I}_S, \pi; i; \rho; \pi^\sim \rangle; (\mathbb{I}_{(L_1 \times L_2) \times S} \times \mathbf{1}_{L_2}))^\sim \\
&= \langle (i; \pi) \times \mathbb{I}_S, \pi; i; \rho; \pi^\sim; \mathbf{1}_{L_2} \rangle^\sim \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; \rho; \pi^\sim; \mathbf{1}_{L_2}; \rho^\sim)^\sim \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; \rho; \pi; \rho^\sim; \pi^\sim; \pi^\sim \sqcap \pi; i; \rho; \pi^\sim; \mathbf{1}_{L_2}; \rho^\sim)^\sim
\end{aligned}$$

where the last = follows from

$$\begin{aligned}
((i; \pi) \times \mathbb{I}_S); \pi^\sim &= (\pi; i; \pi; \pi^\sim \sqcap \rho; \rho^\sim); \pi^\sim \\
&\sqsubseteq \pi; i; \pi; \pi^\sim; \pi^\sim \\
&= \pi; i^\sim; \pi; \pi^\sim; \pi^\sim && \text{partial identity} \\
&\sqsubseteq \pi; \rho; \pi; \rho^\sim; \pi^\sim; \pi; \pi^\sim; \pi^\sim \\
&\sqsubseteq \pi; \rho; \pi; \rho^\sim; \pi^\sim; \pi^\sim && \pi \text{ univalent}
\end{aligned}$$

Together with

$$\begin{aligned}
&\pi^\sim; \mathbf{1}_{L_2} \\
&= \pi^\sim; ((\mathbb{I}_{L_2 \times L_2} \sqcap \pi; \rho^\sim) \times \mathbb{I}_1); \mathbb{I}_{(L_2 \times L_2) \times 1, 1} \\
&= \pi^\sim; (\pi; (\mathbb{I}_{L_2 \times L_2} \sqcap \pi; \rho^\sim); \pi^\sim \sqcap \rho; \rho^\sim); \mathbb{I}_{(L_2 \times L_2) \times 1, 1} \\
&= \pi^\sim; (\pi; \pi^\sim \sqcap \pi; \pi; \rho^\sim; \pi^\sim \sqcap \rho; \rho^\sim); \mathbb{I}_{(L_2 \times L_2) \times 1, 1} && \text{Lemma 1(2)} \\
&= \pi^\sim; (\mathbb{I}_{(L_2 \times L_2) \times 1} \sqcap \pi; \pi; \rho^\sim; \pi^\sim); \mathbb{I}_{(L_2 \times L_2) \times 1, 1} \\
&= (\pi^\sim \sqcap \pi; \rho^\sim; \pi^\sim); \mathbb{I}_{(L_2 \times L_2) \times 1, 1} && \text{Lemma 1(3)} \\
&= (\mathbb{I}_{L_2 \times L_2} \sqcap \pi; \rho^\sim); \pi^\sim; \mathbb{I}_{(L_2 \times L_2) \times 1, 1} && \text{Lemma 1(2)} \\
&= (\mathbb{I}_{L_2 \times L_2} \sqcap \pi; \rho^\sim); \mathbb{I}_{L_2 \times L_2, 1} \\
&= (\rho \sqcap \pi); \mathbb{I}_{L_2 \times L_2, 1} && \text{Lemma 1(1)}
\end{aligned}$$

we obtain

$$\begin{aligned}
& (\mathbb{I}_{(L_1 \times L_2) \times S} \times \mathbf{1}_{L_2}); \alpha; F(i, \pi) \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; \rho; \pi; \rho^\sim; \pi^\sim; \pi^\sim \sqcap \pi; i; \rho; \pi^\sim; \mathbf{1}_{L_2}; \rho^\sim)^\sim \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; \rho; \pi; \rho^\sim; \pi^\sim; \pi^\sim \sqcap \pi; i; \rho; (\rho \sqcap \pi); \mathbb{I})^\sim \quad \rho \text{ total} \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; (\rho; \pi; \rho^\sim; \pi^\sim; \pi^\sim \sqcap i; \rho; (\rho \sqcap \pi); \mathbb{I}))^\sim \quad \text{Lem. 1(2)} \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; (\rho; \pi; \rho^\sim; \pi^\sim; \pi^\sim \sqcap i; \rho; (\rho \sqcap \pi); \mathbb{I}))^\sim \quad \text{Lem. 1(4)} \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; (\rho; \pi \sqcap i; \rho; (\rho \sqcap \pi); \mathbb{I}); \rho^\sim; \pi^\sim; \pi^\sim)^\sim \quad \text{Lem. 1(1)} \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; (\rho; \pi \sqcap i; (\rho; \rho \sqcap \rho; \pi); \mathbb{I}); \rho^\sim; \pi^\sim; \pi^\sim)^\sim \quad \text{Lem. 1(2)} \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; (\rho; \pi \sqcap (\rho; \rho \sqcap i; \rho; \pi); \mathbb{I}); \rho^\sim; \pi^\sim; \pi^\sim)^\sim \quad \text{Lem. 1(4)} \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; (\rho; \pi \sqcap (\rho; \rho \sqcap i; \rho; \pi); \mathbb{I}); \rho^\sim; \pi^\sim; \pi^\sim)^\sim
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \pi; F(\langle \mathbb{I}_{L_1 \times L_2}, \langle \rho, \rho \rangle \rangle; i; \mathbb{I}_S) \\
&= \pi; ((\langle \mathbb{I}_{L_1 \times L_2}, \langle \rho, \rho \rangle \rangle; i) \times \mathbb{I}_S) \\
&= \pi; ((\pi^\sim \sqcap \langle \rho, \rho \rangle; \rho^\sim); i) \times \mathbb{I}_S \\
&= \pi; (\pi; (\pi^\sim \sqcap \langle \rho, \rho \rangle; \rho^\sim); i; \pi^\sim \sqcap \rho; \rho^\sim) \\
&= \pi; (\pi; \pi^\sim; i; \pi^\sim \sqcap \pi; \langle \rho, \rho \rangle; \rho^\sim; i; \pi^\sim \sqcap \rho; \rho^\sim) \quad \text{Lemma 1(2)} \\
&= \pi; (((\pi^\sim; i) \times \mathbb{I}_S) \sqcap \pi; \langle \rho, \rho \rangle; \rho^\sim; i; \pi^\sim) \\
&= \pi; ((\pi^\sim; i) \times \mathbb{I}_S) \sqcap \pi; \pi; \langle \rho, \rho \rangle; \rho^\sim; i; \pi^\sim \quad \text{Lemma 1(2)} \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; \rho; \langle \rho, \rho \rangle^\sim; \pi^\sim; \pi^\sim)^\sim \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; \rho; (\pi; \rho^\sim \sqcap \rho; \rho^\sim); \pi^\sim; \pi^\sim)^\sim \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; (\rho; \pi \sqcap \rho; \rho); \rho^\sim; \pi^\sim; \pi^\sim)^\sim \quad \text{Lemma 1(2)} \\
&= (((i; \pi) \times \mathbb{I}_S); \pi^\sim \sqcap \pi; i; (i; \rho; \pi \sqcap \rho; \rho); \rho^\sim; \pi^\sim; \pi^\sim)^\sim. \quad \text{Lemma 1(4)}
\end{aligned}$$

It remains to show that $\rho; \pi \sqcap (\rho; \rho \sqcap i; \rho; \pi); \mathbb{I} = i; \rho; \pi \sqcap \rho; \rho$. First, we have

$$\begin{aligned}
i; \rho; \pi &= (\mathbb{I}_{(L_1 \times L_2) \times (L_2 \times L_3)} \sqcap \pi; \rho; \pi^\sim; \rho^\sim); \rho; \pi \\
&= (\rho \sqcap \pi; \rho; \pi^\sim); \pi \quad \text{Lemma 1(3)} \\
&= \rho; \pi \sqcap \pi; \rho \quad \text{Lemma 1(3)}
\end{aligned}$$

so that

$$\begin{aligned}
\rho; \pi \sqcap (\rho; \rho \sqcap i; \rho; \pi); \mathbb{I} &= \rho; \pi \sqcap (\rho; \rho \sqcap \rho; \pi \sqcap \pi; \rho); \mathbb{I} \\
&= \rho; \rho \sqcap \rho; \pi \sqcap \pi; \rho \quad \text{Lemma 1(5)} \\
&= \rho; \rho \sqcap i; \rho; \pi
\end{aligned}$$

follows. This completes the proof. $\square$

Notice that the specific α of the previous lemma satisfies even stronger properties. For example, α is a natural transformation in all four parameters and each individual α is an isomorphism.

We model hiding (as well as relabelling) by a partial function $h : L_1 \rightarrow L_2$ on labels, and we define $P \setminus h := P; F(h, \mathbb{I}_S)$ for $P : S \rightarrow F(L, S)$. Notice that this operation is defined by lifting the identity relator using $F(h, \mathbb{I})$. It remains to show that this is a lax natural transformation from $F(L_1, \cdot)$ to $F(L_2, \cdot)$, which follows immediately from

$$F(\mathbb{I}_{L_1}, Q); F(h, \mathbb{I}_{S_1}) = F(h, Q) = F(h, \mathbb{I}_{S_2}); F(\mathbb{I}_{L_2}, Q).$$

Suppose $P_1 : S_1 \rightarrow F(L_1 \times L_2, S)$ and $P_2 : S_2 \rightarrow F(L_2 \times L_3, S_2)$ are representatives of processes in $\text{SProc}(F)$. Its parallel composition $P_1 \parallel P_2 : S_1 \times S_2 \rightarrow F((L_1 \times L_2) \times (L_2 \times L_3), S_1 \times S_2)$ still omits the internal channels from L_2 . The partial function $\text{com} : (L_1 \times L_2) \times (L_2 \times L_3) \rightarrow L_1 \times L_3$ hides those channels and allows an action in the composition if the internal channels match. We define $P_1 \bullet P_2 := (P_1 \parallel P_2) \setminus \text{com}$. Since composition is defined in terms of parallel composition and hiding it is a monotone operation between the corresponding lower semi-lattices.

Lemma 8. *The processes $P_1 \bullet (P_2 \bullet P_3)$ and $(P_1 \bullet P_2) \bullet P_3$ are bisimilar.*

Proof. First of all, we have

$$\begin{aligned} & P_1 \bullet (P_2 \bullet P_3) \\ &= (P_1 \times ((P_2 \times P_3); \alpha; F(\text{com}, \mathbb{I}))); \alpha; F(\text{com}, \mathbb{I}) \\ &= (P_1 \times (P_2 \times P_3)); (\mathbb{I} \times \alpha); (\mathbb{I} \times F(\text{com}, \mathbb{I})); \alpha; F(\text{com}; \mathbb{I}) \\ &= (P_1 \times (P_2 \times P_3)); (\mathbb{I} \times \alpha); (F(\mathbb{I}, \mathbb{I}) \times F(\text{com}, \mathbb{I})); \alpha; F(\text{com}; \mathbb{I}) \\ &= (P_1 \times (P_2 \times P_3)); (\mathbb{I} \times \alpha); \alpha; F(\mathbb{I} \times \text{com}, \mathbb{I}); F(\text{com}; \mathbb{I}) \quad \text{by } (\alpha 2) \\ &= (P_1 \times (P_2 \times P_3)); (\mathbb{I} \times \alpha); \alpha; F((\mathbb{I} \times \text{com}); \text{com}, \mathbb{I}) \end{aligned}$$

and analogously $(P_1 \bullet P_2) \bullet P_3 = ((P_1 \times P_2) \times P_3); (\alpha \times \mathbb{I}); \alpha; F((\text{com} \times \mathbb{I}); \text{com}, \mathbb{I})$. This implies

$$\begin{aligned} & \text{ass}; (P_1 \bullet (P_2 \bullet P_3)) \\ &= \text{ass}; (P_1 \times (P_2 \times P_3)); (\mathbb{I} \times \alpha); \alpha; F((\mathbb{I} \times \text{com}); \text{com}, \mathbb{I}) \\ &= ((P_1 \times P_2) \times P_3); \text{ass}; (\mathbb{I} \times \alpha); \alpha; F((\mathbb{I} \times \text{com}); \text{com}, \mathbb{I}) \quad \text{ass nat. trans.} \\ &= ((P_1 \times P_2) \times P_3); (\alpha \times \mathbb{I}); \alpha; F(\text{ass}, \text{ass}); F((\mathbb{I} \times \text{com}); \text{com}, \mathbb{I}) \quad \text{by } (\alpha 3) \\ &= ((P_1 \times P_2) \times P_3); (\alpha \times \mathbb{I}); \alpha; F(\text{ass}; (\mathbb{I} \times \text{com}); \text{com}, \text{ass}) \\ &= ((P_1 \times P_2) \times P_3); (\alpha \times \mathbb{I}); \alpha; F((\text{com} \times \mathbb{I}); \text{com}, \text{ass}) \quad \text{Lemma 2(1)} \\ &= ((P_1 \times P_2) \times P_3); (\alpha \times \mathbb{I}); \alpha; F((\text{com} \times \mathbb{I}); \text{com}, \mathbb{I}); F(\mathbb{I}, \text{ass}) \\ &= ((P_1 \bullet P_2) \bullet P_3); F(\mathbb{I}, \text{ass}). \end{aligned}$$

From the fact that ass is an isomorphism we get

$$\begin{aligned}
& \text{ass}^\sim; ((P_1 \mathbin{\dot{;}} P_2) \mathbin{\dot{;}} P_3) \\
&= \text{ass}^\sim; ((P_1 \mathbin{\dot{;}} P_2) \mathbin{\dot{;}} P_3); F(\mathbb{I}, \text{ass}; \text{ass}^\sim) && \text{ass isomorphism} \\
&= \text{ass}^\sim; ((P_1 \mathbin{\dot{;}} P_2) \mathbin{\dot{;}} P_3); F(\mathbb{I}, \text{ass}); F(\mathbb{I}, \text{ass}^\sim) \\
&= \text{ass}^\sim; \text{ass}; (P_1 \mathbin{\dot{;}} (P_2 \mathbin{\dot{;}} P_3)); F(\mathbb{I}, \text{ass}^\sim) && \text{see above} \\
&= (P_1 \mathbin{\dot{;}} (P_2 \mathbin{\dot{;}} P_3)); F(\mathbb{I}, \text{ass}^\sim) && \text{ass isomorphism}
\end{aligned}$$

verifying that ass is a bisimulation between the processes $(P_1 \mathbin{\dot{;}} P_2) \mathbin{\dot{;}} P_3$ and $P_1 \mathbin{\dot{;}} (P_2 \mathbin{\dot{;}} P_3)$, which is total and surjective, of course. $\square$

As already mentioned above the process $\mathbf{1} : 1 \rightarrow F(L \times L, 1)$ defined by $\mathbf{1} := \mathbb{I}; F(\mathbb{I} \sqcap \pi; \rho^\sim, \mathbb{I})$ is the identity element for composition.

Lemma 9. $P \mathbin{\dot{;}} \mathbf{1}$ and $\mathbf{1} \mathbin{\dot{;}} P$ are bisimilar to P .

Proof. We are going to show that $\pi : S \times 1 \rightarrow S$ is a bisimulation between $P \mathbin{\dot{;}} \mathbf{1}$ and P . Since 1 is a unit π is an isomorphism, and, hence, total and surjective. The first inclusion follows from

$$\begin{aligned}
\pi; P &= \pi; P; F(\langle \mathbb{I}, \langle \rho, \rho \rangle \rangle; \text{com}, \mathbb{I}) && \text{Lemma 2(2)} \\
&= \pi; P; F(\langle \mathbb{I}, \langle \rho, \rho \rangle \rangle; i; \text{com}, \mathbb{I}) && \text{Lemma 1(4)} \\
&= \pi; P; F(\langle \mathbb{I}, \langle \rho, \rho \rangle \rangle; i, \mathbb{I}); F(\text{com}, \mathbb{I}) \\
&= (P \times \mathbb{I}); \pi; F(\langle \mathbb{I}, \langle \rho, \rho \rangle \rangle; i, \mathbb{I}); F(\text{com}, \mathbb{I}) \\
&= (P \times \mathbb{I}); (\mathbb{I} \times \mathbf{1}); \alpha; F(i, \pi); F(\text{com}, \mathbb{I}) && \text{by } (\alpha 4a) \\
&= (P \times \mathbf{1}); \alpha; F(i; \text{com}, \mathbb{I}); F(\mathbb{I}, \pi) \\
&= (P \times \mathbf{1}); \alpha; F(\text{com}, \mathbb{I}); F(\mathbb{I}, \pi) && \text{Lemma 1(4)} \\
&= (P \mathbin{\dot{;}} \mathbf{1}); F(\mathbb{I}, \pi).
\end{aligned}$$

For the second inclusion consider

$$\begin{aligned}
\pi^\sim; (P \mathbin{\dot{;}} \mathbf{1}) &= \pi^\sim; (P \mathbin{\dot{;}} \mathbf{1}); F(\mathbb{I}, \pi; \pi^\sim) && \pi \text{ isomorphism} \\
&= \pi^\sim; (P \mathbin{\dot{;}} \mathbf{1}); F(\mathbb{I}, \pi); F(\mathbb{I}, \pi^\sim) \\
&= \pi^\sim; \pi; P; F(\mathbb{I}, \pi^\sim) && \text{see above} \\
&= P; F(\mathbb{I}, \pi^\sim). && \pi \text{ isomorphism}
\end{aligned}$$

The fact that $\mathbf{1}$ is also a left identity element is shown analogously using $(\alpha 4b)$ and Lemma 2(3). $\square$

From previous two lemmas we get the main result of this paper as a corollary.

Corollary 3. $\text{SProc}(F)$ is an ordered category.

5 Future Work

As already mentioned in the introduction we see this paper as a starting point of a detailed investigation of the structure of $\text{SProc}(F)$. Additional relational operations such as converse and union will be defined, and we are going to compare their properties with the axioms of an allegory. In addition we want to study time related operations such as the unit delay operation.

Corresponding to guarded processes we are going to study suitable notions of guarded relators on $\text{SProc}(F)$. The existence and the uniqueness of fixed points of such a relator is of special interest for recursively defined processes.

Last but not least, we want to use $\text{SProc}(F)$ to define a denotational semantics for a synchronous version of CCS.

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