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Weak Relational Products

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Abstract. The existence of relational products in categories of relations is strongly connected with the representability of that category. In this paper we propose a canonical weakening of the notion of a relational product. Unlike the strong version any (small) category of relations can be embedded into a suitable category providing all weak relational products. Furthermore, we investigate the categorical properties of the new construction and proof several (weak) versions of propositions well-known for relational products.

1 Introduction

The relational product of two objects A and B in a category of relations is an abstract version of the cartesian product of two sets. It is characterized by an object $A \times B$ together with two projections π and ρ from $A \times B$ to A and B , respectively. A category of relations may provide a relational product for every pair of objects. In this case, it can be shown that the category is representable, i.e. there is an embedding into the category **Rel** of sets and relations. On the other hand, not every reasonable category of relations is representable. This indicates, that one cannot always embed the given structure into a category that provides relational products. This is a mayor disadvantage of this construction since products are usually needed to model certain concepts by relations such as programming languages and most kinds of logics. Other constructions usually required such as sums and powers, i.e. the counterparts of disjoint unions and powersets, can always be created.

In this paper we propose a canonical weakening of the concept of a relational product, the weak relational product. This will be done within the theory of allegories - a categorical model of relations. We will investigate certain properties of the new construction and compare them to those of relational products. In particular, we are interested in the following list of properties. Notice, that those properties are not necessarily independent.

- *Product in $\text{MAP}(\mathcal{R})$:* The given construction may establish a product in the subcategory of mappings (in the sense of category theory). If valid, this

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properties ensures that the corresponding concept is suitable as an abstract version of cartesian products of sets. Therefore, it is essential for any notion of products.

- *$\mathcal{R}$ representable*: A category of relations might be representable, i.e. the morphisms are (up to a suitable mapping) concrete relations between sets. There are non-representable allegories. The existence of **all** possible products may force the allegory to be representable.
- *Unsharpness property*: The unsharpness property is a violation of an equality in terms of relational products. It was claimed that this property may be important to model certain behavior of concurrent processes.
- *Embedding property*: It might be possible to embed a given allegory into another allegory providing all products of a certain kind. With this property we refer to whether this can be always done.
- *Equational theory*: Since the theory of allegories and several of its extensions are/can be defined as an equational theory it is interesting whether a given concept of products can also be expressed by equations.

In the following table we have summarized the validity of the properties above within the concepts of relational and weak relational products.

Property	Relational Product	Weak Relational Product
Product in $\text{MAP}(\mathcal{R})$	+	+
$\mathcal{R}$ representable	+	-
Unsharpness property	-	+
Embedding property	-	+
Equational theory	+	-/+*

* can be defined by equations in division allegories where all partial identities split

Fig. 1. Properties of relational products

In addition the properties of Table 1 we are going to prove several (weak) versions of propositions well-known for relational products.

2 Relational preliminaries

Throughout this paper, we use the following notation. To indicate that a morphism R of a category $\mathcal{R}$ has source A and target B we write $R : A \rightarrow B$. The collection of all morphisms $R : A \rightarrow B$ is denoted by $\mathcal{R}[A, B]$ and the composition of a morphism $R : A \rightarrow B$ followed by a morphism $S : B \rightarrow C$ by $R; S$. Last but not least, the identity morphism on A is denoted by $\mathbb{I}_A$.

We recall briefly some fundamentals on allegories [4] and relational constructions within them. For further details we refer to [4, 8, 9]. Furthermore, we assume that the reader is familiar with the basic notions from category theory such as products and co-products. For unexplained material we refer to [1].

Definition 1. An allegory $\mathcal{R}$ is a category satisfying the following:

1. For all objects A and B the collection $\mathcal{R}[A, B]$ is a meet semi-lattice, whose elements are called relations. Meet and the induced ordering are denoted by $\sqcap$ and $\sqsubseteq$, respectively.
2. There is a monotone operation $\smile$ (called conversion) such that for all relations $Q : A \rightarrow B$ and $R : B \rightarrow C$ the following holds: $(Q; R)^\smile = R^\smile; Q^\smile$ and $(Q^\smile)^\smile = Q$.
3. For all relations $Q : A \rightarrow B$, $R, S : B \rightarrow C$ we have $Q; (R \sqcap S) \sqsubseteq Q; R \sqcap Q; S$.
4. For all relations $Q : A \rightarrow B$, $R : B \rightarrow C$ and $S : A \rightarrow C$ the modular law $Q; R \sqcap S \sqsubseteq Q; (R \sqcap Q^\smile; S)$ holds.

An allegory is called a distributive allegory if

5. the collection $\mathcal{R}[A, B]$ is a distributive lattice with a least element. Join and the least element are denoted by $\sqcup$ and $\perp_{AB}$, respectively.
6. For all relations $Q : A \rightarrow B$ and objects C we have $Q; \perp_{BC} = \perp_{AC}$.
7. For all relations $Q : A \rightarrow B$, $R, S : B \rightarrow C$ we have $Q; (R \sqcup S) = Q; R \sqcup Q; S$.

A distributive allegory is called locally complete iff each $\mathcal{R}[A, B]$ is a complete lattice. Finally, a division allegory is a distributive allegory with a binary operation $/$ satisfying the following:

8. For all relations $R : B \rightarrow C$ and $S : A \rightarrow C$ there is a relation $S/R : A \rightarrow B$ (called the left residual of S and R) such that for all $Q : A \rightarrow B$ the following holds: $Q; R \sqsubseteq S \iff Q \sqsubseteq S/R$.

If $\mathcal{R}[A, B]$ has a greatest element it is denoted by $\top_{AB}$.

Notice, that allegories and distributive allegories are defined by equations. The same can be done for division allegories [4].

The left residual can be used to define another residual operation $Q \backslash S := (S^\smile / Q^\smile)^\smile$, called the right residual of S and Q . A symmetric version, called the symmetric quotient, of the residuals may be defined as

$$\text{syq}(Q, R) := (Q \backslash R) \sqcap (Q^\smile / R^\smile).$$

For further properties of relations in allegories we refer to [4, 8, 9].

In the next lemma we have summarized some properties of the residuals and the symmetric quotient. Proofs can be found in [4, 3, 8, 9].

Lemma 1. Let $\mathcal{R}$ be a division allegory and $Q : A \rightarrow B$, $R : A \rightarrow C$, $S : A \rightarrow D$ be relations, and $f : D \rightarrow A$ be a mapping. Then we have

1. $Q; (Q \backslash R) \sqsubseteq R$,
2. $f; \text{syq}(Q, R) = \text{syq}(Q; f^\smile, R)$,
3. $\text{syq}(Q, R)^\smile = \text{syq}(R, Q)$,
4. $\text{syq}(Q, R); \text{syq}(R, S) \sqsubseteq \text{syq}(Q, S)$,
5. if $\text{syq}(Q, R)$ is total then equality holds in 4.,

6. if $\text{syq}(Q, R)$ is surjective then $Q; \text{syq}(Q, R) = R$.

An important class of relations is given by mappings.

Definition 2. Let $Q : A \rightarrow B$ be a relation. Then we call

1. Q univalent (or functional) iff $Q^\smile; Q \sqsubseteq \mathbb{I}_B$,
2. Q total iff $\mathbb{I}_A \sqsubseteq Q; Q^\smile$,
3. Q a map iff Q is univalent and total,
4. Q injective iff $Q^\smile$ is univalent,
5. Q surjective iff $Q^\smile$ is total.

In the next lemma we have collected several properties of univalent relations used in this paper. A proof can be found in [8, 9].

Lemma 2. Let $\mathcal{R}$ be an allegory so that Π_{AA} exists, $Q : A \rightarrow B$ be univalent, $P : A \rightarrow B$, $R, S : B \rightarrow C$, $T : C \rightarrow B$ and $U : C \rightarrow A$. Then we have

1. $Q; (R \sqcap S) = Q; R \sqcap Q; S$,
2. $(T; Q^\smile \sqcap U); Q = T \sqcap U; Q$,
3. $\Pi_{AA}(Q^\smile \sqcap P) \sqcap Q^\smile = Q^\smile \sqcap P$.

The collection of all mappings of a division allegory $\mathcal{R}$ constitutes a subcategory and is denoted by $\text{MAP}(\mathcal{R})$.

The subcategory of mappings may provide products (in the sense of category theory) for certain objects. As mentioned in the introduction any useful concept of products should establish a categorical product in $\text{MAP}(\mathcal{R})$. Notice, that $\mathcal{R}$ itself may have categorical products. But, contrary to the products in $\text{MAP}(\mathcal{R})$, those products are not suitable to provide an abstract description of pairs. Any allegory is self-dual by the converse operation $\smile$, which implies that products and co-products coincide. Therefore, they are called bi-products, and they are related to the relational sums defined below, which constitutes the abstract counterpart of a disjoint union.

Definition 3. Let $\{A_i \mid i \in I\}$ be a set of objects of a locally complete distributive allegory indexed by some set I . An object $\sum_{i \in I} A_i$, together with relations $\iota_j \in \mathcal{R}[A_j, \sum_{i \in I} A_i]$ for all $j \in I$, is called a relational sum of $\{A_i \mid i \in I\}$ iff for all $i, j \in I$ with $i \neq j$ the following holds

$$\iota_i; \iota_i^\smile = \mathbb{I}_{A_i}, \quad \iota_i; \iota_j^\smile = \perp_{A_i A_j}, \quad \bigsqcup_{i \in I} (\iota_i^\smile; \iota_i) = \mathbb{I}_{\sum_{i \in I} A_i}.$$

$\mathcal{R}$ has (binary) relational sums iff for every (pair) set of objects the relational sum does exist.

The relational sum is a categorical product and co-product, and hence, unique up to isomorphism. In **Rel** the relational sum is given by the disjoint union of sets and the corresponding injection functions.

Definition 4. Let $Q : A \rightarrow A$ be a symmetric idempotent relation, i.e., $Q^\sim = Q$ and $Q; Q = Q$. An object B together with a relation $R : B \rightarrow A$ is called a *splitting of Q* (or R splits Q) iff $R; R^\sim = \mathbb{I}_B$ and $R^\sim; R = Q$.

In **Rel** the splitting of Q is given by the set of existing equivalence classes, and R relates each equivalence class to its elements. A splitting is unique up to isomorphism.

The last construction we want to introduce is the abstract counterpart of a power set - the relational power.

Definition 5. Let $\mathcal{R}$ be a division allegory. An object $\mathcal{P}(A)$, together with a relation $\varepsilon : A \rightarrow \mathcal{P}(A)$ is called a *relational power of A* iff

$$\text{syq}(\varepsilon, \varepsilon) \sqsubseteq \mathbb{I}_{\mathcal{P}(A)} \quad \text{and} \quad \text{syq}(R, \varepsilon) \text{ is total}$$

for all relation $R : B \rightarrow A$. If the relational power does exist for any object $\mathcal{R}$ is called a *power allegory*.

Notice, that $\text{syq}(\varepsilon, \varepsilon) = \mathbb{I}_{\mathcal{P}(A)}$, and that $\text{syq}(R, \varepsilon)$ is, in fact, a mapping. In **Rel** the relation $e_A := \text{syq}(\mathbb{I}_A, \varepsilon) : A \rightarrow \mathcal{P}(A)$ maps each element to the singleton set containing that element. This relation is always (in all allegories) an injective mapping (cf. [9]).

Definition 6. An allegory is called *systemic complete* iff it is a power allegory that has relational sums, and in which all symmetric idempotent relations split.

The univalent part $\text{unp}(R)$ of a relation R was introduced in [8] in the context of (heterogeneous) relation algebras, i.e. division allegories where the order structure is a complete atomic Boolean algebra.

Definition 7. Let $\mathcal{R}$ be a division allegory, and let be $R : A \rightarrow B$ in $\mathcal{R}$. The *univalent part of R* is defined by $\text{unp}(R) := R \sqcap (R^\sim \setminus \mathbb{I}_B)$.

The following lemma was already proved in [8]. The proof provided there makes use of complements, which are not available in an arbitrary division allegory. Here we provide a complement free proof.

Lemma 3. Let $\mathcal{R}$ be a division allegory and $R : A \rightarrow B$. Then we have

1. $\text{unp}(R)$ is univalent and included in R ,
2. $\text{unp}(\text{unp}(R)) = \text{unp}(R)$,
3. R is univalent iff $\text{unp}(R) = R$.

Proof. 1. The second assertion is obvious, and the first is shown by

$$\begin{aligned} \text{unp}(R)^\sim; \text{unp}(R) &= (R \sqcap (R^\sim \setminus \mathbb{I}_B))^\sim; (R \sqcap (R^\sim \setminus \mathbb{I}_B)) \\ &\sqsubseteq R^\sim; (R^\sim \setminus \mathbb{I}_B) \\ &\sqsubseteq \mathbb{I}_B. \end{aligned}$$

2. The inclusion ' $\sqsubseteq$ ' is obvious, and $\text{unp}(R)^\sim; (R^\sim \setminus \mathbb{I}_B) \sqsubseteq R^\sim; (R^\sim \setminus \mathbb{I}_B) \sqsubseteq \mathbb{I}_B$ implies $(R^\sim \setminus \mathbb{I}_B) \sqsubseteq (\text{unp}(R)^\sim \setminus \mathbb{I}_B)$ and hence

$$\text{unp}(R) = \text{unp}(R) \cap (R^\sim \setminus \mathbb{I}_B) \sqsubseteq \text{unp}(R) \cap (\text{unp}(R)^\sim \setminus \mathbb{I}_B) = \text{unp}(\text{unp}(R)).$$

3. This follows immediately from

$$\begin{aligned} R^\sim; R \sqsubseteq \mathbb{I}_B &\Leftrightarrow R \sqsubseteq (\text{unp}(R)^\sim \setminus \mathbb{I}_B) \\ &\Leftrightarrow R \cap (\text{unp}(R)^\sim \setminus \mathbb{I}_B) = R \\ &\Leftrightarrow \text{unp}(R) = R. \end{aligned} \quad \square$$

3 Weak Relational Products

A relational product, as defined in the introduction, is also a categorical product in the subcategory of mappings but not necessarily vice versa. The equation $\pi^\sim; \rho = \Pi_{AB}$ may not be valid. This equation states that the greatest relation is tabular [4]. We will weaken this axiom by requiring that each tabulation is included in this relation.

Definition 8 (Weak relational product). *Let $\mathcal{R}$ be an allegory and A and B objects of $\mathcal{R}$. An object $A \times B$ together with two relations $\pi : A \times B \rightarrow A$ and $\rho : A \times B \rightarrow B$ is called a weak relational product iff*

- (P1) $\pi^\sim; \pi \sqsubseteq \mathbb{I}_A$,
- (P2) $\rho^\sim; \rho \sqsubseteq \mathbb{I}_B$,
- (P3) $\pi; \pi^\sim \cap \rho; \rho^\sim = \mathbb{I}_{A \times B}$,
- (P4) $f^\sim; g \sqsubseteq \pi^\sim; \rho$ for all mappings $f : C \rightarrow A$ and $g : C \rightarrow B$.

$\mathcal{R}$ is called a weak pairing allegory iff the weak relational product for each pair of objects exists.

Example 1. Consider the concrete allegory with one object $A = \{0, 1\}$ and the four relations $\perp_A := \emptyset, \mathbb{I}_A, \bar{\mathbb{I}}_A := \{(0, 1), (1, 0)\}$ and $\Pi_A := A \times A$. It is easy to verify that this structure establishes indeed an allegory with exactly two mappings $\mathbb{I}_A$ and $\bar{\mathbb{I}}_A$. It is well-known that the matrices with entries of a (complete) allegory form an allegory. Mappings in our example are matrices with exactly one entry $\mathbb{I}_A$ or $\bar{\mathbb{I}}_A$ in each row and $\perp_A$ otherwise. The pair

$$\pi := \begin{pmatrix} \mathbb{I}_A & \perp_A \\ \bar{\mathbb{I}}_A & \perp_A \\ \perp_A & \mathbb{I}_A \\ \perp_A & \bar{\mathbb{I}}_A \end{pmatrix} \quad \rho := \begin{pmatrix} \mathbb{I}_A & \perp_A \\ \perp_A & \mathbb{I}_A \\ \bar{\mathbb{I}}_A & \perp_A \\ \perp_A & \bar{\mathbb{I}}_A \end{pmatrix}$$

establishes a weak relational product. Notice, that if we replace $\mathbb{I}_A$ by the greatest element we obtain the well-known matrix representation of the projections (cf. [8]).

The weak version still establishes a categorical product in $\text{MAP}(\mathcal{R})$, and is, therefore, unique (up to isomorphism).

Theorem 1. *Let $\mathcal{R}$ be an allegory. Then a weak relational product $(A \times B, \pi, \rho)$ is a categorical product of A and B in $\text{MAP}(\mathcal{R})$.*

Proof. Let $(A \times B, \pi, \rho)$ be a weak relational product. By P1, P2 and P3 the relations π and ρ are mappings, and hence in $\text{MAP}(\mathcal{R})$. Let $f : C \rightarrow A$ and $g : C \rightarrow B$ be mappings. Then we have

$$\begin{aligned} (f; \pi^\smile \sqcap g; \rho^\smile); \rho &= f; \pi^\smile; \rho \sqcap g && \text{Lemma 2(2)} \\ &= g \end{aligned}$$

where the last equality follows from

$$\begin{aligned} g &\sqsubseteq f; f^\smile; g && f \text{ is total,} \\ &\sqsubseteq f; \pi^\smile; \rho && \text{Axiom P4.} \end{aligned}$$

The equality $(f; \pi^\smile \sqcap g; \rho^\smile); \pi = f$ is shown analogously. Furthermore, the following computation shows that $f; \pi^\smile \sqcap g; \rho^\smile$ is a mapping, and hence an element of $\text{MAP}(\mathcal{R})$

$$\begin{aligned} &(f; \pi^\smile \sqcap g; \rho^\smile)^\smile; (f; \pi^\smile \sqcap g; \rho^\smile) \\ &= (\pi; f^\smile \sqcap \rho; g^\smile); (f; \pi^\smile \sqcap g; \rho^\smile) \\ &\sqsubseteq \pi; f^\smile; f; \pi^\smile \sqcap \rho; g^\smile; g; \rho^\smile \\ &\sqsubseteq \pi; \pi^\smile \sqcap \rho; \rho^\smile && f \text{ and } g \text{ are univalent,} \\ &= \mathbb{I}_{A \times B} && \text{Axiom P3} \\ &(f; \pi^\smile \sqcap g; \rho^\smile); (f; \pi^\smile \sqcap g; \rho^\smile)^\smile \\ &= (f; \pi^\smile \sqcap g; \rho^\smile); (\pi; f^\smile \sqcap \rho; g^\smile) \\ &= (f; \pi^\smile \sqcap g; \rho^\smile); \pi; f^\smile \sqcap (f; \pi^\smile \sqcap g; \rho^\smile); \rho; g^\smile && \text{Lemma 2(1)} \\ &= f; f^\smile \sqcap g; g^\smile && \text{previous computation} \\ &\sqsubseteq \mathbb{I}_C && f \text{ and } g \text{ are total.} \end{aligned}$$

Last but not least, let h be a mapping with $h; \pi = f$ and $h; \rho = g$. Then we conclude

$$\begin{aligned} f; \pi^\smile \sqcap g; \rho^\smile &= h; \pi; \pi^\smile \sqcap h; \rho; \rho^\smile \\ &= h; (\pi; \pi^\smile \sqcap \rho; \rho^\smile) && \text{Lemma 2(1)} \\ &= h && \text{Axiom P3} \end{aligned}$$

This completes the proof. $\square$

Notice, that we have also shown that for a weak relational product the product morphism induced by the mappings $f : C \rightarrow A$ and $g : C \rightarrow B$, i.e. the

unique mapping $h : C \rightarrow A \times B$ satisfying $h; \pi = f$ and $h; \rho = g$, is actually given by the relation $f; \pi^\smile \sqcap g; \rho^\smile$.

Unfortunately, we are just able to prove parts of the converse implication.

Lemma 4. *Let $\mathcal{R}$ be an allegory. Then a categorical product $(A \times B, \pi, \rho)$ of A and B in $\text{MAP}(\mathcal{R})$ fulfils the axioms P1, P2, P4 and the inclusion $\sqsubseteq$ of P3.*

Proof. Suppose $(A \times B, \pi, \rho)$ is a categorical product of A and B in $\text{MAP}(\mathcal{R})$. Axioms P1, P2 and the inclusion $\sqsubseteq$ of P3 are trivial since π and ρ are mappings.

Now, let $f : C \rightarrow A$ and $g : C \rightarrow B$ be mappings. Then there is a unique mapping $h : C \rightarrow A \times B$ with $h; \pi = f$ and $h; \rho = g$. We conclude

$$\begin{aligned} g &= h; \rho \\ &\sqsubseteq h; (\pi; \pi^\smile \sqcap \rho; \rho^\smile); \rho && \text{inclusion } \sqsubseteq \text{ of P3} \\ &\sqsubseteq h; \pi; \pi^\smile; \rho \\ &= f; \pi^\smile; \rho, \end{aligned}$$

which implies $f^\smile; g \sqsubseteq f^\smile; f; \pi^\smile; \rho \sqsubseteq \pi^\smile; \rho$ since f is univalent. $\square$

We can prove the remaining inclusion in the case the allegory provides a suitable splitting.

Lemma 5. *Let $\mathcal{R}$ be an allegory, and let $(A \times B, \pi, \rho)$ be a categorical product in $\text{MAP}(\mathcal{R})$. Furthermore, assume that R splits $\pi; \pi^\smile \sqcap \rho; \rho^\smile$. Then $(A \times B, \pi, \rho)$ is a weak relational product.*

Proof. By Lemma 4 it remains to show the inclusion $\sqsubseteq$ of P3. Let $R : C \rightarrow A \times B$ be the splitting of $\pi; \pi^\smile \sqcap \rho; \rho^\smile$, and define $\tilde{\pi} := R; \pi$ and $\tilde{\rho} := R; \rho$. We want to show that $(C, \tilde{\pi}, \tilde{\rho})$ is a weak relational product of A and B . Once verified Lemma 1 implies that $(C, \tilde{\pi}, \tilde{\rho})$ is another categorical product of A and B in $\text{MAP}(\mathcal{R})$, and hence isomorphic to $A \times B$. It is easy to verify (cf. [1]) that the isomorphism is given by the two mapping $h : C \rightarrow A \times B$ and $k : A \times B \rightarrow C$ fulfilling $h; \pi = \tilde{\pi}$, $h; \rho = \tilde{\rho}$, $k; \tilde{\pi} = \pi$ and $k; \tilde{\rho} = \rho$. Theorem 1 also shows $k = \pi; \tilde{\pi}^\smile \sqcap \rho; \tilde{\rho}^\smile$. Furthermore, in [4] it was shown the inverse of an isomorphism in an allegory is its converse so that $h^\smile = k$ follows. We conclude

$$\begin{aligned} \mathbb{I}_{A \times B} &= k; h && \text{pair of isomorphisms} \\ &= (\pi; \tilde{\pi}^\smile \sqcap \rho; \tilde{\rho}^\smile); h \\ &= (\pi; \pi^\smile; h^\smile \sqcap \rho; \rho^\smile; h^\smile); h \\ &= (\pi; \pi^\smile \sqcap \rho; \rho^\smile); h^\smile; h && \text{Lemma 2(1)} \\ &= (\pi; \pi^\smile \sqcap \rho; \rho^\smile); k; h \\ &= \pi; \pi^\smile \sqcap \rho; \rho^\smile \end{aligned}$$

In order to show that $(C, \tilde{\pi}, \tilde{\rho})$ is a weak relational product we derive Axiom P1 from

$$\begin{aligned}
\tilde{\pi}^\smile; \tilde{\pi} &= \pi^\smile; R^\smile; R; \pi \\
&= \pi^\smile; (\pi; \pi^\smile \sqcap \rho; \rho^\smile); \pi \\
&= \mathbb{I}_A \sqcap \pi^\smile; \rho; \rho^\smile; \pi && \text{Lemma 2(2)} \\
&\sqsubseteq \mathbb{I}_A,
\end{aligned}$$

and Axiom P2 analogously. We get the totality of $\tilde{\pi}$ from $\mathbb{I}_C = R; R^\smile \sqsubseteq R; \pi; \pi^\smile; R^\smile = \tilde{\pi}; \tilde{\pi}^\smile$ and for $\tilde{\rho}$ analogously. Together with

$$\begin{aligned}
&\tilde{\pi}; \tilde{\pi}^\smile \sqcap \tilde{\rho}; \tilde{\rho}^\smile \\
&= R; \pi; \pi^\smile; R^\smile \sqcap R; \rho; \rho^\smile; R^\smile \\
&\sqsubseteq R; (\pi; \pi^\smile; R^\smile \sqcap R^\smile; R; \rho; \rho^\smile; R^\smile) \\
&\sqsubseteq R; (\pi; \pi^\smile \sqcap R^\smile; R; \rho; \rho^\smile; R^\smile; R); R^\smile \\
&\sqsubseteq R; (\pi; \pi^\smile \sqcap (\pi; \pi^\smile \sqcap \rho; \rho^\smile); \rho; \rho^\smile; (\pi; \pi^\smile \sqcap \rho; \rho^\smile)); R^\smile \\
&= R; (\pi; \pi^\smile \sqcap (\pi; \pi^\smile; \rho \sqcap \rho^\smile); (\rho^\smile; \pi; \pi^\smile \sqcap \rho^\smile)); R^\smile && \text{Lemma 2(2)} \\
&\sqsubseteq R; (\pi; \pi^\smile \sqcap \rho; \rho^\smile); R^\smile \\
&= R; R^\smile; R; R^\smile \\
&= \mathbb{I}_C
\end{aligned}$$

we have shown Axiom P3. Last but not least, the computation

$$\begin{aligned}
\tilde{\pi}^\smile; \tilde{\rho} &= \pi^\smile; R^\smile; R; \rho \\
&= \pi^\smile; (\pi; \pi^\smile \sqcap \rho; \rho^\smile); \rho \\
&= \pi^\smile; \rho \sqcap \pi^\smile; \rho && \text{Lemma 2(2)} \\
&= \pi^\smile; \rho
\end{aligned}$$

implies Axiom P4. □

Even though every allegory can be embedded into one providing the splitting required by the last lemma, it is not clear that the given product remains one after embedding.

Now we want to show that for certain allegories the weak relational product can be defined by equations. This seems to be of particular interest because the allegories considered to construct weak relational products (cf. next section) are of that kind.

Lemma 6. *Let $\mathcal{R}$ be a division allegory in which all partial identities split. Then $(A \times B, \pi, \rho)$ is a weak relational product iff the Axioms P1-P3 and*

$$\text{unp}(R)^\smile; \text{unp}(S) \sqsubseteq \pi^\smile; \rho$$

for all relations $R : C \rightarrow A$ and $S : C \rightarrow B$ hold.

Proof. The implication $\Rightarrow$ is trivial since $\text{unp}(f) = f$ for all mappings by Lemma 3 (3). For the converse implication assume P1-P4 and let $R : C \rightarrow A$ and $S : C \rightarrow B$ be arbitrary relations. Now, let $i := \mathbb{I}_C \sqcap \text{unp}(R); \text{unp}(R)^\smile \sqcap \text{unp}(S); \text{unp}(S)^\smile$ and $s : D \rightarrow C$ be its splitting. Then $s; \text{unp}(R)$ is univalent since s and R are. Furthermore, this relation is total because $\mathbb{I}_D = s; s^\smile; s; s^\smile = s; i; s^\smile \sqsubseteq \text{sunp}(R); \text{unp}(R)^\smile; s^\smile$. Analogously, we get that $s; \text{unp}(S)$ is a mapping. Notice, that we have $Q = (\mathbb{I}_A \sqcap Q; Q^\smile); Q$ for arbitrary relations $Q : A \rightarrow B$ and $i; j = i \sqcap j$ for partial identities $i, j : A \rightarrow A$. Proofs of those properties can be found in [4, 8, 9]. We conclude

$$\begin{aligned}
& \text{unp}(R)^\smile; \text{unp}(S) \\
&= \text{unp}(R)^\smile; (\mathbb{I}_D \sqcap \text{unp}(R); \text{unp}(R)^\smile); (\mathbb{I}_D \sqcap \text{unp}(S); \text{unp}(S)^\smile); \text{unp}(S) \\
&= \text{unp}(R)^\smile; (\mathbb{I}_D \sqcap \text{unp}(R); \text{unp}(R)^\smile \sqcap \mathbb{I}_D \sqcap \text{unp}(S); \text{unp}(S)^\smile); \text{unp}(S) \\
&= \text{unp}(R)^\smile; i; \text{unp}(S) \\
&= \text{unp}(R)^\smile; s^\smile; s; \text{unp}(S) \\
&= (s; \text{unp}(R))^\smile; s; \text{unp}(S) \\
&\sqsubseteq \pi^\smile; \rho.
\end{aligned}$$

This completes the proof. $\square$

Example 2. In this example we want to show that, in contrast to relational products, unsharpness may hold for a weak relational product, i.e. there are relations Q, R, S and T with $(Q; \pi^\smile \sqcap R; \rho^\smile); (\pi; S \sqcap \rho; T) \neq Q; S \sqcap R; T$. Our example will show that even under the additional assumption of totality of the relations involved unsharpness is possible. Suppose $\mathcal{R}$ is a weak pairing allegory with a greatest element in $\mathcal{R}[A, B]$ and $\pi^\smile; \rho \neq \mathbb{I}_{AB}$. Then we have

$$\begin{aligned}
(\mathbb{I}_A; \pi^\smile \sqcap \mathbb{I}_{AB}; \rho^\smile); (\pi; \mathbb{I}_{AB} \sqcap \rho; \mathbb{I}_B) &= (\pi^\smile \sqcap \mathbb{I}_{A(A \times B)}); (\mathbb{I}_{(A \times B)B} \sqcap \rho) \\
&= \pi^\smile; \rho \\
&\neq \mathbb{I}_{AB} \\
&= \mathbb{I}_A; \mathbb{I}_{AB} \sqcap \mathbb{I}_{AB}; \mathbb{I}_B.
\end{aligned}$$

One important property of relational products is that one can transform any relation into the abstract counterpart of a set of pairs, i.e. by a vector or a left ideal element $\mathbb{I}_{AA}; v = v$. We want to investigate whether this is also possible for weak relational products. Consider the two operations

$$\tau(R) := \mathbb{I}_{AA}; (\pi^\smile \sqcap R; \rho^\smile) \quad \text{and} \quad \sigma(v) := (\mathbb{I}_{AA}; v \sqcap \pi^\smile); \rho.$$

τ maps relations to vectors and σ vectors to relations.

Lemma 7. *Let $\mathcal{R}$ be a weak pairing allegory with greatest elements, $R : A \rightarrow B$, $v : A \rightarrow A \times B$ be a vector, $Q : C \rightarrow A$ univalent, and $S : C \rightarrow A \times B$. Then we have*

1. $R; \pi^\smile \sqcap (Q; \pi^\smile \sqcap S); \rho; \rho^\smile = Q; \pi^\smile \sqcap S,$
2. $\tau(\sigma(v)) = v,$
3. $\sigma(\tau(R)) \sqsubseteq R$ with $'='$ if $R \sqsubseteq \pi^\smile; \rho.$

Proof. 1.-2. These properties were shown in [8–10] for relational products. The proofs provided there also apply to weak relational products without modifications.

3. Consider the computation

$$\begin{aligned}
\sigma(\tau(R)) &= (\Pi_{AA}; \Pi_{AA}; (\pi^\smile \sqcap R; \rho^\smile) \sqcap \pi^\smile); \rho \\
&= (\Pi_{AA}; (\pi^\smile \sqcap R; \rho^\smile) \sqcap \pi^\smile); \rho \\
&= (\pi^\smile \sqcap R; \rho^\smile); \rho && \text{Lemma 2(3)} \\
&= \pi^\smile; \rho \sqcap R && \text{Lemma 2(2)} \\
&\sqsubseteq R.
\end{aligned}$$

Obviously, we get $'='$ if $R \sqsubseteq \pi^\smile; \rho.$ □

4 Creating Weak Relational Products

The main proposition of this section (Corollary 1) states that any (small) allegory can be embedded into a weak pairing allegory. This theorem is based on the fact that the cartesian product of two power sets can be constructed by the power set of the disjoint union of the sets. An abstract version of this proposition is given in the next lemma and summarized by the following diagram.

Notice, that the constructed weak relational product is not necessarily a relational product [9].

Lemma 8. *Let $\mathcal{R}$ be an allegory, A and B objects of $\mathcal{R}$ so that the relational sum $(A + B, \iota_{AB}, \kappa_{AB})$ and the relational powers $\mathcal{P}(A)$, $\mathcal{P}(B)$ and $\mathcal{P}(A + B)$ exist. Then $(\mathcal{P}(A + B), \text{syq}(\iota_{AB}; \varepsilon_{A+B}, \varepsilon_A), \text{syq}(\kappa_{AB}; \varepsilon_{A+B}, \varepsilon_B))$ is a weak relational product of $\mathcal{P}(A)$ and $\mathcal{P}(B)$.*

Proof. Axiom P1 follows immediately from

$$\begin{aligned}
\pi^\sim; \pi &= \text{syq}(\iota; \varepsilon, \varepsilon)^\sim; \text{syq}(\iota; \varepsilon, \varepsilon) \\
&= \text{syq}(\varepsilon, \iota; \varepsilon); \text{syq}(\iota; \varepsilon, \varepsilon) && \text{Lemma 1(3)} \\
&\sqsubseteq \text{syq}(\varepsilon, \varepsilon) && \text{Lemma 1(4)} \\
&= \mathbb{I}_{\mathcal{P}(A)}.
\end{aligned}$$

Axiom P2 is shown analogously. Since $\text{syq}(\iota; \varepsilon, \varepsilon)$ is total by definition we get

$$\begin{aligned}
\pi; \pi^\sim &= \text{syq}(\iota; \varepsilon, \varepsilon); \text{syq}(\iota; \varepsilon, \varepsilon)^\sim \\
&= \text{syq}(\iota; \varepsilon, \varepsilon); \text{syq}(\varepsilon, \iota; \varepsilon) && \text{Lemma 1(3)} \\
&= \text{syq}(\iota; \varepsilon, \iota; \varepsilon) && \text{Lemma 1(5)}
\end{aligned}$$

and $\rho; \rho^\sim = \text{syq}(\kappa; \varepsilon, \kappa; \varepsilon)$ analogously. Furthermore, we have

$$\begin{aligned}
&\varepsilon; ((\iota; \varepsilon) \setminus (\iota; \varepsilon) \sqcap (\kappa; \varepsilon) \setminus (\kappa; \varepsilon)) \\
&= (\iota^\sim; \iota \sqcup \kappa^\sim; \kappa); \varepsilon; ((\iota; \varepsilon) \setminus (\iota; \varepsilon) \sqcap (\kappa; \varepsilon) \setminus (\kappa; \varepsilon)) \\
&\sqsubseteq \iota^\sim; \iota; \varepsilon; (\iota; \varepsilon) \setminus (\iota; \varepsilon) \sqcup \kappa^\sim; \kappa; \varepsilon; (\kappa; \varepsilon) \setminus (\kappa; \varepsilon) \\
&\sqsubseteq \iota^\sim; \iota; \varepsilon \sqcup \kappa^\sim; \kappa; \varepsilon && \text{Lemma 1(1)} \\
&= (\iota^\sim; \iota \sqcup \kappa^\sim; \kappa); \varepsilon \\
&= \varepsilon
\end{aligned}$$

so that $(\iota; \varepsilon) \setminus (\iota; \varepsilon) \sqcap (\kappa; \varepsilon) \setminus (\kappa; \varepsilon) \sqsubseteq \varepsilon \setminus \varepsilon$ follows. Again, the similar inclusion $(\iota; \varepsilon)^\sim / (\iota; \varepsilon)^\sim \sqcap (\kappa; \varepsilon)^\sim / (\kappa; \varepsilon)^\sim \sqsubseteq \varepsilon^\sim / \varepsilon^\sim$ is shown analogously. Together, we conclude

$$\begin{aligned}
\pi; \pi^\sim \sqcap \rho; \rho^\sim &= \text{syq}(\iota; \varepsilon, \iota; \varepsilon) \sqcap \text{syq}(\kappa; \varepsilon, \kappa; \varepsilon) \\
&= (\iota; \varepsilon) \setminus (\iota; \varepsilon) \sqcap (\iota; \varepsilon)^\sim / (\iota; \varepsilon)^\sim \sqcap (\kappa; \varepsilon) \setminus (\kappa; \varepsilon) \sqcap (\kappa; \varepsilon)^\sim / (\kappa; \varepsilon)^\sim \\
&= \varepsilon \setminus \varepsilon \sqcap \varepsilon^\sim / \varepsilon^\sim \\
&= \text{syq}(\varepsilon, \varepsilon) \\
&= \mathbb{I}_{\mathcal{P}(A+B)}.
\end{aligned}$$

In order to prove P4 let $f : C \rightarrow A$ and $g : C \rightarrow B$ be mappings. The relation $\text{syq}(\iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim, \varepsilon)$ is a mapping by definition, and we have

$$\begin{aligned}
&\text{syq}(\iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim, \varepsilon); \pi \\
&= \text{syq}(\iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim, \varepsilon); \text{syq}(\iota; \varepsilon, \varepsilon) \\
&= \text{syq}(\iota; \varepsilon; \text{syq}(\iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim, \varepsilon)^\sim, \varepsilon) && \text{Lemma 1(2)} \\
&= \text{syq}(\iota; \varepsilon; \text{syq}(\varepsilon, \iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim), \varepsilon) && \text{Lemma 1(3)} \\
&= \text{syq}(\iota; (\iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim), \varepsilon) && \text{Lemma 1(6)} \\
&= \text{syq}(\varepsilon; f^\sim, \varepsilon) \\
&= f; \text{syq}(\varepsilon, \varepsilon) && \text{Lemma 1(2)} \\
&= f.
\end{aligned}$$

$\text{syq}(\iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim, \varepsilon); \rho = g$ is shown analogously. We conclude

$$\begin{aligned}
f^\sim; g &= f^\sim; \text{syq}(\iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim, \varepsilon); \rho \\
&= f^\sim; \text{syq}(\iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim, \varepsilon); (\pi; \pi^\sim \sqcap \rho; \rho^\sim); \rho \\
&\sqsubseteq f^\sim; \text{syq}(\iota^\sim; \varepsilon; f^\sim \sqcup \kappa^\sim; \varepsilon; g^\sim, \varepsilon); \pi; \pi^\sim; \rho \\
&= f^\sim; f; \pi^\sim; \rho \\
&\sqsubseteq \pi^\sim; \rho.
\end{aligned}$$

This completes the proof. $\square$

If the allegory provides splitting of partial identities, the construction described above can be distributed to any pair of objects.

Theorem 2. *Any systemic complete allegory is a weak pairing allegory.*

Proof. Let $\mathcal{R}$ be a systemic complete power allegory, and let A and B be objects of $\mathcal{R}$. By Lemma 8 there is a weak relational product $(\mathcal{P}(A) \times \mathcal{P}(B), \pi, \rho)$ of $\mathcal{P}(A)$ and $\mathcal{P}(B)$. Let be $i := \pi; e_A^\sim; e_A; \pi^\sim \sqcap \rho; e_B^\sim; e_B; \rho^\sim$. The relation i is a partial identity, which is shown as follows

$$\begin{aligned}
i &= \pi; e_A^\sim; e_A; \pi^\sim \sqcap \rho; e_B^\sim; e_B; \rho^\sim \\
&= \pi; \pi^\sim \sqcap \rho; \rho^\sim && e_A \text{ and } e_B \text{ are univalent} \\
&= \mathbb{I}_{\mathcal{P}(A) \times \mathcal{P}(B)}. && \text{Axiom P3}
\end{aligned}$$

Since $\mathcal{R}$ is systemic complete there is an object C and a relation $s : C \rightarrow \mathcal{P}(A) \times \mathcal{P}(B)$ that splits i . Notice, that s is an injective mapping since i is a partial identity. We want to show that C together with the relations $\tilde{\pi} := s; \pi; e_A^\sim$ and $\tilde{\rho} := s; \rho; e_B^\sim$ is a weak relational product of A and B .

Axiom P1 is shown by

$$\begin{aligned}
\tilde{\pi}^\sim; \tilde{\pi} &= (s; \pi; e_A^\sim)^\sim; s; \pi; e_A^\sim \\
&= e_A; \pi^\sim; s^\sim; s; \pi; e_A^\sim \\
&= e_A; \pi^\sim; \pi; e_A^\sim && s \text{ is total and injective} \\
&\sqsubseteq e_A; e_A^\sim && \text{Axiom P1} \\
&= \mathbb{I}_A, && e_A \text{ is total and injective}
\end{aligned}$$

and Axiom P2 follows analogously. The computation

$$\begin{aligned}
\tilde{\pi}; \pi^\sim \sqcap \tilde{\rho}; \tilde{\rho}^\sim &= s; \pi; e_A^\sim; e_A; \pi^\sim; s^\sim \sqcap s; \rho; e_B^\sim; e_B; \rho^\sim; s^\sim \\
&= s; (\pi; e_A^\sim; e_A; \pi^\sim \sqcap \rho; e_B^\sim; e_B; \rho^\sim); s^\sim && \text{Lemma 2(1)} \\
&= s; i; s^\sim \\
&= s; s^\sim; s; s^\sim \\
&= \mathbb{I}_C
\end{aligned}$$

verifies Axiom P3. In order to prove Axiom P4 we first observe

$$\begin{aligned}
\tilde{\pi}^\sim; \tilde{\rho} &= e_A; \pi^\sim; s^\sim; s; \rho; e_B^\sim \\
&= e_A; \pi^\sim; i; \rho; e_B^\sim && s \text{ splits } i \\
&= e_A; \pi^\sim; (\pi; e_A^\sim; e_A; \pi^\sim \sqcap \rho; e_B^\sim; e_B; \rho^\sim); \rho; e_B^\sim \\
&= e_A; (e_A^\sim; e_A; \pi^\sim; \rho \sqcap \pi^\sim; \rho; e_B^\sim; e_B); e_B^\sim && \text{Lemma 2(2)} \\
&= e_A; e_A^\sim; e_A; \pi^\sim; \rho; e_B^\sim \sqcap e_A; \pi^\sim; \rho; e_B^\sim; e_B; e_B^\sim && \text{Lemma 2(2)} \\
&= e_A; \pi^\sim; \rho; e_B^\sim \sqcap e_A; \pi^\sim; \rho; e_B^\sim && e_A, e_B \text{ total and injective} \\
&= e_A; \pi^\sim; \rho; e_B^\sim.
\end{aligned}$$

Now, let $f : D \rightarrow A$ and $g : D \rightarrow B$ be mappings. Then $f; e_A$ and $g; e_B$ are also mappings from C to $\mathcal{P}(A)$ and $\mathcal{P}(B)$, respectively. This implies $e_A^\sim; f^\sim; g; e_B = (f; e_A)^\sim; g; e_B \sqsubseteq \pi^\sim; \rho$. We conclude

$$\begin{aligned}
f^\sim; g &= e_A; e_A^\sim; f^\sim; g; e_B; e_B^\sim && e_A, e_B \text{ total and injective} \\
&\sqsubseteq e_A; \pi^\sim; \rho; e_B^\sim \\
&= \tilde{\pi}^\sim; \tilde{\rho},
\end{aligned}$$

which finally verifies Axiom P4. $\square$

Since the systemic completion of a small allegory is systemic complete ([4] 2.221 and 2.434) we have shown the main proposition of this section.

Corollary 1. *Any small allegory may be faithfully represented in a weak pairing allegory.*

This corollary also shows that there are indeed weak pairing allegories in which the weak relational product is not always a relational product. For example, consider the allegory induced by the non-representable MacKenzie relation algebra. According to Corollary 1 this allegory can be embedded into a weak pairing allegory. This allegory can not have all relational product since then it would be representable [4], which is a contradiction.

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