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Relational semantics through duality

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Abstract. In this paper we show how the classical duality results extended to a Duality via Truth contribute to development of a relational semantics for various modal-like logics. In particular, we present a Duality via Truth for some classes of information algebras and frames. We also show that the full categorical formulation of classical duality extends to a full Duality via Truth.

1 Introduction

In this paper we show how Stone-like or Priestley-like dualities can be split into two parts: one referring to the algebraic aspects and the other to the logical aspects. In this way a relationship between algebraic structures and relational structures (or frames, as they are called in non-classical logics) can be rooted in their common origin as semantic structures of formal languages. We will follow a method called Duality via Truth [OrR05]. This aims to exhibit a relationship between a class of algebras and a class of frames based on their corresponding notions of truth for a formal language. We show that the appropriate elements of Stone-like or Priestley-like dualities can easily be extended to Duality via Truth. Usually, on an algebraic side, we are interested in representation theorems for a class of algebras which involves representing elements of those algebras as subsets of some universal set. On the logical side, we consider a class of frames and prove a completeness of the logic with respect to a class of models determined by those frames. We show that these two approaches can be put together and can be extended to a Duality via Truth which exhibits a principle according to which the two classes of structures are dual.

Given a formal language Lan, a class of frames Frm which determines a frame semantics for Lan, and a class Alg of algebras which determines its algebraic semantics, a Duality via Truth theorem says that these two kinds of semantics are equivalent in the following sense:

DvT A formula $\phi \in \mathsf{Lan}$ is true in every algebra of Alg iff is true in every frame of Frm.

In order to prove such a theorem we proceed as follows. From each algebra $L \in \mathsf{Alg}$ we form a canonical frame $\mathcal{X}(L)$, and from each frame $X \in \mathsf{Frm}$ we form a complex algebra $\mathcal{C}(X)$. Then we prove that $\mathcal{X}(L) \in \mathsf{Frm}$ and $\mathcal{C}(X) \in \mathsf{Alg}$.

Furthermore, we prove what is called a complex algebra theorem:

CA For every frame $X \in \mathsf{Frm}$, a formula $\phi \in \mathsf{Lan}$ is true in X iff ϕ is true in $\mathcal{C}(X)$.

Finally, we prove a representation theorem:

R Every algebra $L \in \mathsf{Alg}$ is isomorphic to a subalgebra of the complex algebra of its canonical frame $\mathcal{C}(\mathcal{X}(L))$.

With a complex algebra theorem and a representation theorem we can prove a Duality via Truth theorem. The right-to-left implication of **DvT** follows from the left-to-right implication of **CA** and the left-to-right implication of **DvT** follows from right-to-left implication of **CA** and **R**.

In Sections 2 and 4 we present Duality via Truth results for modal algebras and modal frames, and for sufficiency algebras and sufficiency frames. In Sections 3 and 5 we consider a duality between some classes of information algebras and information frames. These frames have an indexed family of binary relations satisfying certain properties and were introduced (in [DeO02]) to capture intuitions about relations arising from information systems.

2 Modal algebras and frames

In this section we review Jónsson/Tarski duality for Boolean algebras with operators. This is then used as a case study for illustrating how the Duality via Truth approach extends this duality with dual notions of truth of formulae of a propositional language.

The class of algebras will consist of *modal algebras* (B, f) where B is a Boolean algebra $(B, \vee, \wedge, -, 0, 1)$ and f is an unary operator over B that is additive (i.e. $f(a \vee b) = f(a) \vee f(b)$) and normal (i.e. $f(0) = 0$). The class of frames will consist of *frames* (X, R) where X is a set endowed with a binary relation R over X. Let Alg_M denote the class of modal algebras, and Frm denote the class of frames.

First we show that any frame gives rise to a modal algebra. Let (X, R) be a frame. The binary relation R over X, induces monotone unary operators over 2^X, including, $f_R : 2^X \to 2^X$ defined by

$$f_R(A) = \{x \in X \mid R(x) \cap A \neq \emptyset\} \qquad \text{for } A \subseteq X,$$

and its dual, namely, $f_R^d : 2^X \to 2^X$ defined by

$$f_R^d(A) = \{x \in X \mid R(x) \subseteq A\} = -f_R(-A) \qquad \text{for } A \subseteq X.$$

It is a trivial exercise to show that the operator f_R is normal and (completely) additive, and its dual f_R^d is full (i.e. $f^d(1) = 1$) and (completely) multiplicative (i.e. $f^d(a \wedge b) = f^d(a) \wedge f^d(b)$). (See [BrR01] for further details.) So the powerset Boolean algebra 2^X endowed with the operator f_R is a modal algebra.

Next we show that any modal algebra in turn gives rise to a frame. In the case of a normal and completely additive operator f over a powerset Boolean algebra 2^X, a relation r_f over X may be defined, as in [BrR01,DuO01], by

$$xr_fy \quad \text{iff} \quad x \in f(\{y\}), \qquad \text{for } x, y \in X.$$

For the general case we invoke Stone's representation theorem — i.e., we represent the elements of the Boolean algebra as subsets of some universal set (namely the set of all prime filters), and then define a binary relation over this universe. Let (B, f) be a modal algebra, and let $\mathcal{X}(B)$ be the set of all prime filters in the Boolean algebra B. From the operator f define a binary relation R_f over $\mathcal{X}(B)$ by

$$FR_fG \quad \text{iff} \quad \forall y \in B,\ y \in G \Rightarrow f(y) \in F \quad \text{iff} \quad G \subseteq f^{-1}(F), \quad \text{for } F, G \in \mathcal{X}(B).$$

Note f^{-1} is the inverse image map given by $f^{-1}(F) = \{x \mid f(x) \in F\}$. An exercise in [BrR01] explains that the definition of r_f corresponds to the definition of R_f in the general case.

Lemma 1. *For any frame (X, R) and $F, G \in \mathcal{X}(B)$,*

$$G \subseteq f^{-1}(F) \quad \text{iff} \quad (f^{-1})^d(F) \subseteq G.$$

We now show how a modal algebra can be recovered from the frame it gave rise to. That is, if we start with a modal algebra (B, f), form its canonical frame $(\mathcal{X}(B), R_f)$ and form the complex algebra $(2^{\mathcal{X}(B)}, f_{R_f})$ of that, then this last modal algebra contains an isomorphic copy of the original modal algebra. For this it suffices to show that the Stone mapping $h : B \to 2^{\mathcal{X}(B)}$, which is an embedding of the Boolean algebra B into the Boolean algebra $2^{\mathcal{X}(B)}$, preserves operators over B. That is,

Theorem 2. *For any modal algebra (B, f) and $a \in B$,* $\quad h(f(a)) = f_{R_f}(h(a))$.

Proof: For any $a \in B$,

$$\begin{aligned} f_{R_f}(h(a)) &= \{F \in \mathcal{X}(B) \mid (\exists G \in h(a))[FR_fG]\} \\ &= \{F \in \mathcal{X}(B) \mid (\exists G \in \mathcal{X}(B))[a \in G \text{ and } G \subseteq f^{-1}(F)]\}. \end{aligned}$$

To show that this is equal to $h(f(a)) = \{F \in \mathcal{X}(B) \mid f(a) \in F\}$ we have to show that $f(a) \in F \quad \text{iff} \quad (\exists G \in \mathcal{X}(B))[a \in G \text{ and } G \subseteq f^{-1}(F)]$.
The right-to-left direction is easy, because if $a \in G$ and $G \subseteq f^{-1}(F)$ then $G \subseteq \{x \mid f(x) \in F\}$, and hence $f(a) \in F$. For the left-to-right direction consider the set $Z_f = \{b \in B \mid f^d(b) \in F\}$. Let F' be the filter generated by $Z_f \cup \{a\}$, that is, $F' = \{b \in B \mid \exists a_1, \ldots, a_n \in Z_f, \quad a_1 \wedge \ldots \wedge a_n \wedge a \leq b\}$. Then F' is proper. Suppose otherwise. Then for some $a_1, \ldots, a_n \in Z_f, \quad a_1 \wedge \ldots \wedge a_n \wedge a = 0$, i.e., $a_1 \wedge \ldots \wedge a_n \leq -a$. Since f^d is monotone, $f^d(a_1 \wedge \ldots \wedge a_n) \leq f^d(-a)$, that is, $f^d(a_1) \wedge \ldots \wedge f^d(a_n) \leq f^d(-a)$. By definition of Z_f we have $f^d(a_1), \ldots, f^d(a_n) \in F$ so, since F is a filter, $f^d(a_1) \wedge \ldots \wedge f^d(a_n) \in F$ and hence $f^d(-a) \in F$. Thus

$-a \in Z_f$ which is a contradiction. So, by ([DaP90], p188), there is a prime filter G containing F'. Since $a \in F'$, $a \in G$ and hence $g \in h(a)$. Also $G \subseteq f^{-1}(F)$ since if $y \notin f^{-1}(F)$ then $f(y) \notin F$, i.e., $f^d(-y) \in F$, so $-y \in F' \subseteq G$ and hence $y \notin G$. □

With this result we can prove a representation theorem (see Theorem 4(a)) for modal algebras. For a representation theorem for frames we show how a frame can be recovered from the modal algebra it gave rise to. That is, if we start with a frame (X, R), form its complex algebra $(2^X, f_R)$ and form the canonical frame $(\mathcal{X}(2^X), R_{f_R})$ of that, then this last frame contains an isomorphic copy of the original frame. For this we invoke the one-one correspondence between the elements of X and certain prime filters of 2^X, namely the principal ones given by the mapping $k : X \to \mathcal{X}(2^X)$ where $k(x) = \{A \in 2^X \mid x \in A\}$. It is an easy exercise to show that $k(x)$ is a prime filter. We have to show that this mapping preserves structure. That is,

Theorem 3. *For any frame (X, R) and $x, y \in X$, xRy iff $k(x)R_{f_R}k(y)$.*

Proof: Note, for any $x, y \in X$,

$$\begin{aligned} k(x)R_{f_R}k(y) \quad &\text{iff} \quad k(y) \subseteq (f_R)^{-1}(k(x)) \\ &\text{iff} \quad \{Y \subseteq X \mid y \in Y\} \subseteq \{Z \subseteq X \mid x \in f_R(Z)\}. \end{aligned}$$

We now prove the desired double implication. For the left-to-right direction, suppose xRy. Take any $Y \subseteq X$ with $y \in Y$. Then $R(x) \cap Y \neq \emptyset$ and hence $Y \in \{Z \mid x \in f_R(Z)\}$. Thus $k(x)R_{f_R}k(y)$. For the right-to-left direction, suppose $k(x)R_{f_R}k(y)$. Since $y \in \{y\}$, by the above, $x \in f_R(\{y\})$, that is, xRy. □

A consequence of the above theorems is a Jónsson/Tarski duality between modal algebras and frames.

Theorem 4.

(a) *Every modal algebra (B, F) is isomorphic to a subalgebra of the complex algebra of its canonical frame $(2^{\mathcal{X}(B)}, f_{R_f})$.*
(b) *Every frame (X, R) is isomorphic to a substructure of the canonical frame of its complex algebra $(\mathcal{X}(2^X), R_{f_R})$.*

The final part of the duality consists of establishing a bijective correspondence between maps between modal algebras and maps between frames. In the case of modal algebras (B_1, f_1) and (B_2, f_2) the map is a Boolean algebra homomorphism $l : B_1 \to B_2$ and in the case of frames (X_1, R_1) and (X_2, R_2) the map is a bounded morphism $n : X_1 \to X_2$ (with the properties xR_1y implies $n(x)R_2n(y)$, and if $n(x)R_2y_2$ then for some $y_1 \in X_1$, xR_1y_1 and $f(y_1) = y_2$).

Theorem 5. *Let (B_1, f_1) and (B_2, f_2) be modal algebras and let $l : B_1 \to B_2$ be a homomorphism between them. Let (X_1, R_1) and (X_2, R_2) be frames and let $n : X_1 \to X_2$ be a bounded morphism between them. Then $l^{-1} : \mathcal{X}(B_2) \to \mathcal{X}(B_1)$ is a bounded morphism, and $n^{-1} : 2^{X_2} \to 2^{X_1}$, is a homomorphism.*

It is not difficult to extend Theorem 5 to show that injective/surjective homomorphisms correspond to surjective/injective bounded morphisms and vice versa. Let us use the category-theoretical device of denoting the function m^{-1} by $\mathcal{X}(m)$ (thus invoking a functorial notation), then the duality is finally completed by proving the following results.

Theorem 6. *Let l and n be as in* Theorem 5. *Suppose that the maps $h_{B_1} : B_1 \to 2^{\mathcal{X}(B_1)}$ and $h_{B_2} : B_2 \to 2^{\mathcal{X}(B_2)}$, and $k_{X_1} : X_1 \to \mathcal{X}(2^X)$ and $k_{X_2} : X_2 \to \mathcal{X}(2^Y)$ are the isomorphisms used in Theorem 4. Then*

$$(\mathcal{X}(l))^{-1} \circ h_{B_1} = h_{B_2} \circ l \quad \textit{and} \quad \mathcal{X}(n^{-1}) \circ k_{X_1} = k_{X_2} \circ n.$$

That is, the following diagrams commute:

$$\begin{tikzcd}
B_1 \arrow[r, "h_{B_1}"] \arrow[d, "l"] & 2^{\mathcal{X}(B_1)} \arrow[d, "(\mathcal{X}(l))^{-1}"] \\
B_2 \arrow[r, "h_{B_2}"] & 2^{\mathcal{X}(B_2)}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \arrow[r, "k_{X_1}"] \arrow[d, "n"] & \mathcal{X}(2^{X_1}) \arrow[d, "\mathcal{X}(n^{-1})"] \\
X_2 \arrow[r, "k_{X_2}"] & \mathcal{X}(2^{X_2})
\end{tikzcd}$$

Theorem 7. *Let l, h_{B_1}, h_{B_2} be as in Theorem 6. Then, for any $a \in B_1$,*

$$(\mathcal{X}(l))^{-1}(f_{R_{f_1}}(h_{B_1}(a))) = f_{R_{f_2}}(\mathcal{X}(l))^{-1}h_{B_1}(a)$$

Proof: For any $a \in B_1$,

$$\begin{array}{ll}
\ (\mathcal{X}(l))^{-1}(f_{R_{f_1}}(h_{B_1}(a))) & \\
= (\mathcal{X}(l))^{-1}(h_{B_1}(f_1(a))) & \text{by Theorem 6} \\
= h_{B_2}(l(f_1(a))) & \text{by Theorem 2} \\
= h_{B_2}(f_2(l(a))) & \text{since } l \text{ is a homomorphism} \\
= f_{R_{f_2}}(h_{B_2}(l(a))) & \text{by Theorem 2} \\
= f_{R_{f_2}}(\mathcal{X}(l))^{-1}h_{B_1}(a) & \text{by Theorem 6.}
\end{array}$$

$\square$

Theorem 8. *Let n, k_{X_1}, k_{X_2} be as in Theorem 6. Then, for any $x, y \in X_1$,*

(a) $k_{X_1}(x)R_{f_{R_1}}k_{X_1}(y) \quad \Rightarrow \quad \mathcal{X}(n^{-1})k_{X_1}(x)R_{f_{R_2}}\mathcal{X}(n^{-1})k_{X_1}(y)$

(b) *If $\mathcal{X}(n^{-1})k_{X_1}(x)R_{f_{R_2}}k_{X_2}(y_2)$ then for some $y_1 \in X_1$, $\quad k_{X_1}(x)R_{f_{R_1}}k_{X_1}(y)$ and $\mathcal{X}(n^{-1})(k_{X_1}(y_1)) = k_{X_2}(y_2)$.*

Proof: For any $x, y \in X_1$,

$$\begin{array}{ll}
\ k_{X_1}(x)R_{f_{R_1}}k_{X_1}(y) & \\
\Leftrightarrow xR_1y & \text{by Theorem 3} \\
\Rightarrow n(x)R_2n(y) & \text{since } n \text{ is a bounded morphism} \\
\Leftrightarrow k_{X_2}(n(x))R_{f_{R_2}}k_{X_2}(n(y)) & \text{by Theorem 3} \\
\Leftrightarrow \mathcal{X}(n^{-1})k_{X_1}(x)R_{f_{R_2}}\mathcal{X}(n^{-1})k_{X_1}(y) & \text{by Theorem 6}
\end{array}$$

$$
\begin{array}{ll}
\mathcal{X}(n^{-1})k_{X_1}(x)R_{f_{R_2}}k_{X_2}(y_2) & \\
\Leftrightarrow k_{X_2}(n(x))R_{f_{R_2}}k_{X_2}(y_2) & \text{by Theorem 6} \\
\Leftrightarrow n(x)R_2y_2 & \text{by Theorem 3} \\
\Rightarrow \exists y_1,\ xR_1y_1\ \wedge\ n(y_1) = y_2 & \text{since } n \text{ is a bounded morphism} \\
\Leftrightarrow \exists y_1,\ k_{X_1}(x)R_{f_{R_1}}k_{X_1}(y)\ \wedge & \\
\qquad \mathcal{X}(n^{-1})(k_{X_1}(y_1)) = k_{X_2}(y_2) & \text{by Theorems 3 and 6}
\end{array}
$$

□

All of the preceding results can be cast into a categorical framework as an equivalence between the categories of modal algebras and frames. For example, Theorem 6 is then simply a statement of the definition of the natural transformations involved in an equivalence (or more generally, adjunction).

In order to extend this to a Duality via Truth, we need a logical language. Let Lan_M be a modal language whose formulas are built from propositional variables taken from an infinite denumerable set Var, with the classical propositional operations of negation ($\neg$), disjunction ($\vee$), conjunction ($\wedge$), and with a modal operator ($\Diamond$). We slightly abuse the language by denoting the operations in modal algebras and the classical propositional operations of Lan_M with the same symbols.

The class Alg_M of modal algebras provides an algebraic semantics for Lan_M. Let (B, f) be a modal algebra. A valuation on B is a function $v : Var \to B$ which assigns elements of B to propositional variables and extends homomorphically to all the formulas of Lan_M, that is

$$v(\neg\alpha) = -\alpha,\ v(\alpha \vee \beta) = v(\alpha) \vee v(\beta),\ \ v(\Diamond\alpha) = f(v(\alpha)).$$

The notion of truth determined by this semantics is as follows. A formula α in Lan_M is true in an algebra (B, f) whenever $v(\alpha) = 1$ for every v in B. A formula $\alpha \in \mathsf{Lan}_M$ is true in the class Alg_M iff it is true in every algebra $B \in \mathsf{Alg}_M$.

The class Frm of frames provides a well known frame semantics for Lan_M. A model based on a frame (X, R) is a system $M = (X, R, m)$, where $m : Var \to 2^X$ is a meaning function. The satisfaction relation $\models$ is defined as usual. We say that in a model M state $x \in X$ satisfies a formula whenever the following conditions are satisfied:

$$
\begin{aligned}
M, x \models p &\ \text{ iff }\ x \in m(p),\ \text{ for every } p \in Var \\
M, x \models \alpha \vee \beta &\ \text{ iff }\ M, x \models \alpha \text{ or } M, x \models \beta, \\
M, x \models \neg\alpha &\ \text{ iff }\ \text{ not } M, x \models \alpha, \\
M, x \models \Diamond\alpha &\ \text{ iff }\ \exists y\ \text{ such that } M, y \models \alpha \text{ and } xRy.
\end{aligned}
$$

A notion of truth of formulas based on this semantics is defined as usual. A formula $\alpha \in \mathsf{Lan}_M$ is true in a model M whenever for every $x \in X$ we have $M, x \models \alpha$. A formula $\alpha \in \mathsf{Lan}_M$ is true in a frame (X, R) iff α is true in every model based on this frame. And finally a formula $\alpha \in \mathsf{Lan}_M$ is true in the class Frm of frames iff it is true in every frame $X \in \mathsf{Frm}$.

It is easy to see that the complex algebra theorem holds:

Theorem 9. *A formula $\alpha \in \mathsf{Lan}_M$ is true in every model based on a frame (X, R) iff α is true in the modal complex algebra $(2^X, f_R)$ of that frame.*

Proof: Let (X, R) be any frame. The result is established by taking the meaning function m on any model (X, R, m) based on (X, R) to coincide with the valuation function on the modal complex algebra $(2^X, f_R)$ of (X, R). $\square$

Finally, we prove the Duality via Truth theorem between modal algebras and frames.

Theorem 10. *A formula $\alpha \in \mathsf{Lan}_M$ is true in every algebra of Alg_M iff α is true in every frame of Frm.*

Proof: Let (B, f) be any modal algebra. Then any valuation v on B can be extended to a valuation $h \circ v$ on $2^{\mathcal{X}(B)}$ and thus

$$\begin{aligned} \alpha \text{ is true in } (B, f) \quad & \text{iff} \quad \alpha \text{ is true in } (2^{\mathcal{X}(B)}, f_{R_f}) \\ & \text{iff} \quad \alpha \text{ is true in every model based on } (\mathcal{X}(B), R_f) \\ & \text{iff} \quad \alpha \text{ is true in } (\mathcal{X}(B), R_f). \end{aligned}$$

By the duality, every frame in Frm is of the form $(\mathcal{X}(B), R_f)$ for some modal algebra (B, f) in Alg_M. On the other hand, let (X, R) be any frame. Then

$$\begin{aligned} \alpha \text{ is true in } (X, R) \quad & \text{iff} \quad \alpha \text{ is true in every model based on } (X, R) \\ & \text{iff} \quad \alpha \text{ is true in } (2^X, R_f). \end{aligned}$$

By the duality, every modal algebra in Alg_M is of the form $(2^X, f_R)$ for some frame (X, R) in Frm. $\square$

The final part of the Duality via Truth involves establishing a correspondence between preservation of truth with respect to Alg_M and with respect to Frm_M. As a consequence of Theorem 6, we have

$$(\mathcal{X}(l))^{-1} \circ h_{B_1} \circ v_1 = h_{B_2} \circ v_2 \qquad \text{and} \qquad \mathcal{X}(n^{-1}) \circ \mathcal{X}(m_1) = \mathcal{X}(m_2).$$

That is, the following diagrams commute.

$$\begin{tikzcd} & B_1 \arrow[r, "h_{B_1}"] \arrow[dd, "l"] & 2^{\mathcal{X}(B_1)} \arrow[dd, "(\mathcal{X}(l))^{-1}"] \\ \mathrm{Var} \arrow[ur, "v_1"] \arrow[dr, "v_2"'] & & \\ & B_2 \arrow[r, "h_{B_2}"] & 2^{\mathcal{X}(B_2)} \end{tikzcd} \qquad \begin{tikzcd} & 2^{X_1} \arrow[r, "\mathcal{X}"] \arrow[dd, "n^{-1}"] & \mathcal{X}(2^{X_1}) \arrow[dd, "\mathcal{X}(n^{-1})"] \\ \mathrm{Var} \arrow[ur, "m_1"] \arrow[dr, "m_2"'] & & \\ & 2^{X_2} \arrow[r, "\mathcal{X}"] & \mathcal{X}(2^{X_2}) \end{tikzcd}$$

Hence, we have the following equivalence of preservation of truth.

Theorem 11. *Any homomorphism between algebras in Alg_M preserves truth with respect to Alg_M iff any bounded morphism between frames in Frm preserves truth with respect to Frm.*

Proof: Let $l : B_1 \to B_2$ be a homomorphism between modal algebras (B_1, f_1) and (B_2, f_2). Then $\mathcal{X}(l) : \mathcal{X}(B_2) \to \mathcal{X}(B_1)$ is a bounded morphism between the frames $(\mathcal{X}(B_1), R_{f_1})$ and $(\mathcal{X}(B_2), R_{f_2})$. Suppose l preserves truth in Alg_M, that is, for any formula α and any valuation v_1 on B_1,

$$v_1(\alpha) = 1 \qquad \text{iff} \qquad l \circ v_1(\alpha) = 1.$$

The valuation v_1 can be extended to a meaning function $h_{B_1} \circ v_1$ on $(\mathcal{X}(B_1), R_{f_1})$, and the valuation $l \circ v_1$ can be extended to a meaning function $h_{B_2} \circ l \circ v_1$ on $(\mathcal{X}(B_2), R_{f_2})$. By Theorem 6, it follows that for any formula α and any $F \in \mathcal{X}(B_1)$,

$$F \in h_{B_2} \circ l \circ v_1(\alpha) \quad \text{iff} \quad F \in (\mathcal{X}(l))^{-1} \circ h_{B_1} \circ v_1(\alpha) \quad \text{iff} \quad \mathcal{X}(l)(F) \in h_{B_1} \circ v_1(\alpha).$$

That is, the bounded morphism $\mathcal{X}(l)$ preserves truth with respect to Frm.

On the other hand, let $n : X_1 \to X_2$ be a bounded morphism between frames (X_1, R_1) and (X_2, R_2). Then $n^{-1} : 2^{X_2} \to 2^{X_1}$ is a homomorphism between the modal algebras $(2^{X_1}, f_{R_1})$ and $(2^{X_2}, f_{R_2})$. Suppose n preserves truth with respect to Frm, that is, for any formula α and any meaning function m_1 on X_1,

$$x \in m_1(\alpha) \qquad \text{iff} \qquad n(x) \in n \circ m_1(\alpha).$$

As a valuation function on 2^{X_1} take meaning function m_1 and as a valuation function on 2^{X_2} take the meaning function $n \circ m_1$. Then,

$$n \circ m_1(\alpha) = X_2 \qquad \text{iff} \qquad n^{-1}(n \circ m_1)(\alpha) = n^{-1}(X_2) \qquad \text{iff} \qquad m_1(\alpha) = X_1$$

That is, the homomorphism n^{-1} preserves truth with respect to Alg_M.

$\square$

Another representation of a modal algebra (B, f) is provided in [JoT51] by a canonical extension B^σ of B algebra. The canonical extension of the operator f is a map $f^\sigma : B^\sigma \to B^\sigma$ defined by $f^\sigma(\{y\}) = \bigcap\{h(f(a)) \mid a \in y\},$ for $y \in \mathcal{X}(B)$. It follows that

$$x \in f^\sigma(\{y\}) \quad \text{iff} \quad \forall a,\ a \in y \ \Rightarrow \ f(a) \in x \quad \text{iff} \quad y \subseteq f^{-1}(x).$$

Observe that here in fact we have a definition of a relation on a set of prime filters of B. This is precisely a relation of the canonical frame of the modal algebra. Next, for $Z \in \mathcal{X}(B)$ we define $f^\sigma(Z) = \bigcup\{f^\sigma(\{y\}) : y \in Z\}$. It follows that

$$x \in f^\sigma(h(a)) \quad \text{iff} \quad \exists y,\ a \in y \ \wedge \ x \in f^\sigma(\{y\}) \quad \text{iff} \quad \exists y,\ a \in y \ \wedge \ y \subseteq f^{-1}(x).$$

That is, $f^\sigma(h(a))$ provides a definition of the modal operator in the complex algebra of the canonical frame of (B, f). The canonical extension of the modal algebra (B, f) is then the algebra (B^σ, f^σ). It is known that f^σ is a complete modal operator on B^σ.

3 Information algebras and information frames for reasoning about similarity

In this section we extend the results of Section 2 to a class of information algebras which are extensions of modal algebras with an indexed family of unary operators satisfying certain properties inspired by information systems.

Typically, in an information system objects are described in terms of some attributes and their values. The queries to an information system often have the form of a request for finding a set of objects whose sets of attribute values satisfy some conditions. This leads to the notion of information relation determined by a set of attributes. Let $a(x)$ and $a(y)$ be sets of values of an attribute a of the objects x and y. We may want to know a set of those objects from an information system whose sets of values of all (or some) of the attributes from a subset A of attributes are equal (or disjoint, or overlap etc.). To represent such queries we define, first, information relations on the set of objects and, second, information operators determined by those relations. For example, a relation of similarity of objects is defined as:

$$(x, y) \in \mathrm{sim}(a) \quad \text{iff} \quad a(x) \cap a(y) \neq \emptyset.$$

Next, we can extend this relation to any subset A of attributes so that a quantification over A is added:

$$(x, y) \in \mathrm{sim}(A) \quad \text{iff} \quad a(x) \cap a(y) \neq \emptyset \qquad \text{for all (some) } a \in A.$$

Relations defined with the universal (existential) quantifier are referred to as strong (weak) relations.

In an abstract setting as an index set we take a set of sets 2^{Par}, where each set $P \subseteq \mathrm{Par}$ is intuitively viewed as a set of attributes of objects in an information system. Then strong or weak relations are defined axiomatically.

An *information frame of weak similarity* (denoted $\mathsf{FW\text{-}SIM}$ in [DeO02]) is a binary relational structure $(X, \{R_P \mid P \subseteq \mathrm{Par}\})$ where the binary relations $R_P \subseteq X \times X$ (for each $P \subseteq \mathrm{Par}$) satisfy the following properties:

MF1 $R_{P \cup Q} = R_P \cup R_Q$
MF2 $R_\emptyset = \emptyset$
MF3 R_P is weakly reflexive (i.e., $\forall x, \forall y,\ xRy \Rightarrow xRx$)
MF4 R_P is symmetric (i.e., $\forall x, \forall y,\ xRy \Rightarrow yRx$)

Properties MF1 and MF2 reflect the intuition of weak relations; properties MF3 and MF4 are the abstract characterisation of similarity relations derived from an information system. By $\mathsf{Frm}_{\mathsf{WSIM}}$ we denote the class of weak similarity frames.

An *information algebra of weak similarity* (denoted $\mathsf{AW\text{-}SIM}$ in [DeO02]) is a Boolean algebra B with a family $\{f_P \mid P \subseteq \mathrm{Par}\}$ of additive normal unary operators satisfying the following additional properties:

MA1 $f_{P \cup Q}(x) = f_P(x) \vee f_Q(x)$
MA2 $f_\emptyset(x) = 0$
MA3 $x \wedge f_P(1) \leq f_P(x)$
MA4 $x \leq f_P^d f_P(x)$

Properties MA1-MA4 will be shown below to be the algebraic counterparts of the properties MF1-MF4 on information relations.

Lemma 12. *Let* $(B, \{f_P \mid P \subseteq \mathrm{Par}\})$ *be an information algebra of weak similarity. For each operator* f_P $(P \subseteq \mathrm{Par})$*, the corresponding binary relation* R_{f_P} *over* $\mathcal{X}(B)$ *satisfies properties* MF1 - MF4*.*

Proof: We prove MF1-MF3; MF4 is well known from modal correspondence theory [vaB84].

MF1 For any $F, G \in \mathcal{X}(B)$, $FR_{f_{P\cup Q}}G$ iff $G \subseteq f^{-1}_{P\cup Q}(F)$ iff $(f^d_{P\cup Q})^{-1}(F) \subseteq G$. But $(f^d_{P\cup Q})^{-1}(F) = (f^d_P)^{-1}(F) \cap (f^d_Q)^{-1}(F)$. The propositional logic formulae $\alpha \wedge \beta \rightarrow \gamma$ and $\alpha \rightarrow \gamma \ \vee \ \beta \rightarrow \gamma$ are equivalent. Thus, $(f^d_P)^{-1}(F) \subseteq G$ or $(f^d_Q)^{-1}(F) \subseteq G$, that is, $G \subseteq (f_P)^{-1}(F)$ or $G \subseteq (f_Q)^{-1}(F)$. Thus $FR_{f_P}G$ or $FR_{f_Q}G$.

MF2 $FR_{f_\emptyset}G$ iff $G \subseteq f^{-1}_\emptyset(F)$ iff $F \subseteq \emptyset$. The latter is always false since F is a prime filter. Thus $R_{f_\emptyset} = \emptyset$.

MF3 Take any $F, G \in \mathcal{X}(B)$ with $FR_{f_P}G$ and $G - R_{f_P}G$. Then $G \subseteq f_P^{-1}(F)$ and $G \not\subseteq f_P(G)$. So, since $f_P(F)$ and G are non-empty, $g_P(G) = \emptyset$ and $g_P(G) \neq \emptyset$, which provides the required contradiction.

□

Lemma 13. *Let* $(X, \{R_P \mid P \subseteq \mathrm{Par}\})$ *be an information frame. For each binary relation* R_P $(P \subseteq \mathrm{Par})$*, the corresponding unary operator* f_{R_P} *over* 2^X *satisfies properties* MA1 - MA4*.*

Proof: We prove MA1-MA3; MA4 is well known from modal correspondence theory [vaB84].

MA1 For $A \subseteq X$, $x \in f_{R_{P\cup Q}}(A)$ iff $R_{P\cup Q}(x) \cap A \neq \emptyset$ iff $R_{P\cup Q}(x) \not\subseteq -A$. But $R_{P\cup Q}(x) = R_P(x) \cup R_Q(x)$, so $R_P(x) \not\subseteq -A$ or $R_Q(x) \not\subseteq -A$, that is, $R_P(x) \cap A \neq \emptyset$ or $R_Q(x) \cap A \neq \emptyset$. Thus $x \in f_{R_P}(A)$ or $x \in f_{R_Q}(A)$.

MA2 $x \in f_{R_\emptyset}(A)$ iff $A \subseteq R_\emptyset(x)$ iff $A \subseteq X$. The latter is always true so $f_{R_\emptyset}(A) = X$.

MA3 Take any $x \in X$ such that $x \in A \cap f_{R_P}(A)$ and $x \notin f_{R_P}(X)$. Then $x \in A$ and $A \subseteq R_P(x)$ and $X \not\subseteq R_P(x)$. So xR_Px and for some $y \in X$ not xR_Py. Thus, xR_Px and, by property MF3 of R_P, not xRx, which provides the required contradiction.

□

A representation theorem analogous to Theorem 4 holds for weak similarity algebras and weak similarity frames. Also Theorems 6, 7, 8 are applicable to weak similarity algebras and weak similarity frames, these being special modal algebras and special frames, respectively.

The language $\mathsf{Lan}_{\mathsf{WSIM}}$ relevant for algebras and frames of weak similarity is an extension of the modal language Lan_M with a family of $\{\langle R_P\rangle \mid P \subseteq \mathrm{Par}\}$ of modal operators. Algebraic semantics of the language is provided by the class $\mathsf{Alg}_{\mathsf{WSIM}}$ and the frame semantics by the class $\mathsf{Frm}_{\mathsf{WSIM}}$. The notion of a model based on a frame of $\mathsf{Frm}_{\mathsf{WSIM}}$, satisfaction relation, and the notions of truth in a model, in a frame and in a class of frames are analogous to the respective notions in Section 2. In view of Lemma 13 the complex algebra theorem (CA) holds for $\mathsf{Frm}_{\mathsf{WSIM}}$. From the representation theorem and (CA) we obtain a Duality via Truth theorem, and also the equivalence of preservation of truth in Theorem 11.

4 Sufficiency algebras and frames

As second case study for the Duality via Truth approach we consider, in this section, a duality between sufficiency algebras and frames. These algebras were introduced in [DuO01] for reasoning about incomplete information and expressing algebraically certain properties of binary relations, such as irreflexivity or co-reflexivity, defined in terms of the complement of the relation.

A *sufficiency algebra* (B, g) is a Boolean algebra B endowed with an unary operator g over B that is co-additive (i.e. $g(a \vee b) = g(a) \wedge g(b)$) and co-normal (i.e. $g(0) = 1$). Let Alg_S denote the class of sufficiency algebras. The class Frm of frames adequate for providing Duality via Truth for sufficiency algebras is the same as in the case of modal algebras.

Given any frame (X, R), the binary relation R over X induces antitone operators, including $g_R : 2^X \to 2^X$ defined by

$$g_R(A) = \{x \in X \mid R(x) \cup A \neq X\} \qquad \text{for } A \subseteq X,$$

and its dual, namely, $g_R^d : 2^X \to 2^X$ defined by

$$g_R^d(A) = \{x \in X \mid A \subseteq R(x)\} \qquad \text{for } A \subseteq X.$$

Observing that these operators may be defined in terms of the monotone operators in Section 2 by $g_R(A) = f_{-R}(-A)$ and $g_R^d(A) = -f_{-R}(A)$ it follows that g_R is co-normal and co-additive, and g_R^d is co-full (i.e. $g^d(1) = 0$) and co-multiplicative (i.e. $g^d(a \wedge b) = g^d(a) \vee g^d(b)$). Hence, from a frame (X, R) we may define a sufficiency algebra $(2^X, g_R)$.

Next we show that any sufficiency algebra in turn gives rise to a frame. In the case of a co-normal and completely co-additive operator g over a powerset Boolean algebra 2^X, a relation r_g over X may be defined, as in [DuO01], by

$$x r_g y \quad \text{iff} \quad x \in g(\{y\}), \qquad \text{for } x, y \in X.$$

In general, as in Section 2 we invoke Stone's representation theorem and then define a binary relation over $\mathcal{X}(B)$. Let (B, g) be a sufficiency algebra. From the operator g define a binary relation R_g over $\mathcal{X}(B)$ by

$$F R_g G \quad \text{iff} \quad g(G) \cap F \neq \emptyset, \quad \text{for } F, G \in \mathcal{X}(B).$$

It is an easy exercise to show that the definition of r_g corresponds to that of R_g in the general case.

We now show how a sufficiency algebra can be recovered from the frame it gave rise to. That is, if we start with a sufficiency algebra (B, g), form its canonical frame $(\mathcal{X}(B), R_g)$ and form the complex algebra $(2^{\mathcal{X}(B)}, g_{R_g})$ of that, then this last sufficiency algebra contains an isomorphic copy of the original sufficiency algebra. For this it suffices to show that the Stone mapping $h : B \to 2^{\mathcal{X}(B)}$ preserves operators g over B. That is,

Theorem 14. *For any sufficiency algebra (B, g) and $a \in B$, $h(g(a)) = g_{R_g}(h(a))$.*

Proof: For any $a \in B$,

$$\begin{aligned} g_{R_g}(h(a)) &= \{F \in \mathcal{X}(B) \mid h(a) \subseteq R_g(F)]\} \\ &= \{F \in \mathcal{X}(B) \mid (\forall G \in \mathcal{X}(B))[a \in G \text{ and } g(G) \cap F \neq \emptyset]\}. \end{aligned}$$

To show that this is equal to $h(g(a)) = \{F \in \mathcal{X}(B) \mid g(a) \in F\}$ we have to show that $g(a) \notin F \quad \text{iff} \quad (\exists G \in \mathcal{X}(B))[a \in G \text{ and } g(G) \cap F = \emptyset]$.
The right-to-left direction is easy, because if $a \in G$ and $g(G) \cap F = \emptyset$ then $g(a) \in g(G)$ so $g(a) \notin F$. For the left-to-right direction, assume $g(a) \in F$. Consider the set $Z_g = \{b \in B \mid g^d(b) \notin F\}$. Let F' be the filter generated by $Z_g \cup \{a\}$, that is, $F' = \{b \in B \mid \exists a_1, \ldots, a_n \in Z_g, \quad a_1 \wedge \ldots \wedge a_n \wedge a \leq b\}$. Then F' is proper. Suppose otherwise. Then for some $a_1, \ldots, a_n \in Z_g, \quad a_1 \wedge \ldots \wedge a_n \wedge a = 0$, i.e., $a \leq -(a_1 \wedge \ldots \wedge a_n) = -a_1 \vee \ldots \vee -a_n$. Since g is antitone, $g(-a_1 \vee \ldots \vee -a_n) \leq g(a)$. Thus $g(-a_1) \wedge \ldots \wedge g(-a_n) \leq g(a)$, that is, $-g^d(a_1) \wedge \ldots \wedge -g^d(a_n) \leq g(a)$. By definition of Z_g we have $g^d(a_1), \ldots, g^d(a_n) \notin F$ so $-g^d(a_1), \ldots, -g^d(a_n) \in F$. Since F is a filter, $-g^d(a_1) \wedge \ldots \wedge -g^d(a_n) \in F$ and hence $g(a) \in F$ which contradicts the original assumption. So, by ([DaP90], p188), there is a prime filter G containing F'. Since $a \in F'$, $a \in G$ and hence $g \in h(a)$. Also $g(G) \cap F = \emptyset$ since if there is some $b \in B$ with $b \in g(G)$ and $b \in F$, then $b = g(c)$ for some $c \in G$ and thus $g(c) \in F$, so $g^d(-c) \notin F$ hence $-c \in Z_g \subseteq F' \subseteq G$ and thus $c \notin G$, which is a contradiction. $\square$

On the other hand a frame can be recovered from the sufficiency algebra it gave rise to. That is, if we start with a frame (X, R), form its complex algebra $(2^X, g_R)$ and form the canonical frame $(\mathcal{X}(2^X), R_{g_R})$ of that, then this last frame contains an isomorphic copy of the original frame. For this we show the mapping $k : X \to \mathcal{X}(2^X)$ preserves structure. That is,

Theorem 15. *For any frame (X, R) and any $x, y \in X$, xRy iff $k(x)R_{g_R}k(y)$.*

Proof: Note, for any $x, y \in X$,

$$k(x)R_{g_R}k(y) \quad \text{iff} \quad g_R(k(y)) \cap k(x) \neq \emptyset \quad \text{iff} \quad \{g_R(Y) \mid y \in Y\} \cap \{Z \mid x \in Z\} \neq \emptyset.$$

We now prove the desired double implication. For the left-to-right direction, suppose xRy. Then $\{y\} \subseteq R(x)$, so $x \in g_R(\{y\})$. Hence $g_R(\{y\}) \in g_R(k(y)) \cap k(x)$. Thus $k(x)R_{g_R}k(y)$. For the right-to-left direction, suppose $k(x)R_{g_R}k(y)$. Since $y \in \{y\}$, by the above, $g_R(\{y\}) \in \{Z \mid x \in Z\}$. Thus $x \in g_R(\{y\})$, that is, $\{y\} \subseteq R(x)$, that is, xRy. $\square$

Therefore, we have a Jónsson/Tarski duality between sufficiency algebras and frames.

Theorem 16.

(a) *Any sufficiency algebra (B, g) is isomorphic to a subalgebra of the complex algebra of its canonical frame $(2^{\mathcal{X}(B)}, g_{R_g})$.*
(b) *Any frame (X, R) is isomorphic to a substructure of the canonical frame of its complex algebra $(\mathcal{X}(2^X), R_{g_R})$.*

Analogous results to Theorems 6, 7, 8 can be proved for sufficiency algebras and frames by invoking Theorems 14 and 15.

The language adequate for discussing a duality between sufficiency algebras Alg_S and frames of Frm is a propositional language Lan_S whose formulas are built with classical propositional connectives and the sufficiency operator $[\![\ ]\!]$. The frame semantics for Lan_S is defined as for the modal language. Let (X, R) be a frame and let $M = (X, R, m)$ be a model based on that frame. The satisfaction relation extends to the formulas with the sufficiency operator as follows:

$$M, x \models [\![R]\!]\alpha \quad \text{iff} \quad \forall y \ \text{ if } M, y \models \alpha \text{ then } xRy.$$

The notions of truth of a formula in a model, in a frame, and in a class of frames are defined as in the case of the modal logic. Using analogous reasoning to that for modal logic, we can prove the complex algebra theorem and Duality via Truth theorem, and also the equivalence of preservation of truth.

Theorem 17. *A formula $\alpha \in \mathsf{Lan}_S$ is true in every model based on a frame (X, R) iff α is true in the sufficiency complex algebra $(2^X, g_R)$ of that frame.*

Theorem 18. *A formula $\alpha \in \mathsf{Lan}_S$ is true in every algebra of Alg_S iff α is true in every frame of Frm.*

Theorem 19. *Any homomorphism between algebras in Alg_S preserves truth with respect to Alg_S iff any bounded morphism between frames in Frm preserves truth with respect to Frm.*

The canonical extension of a sufficiency algebra (B, g) is defined as follows. Let B^σ be the canonical extension of the Boolean algebra B and let h be the Stone embedding. Then the canonical extension of the operator g is a map $g^\sigma : B^\sigma \to B^\sigma$ defined by $g^\sigma(\{y\}) = \bigcup\{h(g(a)) \mid a \in y\}$, for $y \in \mathcal{X}(B)$. We have that

$$x \in g^\sigma(\{y\}) \quad \text{iff} \quad \exists a,\ a \in y \ \wedge\ g(a) \in x \quad \text{iff} \quad y \cap g(x) \neq \emptyset.$$

As in the case of modal algebras, this provides a definition of a relation on $\mathcal{X}(B)$. Next, for $Z \in \mathcal{X}(B)$ we define $g^\sigma(Z) = \bigcap\{g^\sigma(\{y\}) : y \in Z\}$. It follows that

$$x \in g^\sigma(h(a)) \quad \text{iff} \quad \forall y,\ a \in y \ \Rightarrow\ x \in g^\sigma(\{y\}) \quad \text{iff} \quad \forall y,\ a \in y \ \Rightarrow\ y \cap g(x) \neq \emptyset.$$

That is, $g^\sigma(h(a))$ provides a definition of the sufficiency operator in the complex algebra of the canonical frame of (B, g). The canonical extension of the sufficiency algebra (B, g) is then the algebra (B^σ, g^σ). It is known that g^σ is a completely co-additive sufficiency operator on B^σ.

5 Information algebras and information frames of strong right orthogonality

As before the representation results of Theorem 16 can be extended to some information algebras based on sufficiency algebras. Here the relations derived from an information systems are strong relations of right orthogonality defined as follows. For objects x and y of an information system and an attribute a,

$$(x, y) \in \mathrm{rort}(a) \quad \text{iff} \quad a(x) \subseteq -a(y).$$

For a subset A of attributes we may define strong (weak) relations by

$$(x, y) \in \mathrm{rort}(A) \quad \text{iff} \quad a(x) \subseteq -a(y) \qquad \text{for all (some) } a \in A.$$

An abstract characterisation of strong relations of right orthogonality derived from an information system may be defined as follows. An *information frame of strong right orthogonality* (denoted FS-RORT in [DeO02]) is a binary relational structure $(X, \{R_P \mid P \subseteq \mathrm{Par}\})$ where the binary relations $R_P \subseteq X \times X$ (for each $P \subseteq \mathrm{Par}$) satisfy the following properties:

SF1 $R_{P \cup Q} = R_P \cap R_Q$
SF2 $R_\emptyset = X \times X$
SF3 R_P is co-weakly reflexive (i.e., $\forall x, \forall y,\ x(-R)y \Rightarrow x(-R)x$)
SF4 R_P is symmetric (i.e., $\forall x, \forall y,\ xRy \Rightarrow yRx$)

Let $\mathsf{Frm}_{\mathsf{AS-RORT}}$ denote the class of all information frames of strong right orthogonality. On the other hand an *information algebra of strong right orthogonality* (denoted AS-RORT in [DeO02]) is a Boolean algebra B with a family $\{g_P \mid P \subseteq \mathrm{Par}\}$ of sufficiency operators satisfying the following additional properties:

SA1 $g_{P \cup Q}(x) = g_P(x) \wedge g_Q(x)$
SA2 $g_\emptyset(x) = 1$
SA3 $x \wedge g_P(x) \leq g_P(1)$
SA4 $x \leq g_P g_P(x)$

Let $\mathsf{Alg}_{\mathsf{AS-RORT}}$ denote the class of all information algebras of strong right orthogonality.

Lemma 20. *Let $(B, \{g_P \mid P \subseteq \mathrm{Par}\})$ be an information algebra of strong right orthogonality. For each g_P ($P \subseteq \mathrm{Par}$), the corresponding binary relation R_{g_P} over $\mathcal{X}(B)$ satisfies properties* SF1 - SF4.

Proof: We prove SF1-SF3; SF4 is shown in [DeO02].

SF1 $FR_{g_{P \cup Q}}G$ iff $g_{P \cup Q}(G) \cap F \neq \emptyset$ iff $g_P(G) \cap g_Q(G) \cap F \neq \emptyset$ iff $g_P(G) \cap F \neq \emptyset$ and $g_Q(G) \cap F \neq \emptyset$ iff $FR_{g_P}G$ and $F_{g_Q}G$.
SF2 $FR_{g_\emptyset}G$ iff $g_\emptyset(G) \cap F \neq \emptyset$ iff $2^X \cap F \neq \emptyset$ iff $F \neq \emptyset$. The latter is always true since F is a prime filter. Thus $R_{g_\emptyset} = \mathcal{X}(B) \times \mathcal{X}(B)$.
SF3 Take any $F, G \in \mathcal{X}(B)$ with $F(-R_{g_P})G$ and $GR_{g_P}G$. Then $g_P(G) \cap F = \emptyset$ and $g_P(G) \cap G \neq \emptyset$. So, since F and G are non-empty, $g_P(G) = \emptyset$ and $g_P(G) \neq \emptyset$, which provides the required contradiction.

$\square$

Lemma 21. *Let* $(X, \{R_P \mid P \subseteq \mathrm{Par}\})$ *be an information frame of strong right orthogonality. For each binary relation* R_P $(P \subseteq \mathrm{Par})$*, the corresponding sufficiency operator* g_{R_P} *over* 2^X *satisfies properties* SA1 - SA4.

Proof: We prove SA1-SA3; SA4 is shown in [DeO02].

SA1 $x \in g_{R_{P\cup Q}}(A)$ iff $A \subseteq R_{P\cup Q}(x)$ iff $A \subseteq R_P(x) \cap R_Q(x)$ iff $A \subseteq R_P(x)$ and $A \subseteq R_Q(x)$ iff $x \in g_{R_P}(A) \cap g_{R_Q}(A)$, where the third double implication holds by definition of intersection and greatest lower bound.

SA2 $x \in g_{R_\emptyset}(A)$ iff $A \subseteq R_\emptyset(x)$ iff $A \subseteq X$. The latter is always true so $g_{R_\emptyset}(A) = X$.

SA3 Take any $x \in X$ such that $x \in A \cap g_{R_P}(A)$ and $x \notin g_{R_P}(X)$. Then $x \in A$ and $A \subseteq R_P(x)$ and $X \not\subseteq R_P(x)$. So xR_Px and for some $y \in X$ $x(-R)_Py$. Thus, xR_Px and, by property SF3 of R_P, $x(-R)x$, which provides the required contradiction.

□

A representation theorem analogous to Theorem 16 holds for algebras and frames of strong right orthogonality. Also Theorems 6, 7, 8 are applicable to algebras and frames of strong right orthogonality, these being special sufficiency algebras and special frames, respectively.

The language $\mathsf{Lan}_{\mathsf{AS-RORT}}$ relevant for algebras and frames of strong right orthogonality is an extension of the modal language Lan_M with a family of $\{[\![R_P]\!] \mid P \subseteq \mathrm{Par}\}$ of sufficiency operators. Algebraic semantics of the language is provided by the class $\mathsf{Alg}_{\mathsf{AS-RORT}}$ and the frame semantics by the class $\mathsf{Frm}_{\mathsf{AS-RORT}}$. The notion of a model based on a frame of $\mathsf{Frm}_{\mathsf{AS-RORT}}$, satisfaction relation, and the notions of truth in a model, in a frame and in a class of frames are analogous to the respective notions in Section 2. In view of Lemma 21 the complex algebra theorem (CA) holds for $\mathsf{Frm}_{\mathsf{AS-RORT}}$. From the representation theorem and (CA) we obtain a Duality via Truth theorem, and also the equivalence of preservation of truth.

6 Conclusion

We presented a Duality via Truth results for modal algebras, sufficiency algebras and for two classes of information algebras based on modal or sufficiency algebras, respectively. The main idea of Duality via Truth is to 'lift' the concepts of complex algebra and canonical frame so that they are assigned to an abstract frame and a general algebra, not only to a canonical frame and complex algebra. Once a Duality via Truth is established for a formal language with algebraic and frame semantics, a natural question arises regarding a suitable deduction mechanism for the language. Duality via Truth theorem guarantees that once we prove a completeness theorem with respect to one of the semantics, then we get it with respect to the other semantics too. Often, once the algebraic semantics of the language is given, a Hilbert-style axiomatisation can be derived from it. However, in the paper we do not consider any deduction methods for the presented languages.

Other Duality via Truth results can be found in [OrV03] (for lattice based languages with modal, sufficiency, necessity and dual sufficiency operators). The complete proof systems for these languages are also presented there. Duality via Truth results for a language of lattice-based relation algebras and for languages of substructural logics can be easily developed based on the representation results presented in [OrR05,DOR03,Rew03,DORV05]

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