Brock University

Department of Computer Science

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M. Winter
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Brock University
Department of Computer Science
St. Catharines, Ontario
Canada L2S 3A1
www.cosc.brocku.ca

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Michael Winter*

Department of Computer Science,
Brock University,
St. Catharines, Ontario, Canada, L2S 3A1
mwinter@brocku.ca

Abstract. In this paper we focus on a time-extended theory of contact. It turns out that a suitable theory can be defined using $\mathcal{L}$-valued or $\mathcal{L}$-fuzzy version of a contact relation. We study this structure in the context of Goguen categories - a suitable categorical formalization of $\mathcal{L}$-valued/fuzzy relations.

1 Introduction

Contact relations have been studied in the context of qualitative or 'pointless' geometry since the early 1920's and nowadays in qualitative spatial reasoning.

In this paper we want to recall a suitable set of axioms for contact in a static world (c.f. [3]), i.e., in a world without any notion of time. In such a situation the contact relation C is a regular binary relation taking values of the Boolean lattice with two elements, true (or 1) and false (or 0), indicating that the pair of regions in question is either in contact or not.

Then we switch to regions moving in time. It turns out that a convenient definition of contact uses an $\mathcal{L}$-valued/fuzzy relation. The underlying concept is a generalization of fuzzy relations using values of an arbitrary complete Brouwerian lattice $\mathcal{L}$ instead of the unit interval $[0, 1]$ of the real numbers [6, 21]. In our application the degree of 'x is in contact to y' is given by the time x and y are indeed in contact. Since every complete Brouwerian lattice has a least element 0 and a greatest element 1 a (static) contact relation can be seen as a so-called crisp $\mathcal{L}$-fuzzy relation.

In order to find suitable axioms in this situation, which coincide with the original axioms in the static case, we use the theories of Dedekind and Goguen categories. Goguen categories are a categorical formalization of $\mathcal{L}$-valued/fuzzy relations as Dedekind categories or allegories are of regular relations [4, 17–20]. We first translate the original axioms for contact into the language of Dedekind categories. Since the language of Dedekind categories is a subset of the language of Goguen categories we are able to investigate those axioms in the $\mathcal{L}$-valued/fuzzy world. It turns out that the same axioms can be used as a suitable theory of time-dependent contact in an arbitrary Goguen category.

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Last but not least, we focus on the notion of (relative) movement and coordinate systems in such a time-dependent contact structure. It is not surprising that we need the additional operations of a Goguen category in order to grasp those notions.

2 Relations and Dedekind categories

If R is a concrete relation between to sets A and B, i.e., $R \subseteq A \times B$, we use the notation xRy instead of $(x, y) \in R$ to indicate that x and y are in relation R. A suitable categorical description of relations is given by Dedekind categories [10, 11]. This kind of categories are called locally complete division allegories in [4].

Throughout this paper, we use the following notations. To indicate that a morphism R of a category $\mathcal{R}$ has source A and target B we write $R : A \to B$. The collection of all morphisms $R : A \to B$ is denoted by $\mathcal{R}[A, B]$ and the composition of a morphism $R : A \to B$ followed by a morphism $S : B \to C$ by $R; S$. Last but not least, the identity morphism on A is denoted by $\mathbb{I}_A$.

Definition 1. *A Dedekind category $\mathcal{R}$ is a category satisfying the following:*

1. *For all objects A and B the collection $\mathcal{R}[A, B]$ is a complete Brouwerian lattice. The elements of $\mathcal{R}[A, B]$ are called (abstract) relations (with source A and target B). Meet, join, the induced ordering, the least and the greatest element are denoted by $\sqcap, \sqcup, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$, respectively.*
2. *There is a monotone operation $^\smile$ (called converse) mapping a relation $Q : A \to B$ to $Q^\smile : B \to A$ such that for all relations $Q : A \to B$ and $R : B \to C$ the following holds: $(Q; R)^\smile = R^\smile; Q^\smile$ and $(Q^\smile)^\smile = Q$.*
3. *For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law $(Q; R) \sqcap S \sqsubseteq Q; (R \sqcap (Q^\smile; S))$ holds.*
4. *For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) such that for all $X : A \to B$ the following holds: $X; R \sqsubseteq S \iff X \sqsubseteq S/R$.*

Notice, that by convention composition binds more tightly than meet. Therefore, Axiom 3 may be written as $Q; R \sqcap S \sqsubseteq Q; (R \sqcap Q^\smile; S)$.

Corresponding to the left residual, we define the right residual by $Q\backslash R := (R^\smile/Q^\smile)^\smile$. This relation is characterized by $Q; Y \sqsubseteq R \iff Y \sqsubseteq Q\backslash R$.

Because the so-called Tarski rule

$$R \neq \perp\!\!\!\perp_{AB} \implies \top\!\!\!\top_{CA}; R; \top\!\!\!\top_{BD} = \top\!\!\!\top_{CD} \quad \text{for all objects } C \text{ and } D$$

is equivalent to a generalized version of the notion of simplicity known from universal algebra, we call a Dedekind category simple iff the Tarski rule is valid.

Recall that a complete Brouwerian lattice has relative pseudo complements, i.e., for all elements x, y there is an element $x \to y$ so that $x \sqcap z \sqsubseteq y$ iff $z \sqsubseteq x \to y$. Of course, this applies to the sets $\mathcal{R}[A, B]$ for all objects A and B of a Dedekind category. Furthermore, recall that in the case of a Boolean algebra the relative pseudo complement $x \to y$ is given by $\bar{x} \sqcup y$ where $\bar{x}$ denotes the complement of x.

In a Dedekind category a subset can represented by vector, i.e., a relation $v : A \to A$ with $\top\!\!\!\top_{AA}; v = v$. Furthermore, a relation $Q : A \to B$ is called total iff $Q;\top\!\!\!\top_{BB} = \top\!\!\!\top_{AB}$. Notice, that in a simple Dedekind category all vectors $v \neq \perp\!\!\!\perp_{AA}$ are total since $\top\!\!\!\top_{AA} = \top\!\!\!\top_{AA}; v;\top\!\!\!\top_{AA} = v;\top\!\!\!\top_{AA}$.

An important concept in Dedekind categories are relational products, i.e., the abstract version of a Cartesian product. A relational product of two objects A and B is an object $A \times B$ together with two relation $\pi : A \times B \to A$ and $\rho : A \times B \to B$ such that

$$\pi^{\smile}; \pi \sqsubseteq \mathbb{I}_A, \quad \rho^{\smile}; \rho \sqsubseteq \mathbb{I}_B \quad \pi^{\smile}; \rho = \top\!\!\!\top_{AB}, \quad \pi; \pi^{\smile} \sqcap \rho; \rho^{\smile} = \mathbb{I}_{A \times B}.$$

A relation $P : A \to A$ of a Dedekind category is called a pre-order iff P is reflexive $\mathbb{I}_A \sqsubseteq P$ and transitive $P; P \sqsubseteq P$. If P is, in addition, antisymmetric $P \sqcap P^{\smile} \sqsubseteq \mathbb{I}_A$ it is called an ordering. The upper/lower bounds with respect to a pre-order P of subsets given by a relation $R : B \to A$ are computed by $\mathrm{ub}_P(R) := R^{\smile}\backslash P$ and $\mathrm{lb}_P(R) = \mathrm{ub}_{E^{\smile}}(P) = R^{\smile}\backslash P^{\smile}$, respectively. Consequently, the least upper bound is given by $\mathrm{lub}_P(R) = \mathrm{ub}_P(R) \sqcap \mathrm{lb}_P(\mathrm{ub}_P(R))$. If $\mathrm{lub}_P(R) \neq \perp\!\!\!\perp_{BA}$ this relation computes the equivalence class (with respect to the equivalence relation $P \sqcap P^{\smile}$ induced by P) of least upper bounds. If P is an ordering this class is a singleton. For further details on the relational description of bounds we refer to [13, 14, 16].

A pre-order P is called an upper semi-prelattice iff $J_P := \mathrm{lub}_P(\pi \sqcup \rho) : A \times A \to A$ is total, i.e., if there is an equivalence class of least upper bounds for every pair of elements. If P is an ordering we call it an upper semi-lattice. Notice, that it can be shown that $\mathrm{lub}_P(\pi \sqcup \rho) = \pi; P \sqcap \rho; P \sqcap ((\pi; P \sqcap \rho; P)^{\smile}\backslash P^{\smile})$. P is said to provide a least element iff the vector $0_P := \mathrm{lub}_P(\perp\!\!\!\perp_{AA})$ is total. If it is clear from the context we omit the index P.

3 $\mathcal{L}$-valued/fuzzy relations

In fuzzy theory usually relations taking values form the unit interval $[0, 1]$ of the real numbers are considered. A more general approach uses $\mathcal{L}$-valued/fuzzy relations, e.g., relations taking values from an arbitrary complete Brouwerian lattice $\mathcal{L}$.

Definition 2. *Let $\mathcal{L}$ be a complete Brouwerian lattice. Then the structure of $\mathcal{L}$-fuzzy relations is defined as follows:*

1. *The objects are sets.*
2. *A relation $Q : A \to B$ between two sets A and B is function from $A \times B$ to $\mathcal{L}$.*
3. *For $Q : A \to B$ and $R : B \to C$ composition is defined by*

$$(Q; R)(x, z) := \bigsqcup_{y \in B} (Q(x, y) \sqcap R(y, z)).$$

4. *For $Q, R : A \to B$ the join and meet operations are defined by $(Q \sqcup R)(x, y) := Q(x, y) \sqcup R(x, y)$ and $(Q \sqcap R)(x, y) := Q(x, y) \sqcap R(x, y)$, respectively.*
5. *For $Q : A \to B$ the converse relation is defined by $Q^{\smile}(y, x) := Q(x, y)$.*
6. *The identity, zero and universal elements are defined by*

$$\mathbb{I}_A(x, y) := \begin{cases} 0 : x \neq y \\ 1 : x = y, \end{cases} \qquad \perp\!\!\!\perp_{AB}(x, y) := 0, \\ \top\!\!\!\top_{AB}(x, y) := 1.$$

It is easy to verify that the structure of $\mathcal{L}$-fuzzy relations defined above is a Dedekind category. The residual operation is given by

$$(Q \backslash R)(x, y) = \prod_z (Q(z, x) \to R(z, y))$$

If R is a concrete $\mathcal{L}$-valued/fuzzy relation between two sets A and B we usually write xRy instead of $R(x, y)$. The element $xRy \in \mathcal{L}$ is interpreted as the degree of validity of the property 'x and y are in relation R'. Notice, that a regular relation is a special case of an $\mathcal{L}$-valued/fuzzy relation where $\mathcal{L}$ is the Boolean algebra with two elements.

On the other hand, the set of regular relations is embedded (up to isomorphism) in the set of $\mathcal{L}$-valued/fuzzy relations. This substructure is given by the so-called 0-1 crisp relations, i.e., the relations R with $xRy \in \{0, 1\}$ for all x and y. According to this substructure two additional operations on the set of $\mathcal{L}$-valued/fuzzy relations can be defined. Notice, that for regular relations these operations are trivial.

$$yR^{\downarrow}z := \begin{cases} 1 \text{ iff } yRz = 1 \\ 0 \text{ iff } yRz \neq 1 \end{cases} \qquad yR^{\uparrow}z := \begin{cases} 1 \text{ iff } yRz \neq 0 \\ 0 \text{ iff } yRz = 0 \end{cases}$$

The operations above map a relation R to the greatest 0-1 crisp relation R contains and to the least 0-1 crisp relation R is included in, respectively.

4 Goguen categories

In some sense a relation of a Dedekind category may be seen as an $\mathcal{L}$-valued relation. The lattice $\mathcal{L}$ may be characterized by scalar relations, i.e., relations $\alpha : A \to A$ satisfying $\alpha_A \sqsubseteq \mathbb{I}_A$ and $\top\!\!\!\top_{AA};\alpha_A = \alpha_A;\top\!\!\!\top_{AA}$. We will denote the set of scalar relations in $\mathcal{R}$ on A by $\mathrm{Sc}_{\mathcal{R}}(A)$. Notice, that if $\mathcal{R}$ is simple then $\mathrm{Sc}_{\mathcal{R}}(A) = \{\perp\!\!\!\perp_{AA}, \mathbb{I}_A\}$, i.e., the relations are based on the Boolean algebra with two elements.

Several notions of crispness within a Dedekind category were introduced and discussed in [5, 7, 8]. In [17] it was shown that the theory of Dedekind categories is too weak to express basic notions of $\mathcal{L}$-fuzzy relations such as crispness. Therefore, an extended categorical structure – Goguen Categories – was introduced. This approach adds abstract versions of the two operations $R^{\downarrow}$ and $R^{\uparrow}$ to Dedekind categories.

Definition 3. *A Goguen category $\mathcal{G}$ is a Dedekind category with $\top\!\!\top_{AB} \neq \perp\!\!\perp_{AB}$ for all objects A and B together with two operations $^{\uparrow}$ and $^{\downarrow}$ satisfying the following:*

1. *$R^{\uparrow}, R^{\downarrow} : A \to B$ for all $R : A \to B$.*
2. *$(^{\uparrow}, ^{\downarrow})$ is a Galois correspondence.*
3. *$(R^{\smile}; S^{\downarrow})^{\uparrow} = R^{\uparrow\smile}; S^{\downarrow}$ for all $R : B \to A$ and $S : B \to C$.*
4. *If $\alpha_A \neq \perp\!\!\perp_{AA}$ is a non-zero scalar then $\alpha_A^{\uparrow} = \mathbb{I}_A$.*
5. *For all antimorphisms[1] $f : \mathrm{Sc}_{\mathcal{G}}(A) \to \mathcal{G}[A, B]$ such that $f(\alpha_A)^{\uparrow} = f(\alpha_A)$ for all $\alpha_A \in \mathrm{Sc}_{\mathcal{G}}(A)$ and all $R : A \to B$ the following equivalence holds*

$$R \sqsubseteq \bigsqcup_{\alpha_A \in \mathrm{Sc}_{\mathcal{G}}(A)} (\alpha_A; f(\alpha_A)) \iff$$
$$(\alpha_A \backslash R)^{\downarrow} \sqsubseteq f(\alpha_A) \text{ for all } \alpha_A \in \mathrm{Sc}_{\mathcal{G}}(A).$$

Again, it is not hard to verify that the Dedekind category of $\mathcal{L}$-fuzzy relations with $^{\uparrow}$ and $^{\downarrow}$ establishes a Goguen category.

Notice, that we do not need Axiom 5 in this paper so that the weaker theory of arrow categories, which does not require a second order axiom, would be sufficient.

In general, we have $R^{\downarrow} \sqsubseteq R \sqsubseteq R^{\uparrow}$. Consequently, we call a relation $R : A \to B$ of a Goguen category crisp iff $R^{\uparrow} = R$. Notice, that a relation is crisp iff $R^{\downarrow} = R$ iff $R^{\uparrow} = R^{\downarrow}$. The crisp fragment $\mathcal{G}^{\uparrow}$ of $\mathcal{G}$ is defined as the collection of all crisp relations of $\mathcal{G}$. This structure together with the inherited operations and constants is simple Dedekind category, i.e, an abstract counterpart of regular binary relations.

The projections π and ρ of a relational product in a Goguen category need not to be crisp. Under a condition of the set of scalar elements (or the underlying lattice $\mathcal{L}$) it can be shown that crisp versions actually exist [19]. The required condition is not very strong (and fulfilled in all examples considered) so that we assume the projections to be crisp in this paper.

Furthermore, it can be shown that the complete Brouwerian lattices $\mathrm{Sc}_{\mathcal{G}}(A)$ are isomorphic for all objects A and hence that the elements of a Goguen category are indeed $\mathcal{L}$-fuzzy relations based on a single underlying lattice $\mathrm{Sc}_{\mathcal{G}}(A)$.

In a Goguen category we may consider two different types of pre-orders. The induced equivalence relation $P \sqcap P^{\smile}$ may relate a pair of different elements with 'full' degree 1. In this situation there are equivalent elements even in a crisp interpretation. On the other hand, each pair of different elements could be related in $P \sqcap P^{\smile}$ with a degree strictly less than 1. A crisp interpretation might conclude that we do not have equivalent elements (in a crisp sense). In the language of Goguen categories this is expressed by validity of $(P \sqcap P^{\smile})^{\downarrow}$ being included in $\mathbb{I}$. We call a pre-order P with $(P \sqcap P^{\smile})^{\downarrow} \sqsubseteq \mathbb{I}$ weakly antisymmetric.

[1] A function $f : \mathcal{L}_1 \to \mathcal{L}_2$ between complete lattices is called an antimorphism, iff $f(\bigsqcup M) = \bigsqcap f(M)$ holds for all subsets M of $\mathcal{L}_1$.

Notice, that if the underlying lattice $\mathcal{L}$ is interpreted as time periods P is weakly antisymmetric if two different elements are not equivalent during the whole time considered.

Last but not least, in the next lemma we have summarized some basic properties of relations in a Goguen category.

Lemma 1. *Let $\mathcal{G}$ be a Goguen category. Then we have for all $Q, R : A \to B$*

1. $Q^{\downarrow\uparrow} = Q^{\downarrow}$ *and* $Q^{\uparrow\downarrow} = Q^{\uparrow}$,
2. $\left(\bigsqcup_{i\in I} R_i\right)^{\uparrow} = \bigsqcup_{i\in I} R_i^{\uparrow}$ *and* $\left(\bigsqcap_{i\in I} R_i\right)^{\downarrow} = \bigsqcap_{i\in I} R_i^{\downarrow}$,
3. $\left(Q \sqcap R^{\uparrow}\right)^{\uparrow} = Q^{\uparrow} \sqcap R^{\uparrow}$ *and* $\left(Q \sqcap R^{\downarrow}\right)^{\uparrow} = Q^{\uparrow} \sqcap R^{\downarrow}$,
4. $Q^{\uparrow\smile} = Q^{\smile\uparrow}$ *and* $Q^{\downarrow\smile} = Q^{\smile\downarrow}$,
5. $R = \bigsqcup_{\alpha_A \in \mathrm{Sc}_{\mathcal{G}}(A)} (\alpha_A; (\alpha_A \backslash R)^{\downarrow})$.

A proof may be found in [17]. Notice, that the last property is an abstract version of the α-cut Theorem known from fuzzy theory.

5 Static contact and time-dependent regions

Several theories of contact are proposed in the literature [1, 2, 9, 12, 15]. In this paper we want to use a very general and, therefore, basic theory. The order structure that is assumed on the set of regions throughout the different proposals is at least an upper semi-lattice (c.f. [3]).

Let C be a relation on an upper semi-lattice $(L, +, \leq)$ with a least element 0. Then C is called a contact relation iff it fulfills the following axioms:

C0: $\forall x : \neg(xC0)$, *"No region is in contact to the empty region."*
C1: $\forall x : x \neq 0 \Rightarrow xCx$, *"Contact is reflexive for non-empty regions."*,
C2: $\forall x, y : xCy \Rightarrow yCx$, *"Contact is symmetric."*,
C3: $\forall x, y, z : xCy$ and $y \leq z \Rightarrow xCz$, *"Contact is monotonic."*,
C4: $\forall x, y, z : (x + y)Cz \Rightarrow xCz$ or yCz, *"Contact fulfils the sum axiom."*,
C5: $\forall x, y : (\forall z : xCz \Leftrightarrow yCz) \Rightarrow x = y$. *"Contact is extensional."*.

Notice, that using C4 it can be shown that C5 is equivalent to the so-called compatibility axiom

C5': $\forall x, y : (\forall z : xCz \Rightarrow yCz) \Rightarrow x \leq y$.

Later on, we will generalize this approach by requiring just a pre-order instead of the ordering $\leq$. In that case C5' seems to be more suitable since C5 implies that the equivalence relation induced by the pre-order is the identity, i.e., the pre-order is in fact an ordering.

A traditional example of a contact structure is the set of regular closed sets of a connected regular T_0 space together with Whitehead's contact relation xCy iff $x \cap y \neq \emptyset$. This model and hence the whole theory is static, i.e., it does not use any notion of time.

By translating the first-order axioms given above into equations of the relational language we are able to define an abstract notion of a contact structure in an arbitrary Dedekind category. As mentioned above we take a more general approach by using a pre-order instead of an ordering.

Definition 4. *Let $\mathcal{R}$ be a Dedekind category, $P : A \to A$ be an upper semi-prelattice with a least element, $C : A \to A$ be a relation, and $(A \times A, \pi, \rho)$ a relational product. Then C is called a contact relation on (A, P) iff*

C0: $C \sqcap 0_P = \perp\!\!\!\perp_{AA}$,
C1: $\mathbb{I}_A \sqsubseteq C \sqcup 0_P$,
C2: $C^{\smile} \sqsubseteq C$,
C3: $C; P \sqsubseteq C$,
C4: $J_P; C \sqsubseteq (\pi \sqcup \rho); C$,
C5: $C \backslash C \sqsubseteq P$.

The pair (P, C) is also called a contact structure. Furthermore (P, C) is called antisymmetric if P is, i.e., if P is an ordering.

Now, we want to switch to regions moving in time. We will establish a notion of a standard model and derive the abstract properties thereof. A region in this context is a function mapping points in time to fixed set of static regions. The standard example will be a subset R of all functions from $\mathbb{R}$ to the regular closed sets of a connected regular T_0 space T, for instance the Euclidean plane. Probably we will require that the elements in R fulfill some kind of a continuity property so that R is not necessarily the set of all function. The only additional property we want to add is that at any point in time every (static) region is covered by a moving region, i.e., that for all $t \in \mathbb{R}$ and regular closed set r of T there is a function $f \in R$ with $f(t) = r$. We may, for example, require that the set of constant-valued functions (which are, of course, continuous) is a subset of R.

Definition 5. *Let X and Y be sets and F be a subset of all functions from X to Y. Then F is called dense iff for all $x \in X$ and $y \in Y$ there is a function $f \in F$ with $f(x) = y$.*

First of all, we want to consider an order structure induced on R. If $\mathcal{P}$ denotes the powerset operation we define an $\mathcal{P}(\mathbb{R})$-valued/fuzzy relation P. The value fPg can be interpreted as the degree of f being less than g and is given by the set of all points in time $t \in \mathbb{R}$ where $f(t)$ is indeed less than $g(t)$.

Proposition 1. *Let X be a set, $P : Y \to Y$ be a pre-order and F be a subset of all functions from X to Y. Then the relation $\hat{P}$ defined by*

$$f\hat{P}g := \{x \in X \mid f(x)Pg(x)\}$$

is an $\mathcal{P}(X)$-valued/fuzzy pre-order on F. If P is an ordering then $\hat{P}$ is weakly antisymmetric.

Proof. Since $f(x)Pf(x)$ for all x we have $f\hat{P}f = X$ and hence $\mathbb{I} \sqsubseteq \hat{P}$. Now, suppose $x \in (f\hat{P}g \cap g\hat{P}h)$. Then we have $f(x)Pg(x)$ and $g(x)Ph(x)$ and we conclude $f(x)Ph(x)$ by the transitivity of P. This implies $f\hat{P}g \cap g\hat{P}h \subseteq f\hat{P}h$ and hence $f(\hat{P};\hat{P})h = \bigcup_g (f\hat{P}g \cap g\hat{P}h) \subseteq f\hat{P}h$ for all f and h or equivalently $\hat{P};\hat{P} \sqsubseteq \hat{P}$. Finally, suppose P is antisymmetric and $f(\hat{P} \sqcap \hat{P}^{\smile})^{\downarrow} g \neq \emptyset$. Then we have $f(\hat{P} \sqcap \hat{P}^{\smile})g = X$ and hence $f(x)Pg(x)$ and $g(x)Pf(x)$ for all $x \in X$. This implies $f = g$ and hence $(\hat{P} \sqcap \hat{P}^{\smile})^{\downarrow} \sqsubseteq \mathbb{I}$. $\square$

In addition, we want to define a join operation $J_{\hat{P}}$ on R with respect to the pre-ordering $\hat{P}$. $J_{\hat{P}}$ will be a $\mathcal{P}(\mathbb{R})$-valued/fuzzy relation between pairs of elements of R and R. The value $(f,g)J_{\hat{P}}h$ can be interpreted as the degree of h being the join of f and g and is given by the set of all points in time $t \in \mathbb{R}$ where $h(t)$ is indeed the join of $f(t)$ and $g(t)$.

Proposition 2. *Let X be a set, (Y,P) be an upper semi-prelattice and F be a dense subset of functions from X to Y. Then the relation $J_{\hat{P}}$ defined by*

$$(f,g)J_{\hat{P}}h := \{x \in X \mid (f(x),g(x))J_P h(x)\}$$

is total and equal to $\mathrm{lub}_{\hat{P}}(\pi \sqcup \rho)$*, i.e., $\hat{P}$ is an upper semi-prelattice on F.*

Proof. Since (Y,P) is an upper semi-prelattice there is an r with $(f(x),g(x))J_P r$ for all $x \in X$ and every pair $f(x)$ and $g(x)$. By the density of F there is a function $k \in R$ with $k(x) = r$ and hence $(f(x),g(x))J_P k(x)$ or equivalently $x \in (f,g)J_{\hat{P}}k$. We conclude $x \in \bigcup_k ((f,g)J_{\hat{P}}k \cap k\top\!\!\top_{AA}h) = (f,g)(J_{\hat{P}};\top\!\!\top_{AA})h$ for all $f,g,h \in F$ and $x \in X$ so that $J_{\hat{P}};\top\!\!\top_{AA} = \top\!\!\top_{A\times A\,A}$ and hence the totality of $J_{\hat{P}}$ follows. Now, consider the following computation

$$\begin{aligned}
& x \in (f,g)J_{\hat{P}}h \\
\Leftrightarrow\ & (f(x),g(x))J_P h(x) \\
\Leftrightarrow\ & f(x)Ph(x) \text{ and } g(x)Ph(x) \text{ and} \\
& f(x)Pk(x) \text{ and } g(x)Pk(x) \text{ implies } h(x)Pk(x) \text{ for all } k \\
\Leftrightarrow\ & x \in (f,g)(\pi;\hat{P} \sqcap \rho;\hat{P})h \text{ and} \\
& x \in (f,g)(\pi;\hat{P} \sqcap \rho;\hat{P})k \text{ implies } x \in h\hat{P}k \text{ for all } k \\
\Leftrightarrow\ & x \in (f,g)(\pi;\hat{P} \sqcap \rho;\hat{P})h \text{ and} \\
& x \notin (f,g)(\pi;\hat{P} \sqcap \rho;\hat{P})k \text{ or } x \in h\hat{P}k \text{ for all } k \\
\Leftrightarrow\ & x \in (f,g)(\pi;\hat{P} \sqcap \rho;\hat{P})h \text{ and} \\
& x \in (f,g)(\pi;\hat{P} \sqcap \rho;\hat{P})k \to h\hat{P}k \text{ for all } k \\
\Leftrightarrow\ & x \in (f,g)(\pi;\hat{P} \sqcap \rho;\hat{P})h \text{ and } x \in \bigcap_k ((f,g)(\pi;\hat{P} \sqcap \rho;\hat{P})k \to h\hat{P}k) \\
\Leftrightarrow\ & x \in (f,g)(\pi;\hat{P} \sqcap \rho;\hat{P})h \text{ and } x \in (f,g)((\pi;\hat{P} \sqcap \rho;\hat{P})^{\smile} \backslash \hat{P}^{\smile})h \\
\Leftrightarrow\ & x \in (f,g)(\pi;\hat{P} \sqcap \rho;\hat{P} \sqcap ((\pi;\hat{P} \sqcap \rho;\hat{P})^{\smile} \backslash \hat{P}^{\smile}))h \\
\Leftrightarrow\ & x \in (f,g)\mathrm{lub}_{\hat{P}}(\pi \sqcup \rho)h,
\end{aligned}$$

which shows $J_{\hat{P}} = \mathrm{lub}_{\hat{P}}(\pi \sqcup \rho)$. $\square$

If P has a least element we may define a relation $0_{\hat{P}}$ on F by $f0_{\hat{P}}g = \{x \in X \mid g(x) = 0\}$. Similar to the last proposition it can be shown that $0_{\hat{P}}$ is total and $0_{\hat{P}} = \mathrm{lub}_{\hat{P}}(\perp\!\!\!\perp_{AA})$, i.e., that $0_{\hat{P}}$ is the least elements of $\hat{P}$. We omit the proof and just state the following proposition.

Proposition 3. *Let X be a set, (Y, P) be an upper semi-prelattice with a least element and F be a dense subset of all functions from X to Y. Then $\hat{P}$ is an upper semi-prelattice with least element. If P is an ordering then $\hat{P}$ is weakly antisymmetric.*

According to our standard model a time-dependent contact structure will be based on a weakly antisymmetric upper semi-prelattice with a least element.

We define that two such regions $r_1, r_2 \in R$ are in contact at a time $t \in \mathbb{R}$ if $r_1(t)Cr_2(t)$ where C is the (static) Whitehead's contact relation on T. As above, this allows us to define an $\mathcal{P}(\mathbb{R})$-valued/fuzzy contact relation on R by $r_1\hat{C}r_2 := \{t \in \mathbb{R} \mid r_1(t)Cr_2(t)\}$. The degree of the property 'r_1 is in contact with r_2' is exactly the set of all points in time when they are actually in (static) contact.

Proposition 4. *Let X be a set, (Y, P, C) be a (concrete) contact structure on the set Y and F be a dense subset of functions from X to Y. Then the relation $\hat{C}$ defined by*

$$f\hat{C}g := \{x \in X \mid f(x)Cg(x)\}$$

together with $\hat{P}$ is a contact structure on F.

Proof. By Proposition 3 it remains to show that $\hat{C}$ is a contact relation.

C0: Suppose $x \in f(\hat{C} \sqcap 0_P)g$. Then we have $f(x)Cg(x)$ and $x \in f0_{\hat{P}}g$. The latter is equivalent to $g(x) = 0$, a contradiction, and hence $f(\hat{C} \sqcap 0_{\hat{P}})g = \emptyset$ for all f and g or equivalently $\hat{C} \sqcap 0_{\hat{P}} = \perp\!\!\!\perp_{AA}$.
C1: Suppose $f(x) = 0$. Then we have $x \in f0_{\hat{P}}f$. If $f(x) \neq 0$ we have $f(x)Cf(x)$ and hence $x \in f\hat{C}f$. We conclude $x \in f(\hat{C} \sqcup 0_{\hat{P}})f$ for all $x \in X$ and $f \in F$ which shows C1.
C2: Suppose $x \in f\hat{C}^{\smile}g = g\hat{C}f$. Then we have $g(x)Cf(x)$ so that $f(x)C^{\smile}g(x)$ follows. We conclude $f(x)Cg(x)$ and hence $x \in f\hat{C}g$ which shows C2.
C3: Suppose $x \in f(\hat{C};\hat{P})h = \bigcup_g (f\hat{C}g \cap g\hat{P}h)$. Then there is a g so that $f(x)Cg(x)$ and $g(x)Ph(x)$. This implies $f(x)Ch(x)$ and hence $x \in f\hat{C}h$.
C4: Suppose $x \in (f,g)(J_{\hat{P}};\hat{C})h = \bigcup_k ((f,g)J_{\hat{P}}k \cap k\hat{C}h)$. Then there is a function k so that $(f(x), g(x))J_Pk(x)$ and $k(x)Ch(x)$. This implies $f(x)Ch(x)$ or $g(x)Ch(x)$ which is equivalent to $x \in (f,g)((\pi \sqcup \rho);\hat{C})h$.
C5: Finally, suppose $x \in f(\hat{C}\backslash\hat{C})h = \bigcap_g (g\hat{C}f \to g\hat{C}h)$. Then $x \in g\hat{C}f \to g\hat{C}h$ for all $g \in F$ which is equivalent to $x \notin g\hat{C}f$ or $x \in g\hat{C}h$. We conclude that either not $g(x)Cf(x)$ or $g(x)Ch(x)$ holds or equivalently that $g(x)Cf(x)$

implies $g(x)Ch(x)$ for all $g \in F$. Assume $r \in Y$ is an arbitrary region with $rCf(x)$. By the density of F there is a function $g_r \in F$ with $g_r(x) = r$. Together we conclude $r = g_r(x)Ch(x)$ and hence $rCf(x)$ implies $rCh(x)$ for all r. Consequently, we have $f(x)(C\backslash C)h(x)$. Since C is a (concrete) contact relation this implies $f(x)Ph(x)$ and hence $x \in f\hat{P}h$. □

Notice, that the two contact structures (Y, P, C) and $(F, \hat{P}, \hat{C})$, even though they fulfil the same set of axioms, are based on a different notion of a relation and hence the corresponding operations are defined differently.

In the standard example $\hat{P}$ is weakly antisymmetric. Recall that in this case two different regions cannot be equivalent during the whole time considered.

6 Coordinate Systems

In this section we want to introduce coordinate systems within an arbitrary contact structure. Such a system in our sense is given by a set of regions with the following properties. First of all, elements of a coordinate system are considered to be at rest, i.e., they are not moving. The system itself should consists of all such regions. The problem here is that we do not have any notion of non-moving regions since this would require recursively a fixed coordinate system. In our approach we are just able to compare two regions with respect to contact which can be used to introduce a notion of those regions being in relative rest to each other.

If two regions r_1 and r_2 are not in contact at any point in time or if the are in contact at all points in time we can safely assume that they are in relative rest to each other. The reason simply is that one possible interpretation of this situation is that r_1 and r_2 are not moving at all. All other regions are moving around (or within) r_1 and r_2.

The observations above lead immediately to the following definition.

Definition 6. *Let $\mathcal{G}$ be a Goguen category and (P, C) be a contact structure. Then we define the 'relative at rest' relation RR by $RR := C^{\uparrow} \to (C \sqcup \mathbb{I}_A)^{\downarrow}$.*

In the next lemma we have summarized some basic properties of the relation RR.

Lemma 2. *1. RR is crisp, reflexive and symmetric.*
2. If C is crisp then $RR = \top\!\!\top_{AA}$.

Proof. 1. In order to prove that RR is crisp it is sufficient to show $RR^{\uparrow} \sqsubseteq RR$ which follows from $RR^{\uparrow} \sqcap C^{\uparrow} = (RR \sqcap C^{\uparrow})^{\uparrow} \sqsubseteq (C \sqcup \mathbb{I}_A)^{\downarrow\uparrow} = (C \sqcup \mathbb{I}_A)^{\downarrow}$ using Lemma 1(1 & 3). From $\mathbb{I}_A \sqcap C^{\uparrow} \sqsubseteq \mathbb{I}_A \sqsubseteq (C \sqcup \mathbb{I}_A)^{\downarrow}$ we immediately conclude that RR is reflexive. The last assertion follows from

$$\begin{aligned}
RR^{\smile} \sqcap C^{\uparrow} &= (RR \sqcap C^{\uparrow\smile})^{\smile} && \\
&= (RR \sqcap C^{\smile\uparrow})^{\smile} && \text{Lemma 1(4)} \\
&\sqsubseteq (RR \sqcap C^{\uparrow})^{\smile} && \text{Axiom C2} \\
&\sqsubseteq (C \sqcup \mathbb{I}_A)^{\downarrow\smile} && \\
&= (C^{\smile} \sqcup \mathbb{I}_A)^{\downarrow} && \text{Lemma 1(4)} \\
&\sqsubseteq (C \sqcup \mathbb{I}_A)^{\downarrow} && \text{Axiom C2}
\end{aligned}$$

2. This follows immediately from $\top\!\!\!\top_{AA} \sqcap C^{\uparrow} = C^{\uparrow} = C = C^{\downarrow} \sqsubseteq (C \sqcup \mathbb{I}_A)^{\downarrow}$. □

Notice, that 2. of the last lemma indicates that in a static model all regions are in relative rest to each other.

Using this relation it is now possible to define a coordinate system. Such a system is given by a maximal vector of elements in relative rest to each other.

Definition 7. *Let $\mathcal{G}$ be a Goguen category and (P, C) be a contact structure. A crisp vector v is called a coordinate system (for (P, C)) iff it is maximal with respect to the property $v^{\smile}; v \sqsubseteq RR$.*

Using the Lemma of Zorn it can be shown that coordinate systems exist.

Proposition 5. *Let $\mathcal{G}$ be a Goguen category and (P, C) be a contact structure. Then there is a coordinate system for (P, C).*

Proof. Consider the set $M := \{v : A \to A \mid v \text{ is a crisp vector and } v^{\smile}; v \sqsubseteq RR\}$. Then M is not empty since $\perp\!\!\!\perp_{AA} \in M$. Let $v_1 \sqsubseteq v_2 \sqsubseteq \dots$ be a chain in M. Then $\bigsqcup_i v_i$ is a vector since $\top\!\!\!\top_{AA}; (\bigsqcup_i v_i) = \bigsqcup_i (\top\!\!\!\top_{AA}; v_i) = \bigsqcup_i v_i$ and crisp since $(\bigsqcup_i v_i)^{\uparrow} = \bigsqcup_i {v_i}^{\uparrow} = \bigsqcup_i v_i$. Furthermore, we have

$$(\bigsqcup_i v_i)^{\smile}; (\bigsqcup_i v_i) = \bigsqcup_{i,j} ({v_i}^{\smile}; v_j) \sqsubseteq \bigsqcup_i ({v_i}^{\smile}; v_i) \sqsubseteq RR$$

so that M is closed under unions of chains. The Lemma of Zorn implies that M has a maximal element which is by definition a coordinate system. □

The question arises whether a region can be uniquely represented by its coordinates. This property can be expressed in the language of relations by the injectivity (up to the equivalence relation induced by the pre-order P) of the function r mapping each region to the set of coordinate elements included in. This construction uses relational powers; the abstract counterpart of powersets. It can be shown that the function r is indeed injective if $P \sqsubseteq P; (\mathbb{I}_A \sqcap v^{\smile}; v); P$ holds, i.e., if at any time there is an element of the coordinate system in between every pair of regions. Furthermore, the regions that cannot be distinguished by that representation are exactly the regions that are equivalent with respect to P (and/or C), i.e., we have $r; r^{\smile} = P \sqcap P^{\smile}$. If P is weakly antisymmetric we get $(r; r^{\smile})^{\downarrow} = (P \sqcap P^{\smile})^{\downarrow} = \mathbb{I}_A$. Due to lack of space we cannot present the details in this paper.

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