



Brock University

Department of Computer Science

**A Representation Theorem for Boolean Contact Algebras**

Ivo Düntsch and Michael Winter  
Technical Report # CS-03-08  
September 2003

Brock University  
Department of Computer Science  
St. Catharines, Ontario  
Canada L2S 3A1  
[www.cosc.brocku.ca](http://www.cosc.brocku.ca)

---

# A Representation Theorem for Boolean Contact Algebras

Ivo Düntsch<sup>\*†</sup> & Michael Winter<sup>†</sup>  
Department of Computer Science  
Brock University  
St. Catharines, Ontario, Canada, L2S 3A1  
{duentsch|mwinter}@cosc.brocku.ca

September 25, 2003

## Abstract

We prove a representation theorem for Boolean contact algebras which implies that the axioms for the Region Connection Calculus [20] (RCC) are complete for the class of subalgebras of the algebras of regular closed sets of weakly regular connected  $T_1$  spaces.

## 1 Introduction

The main aim of the present paper is to establish a representation theorem for certain *contact structures* which have arisen in various fields such as qualitative spatial reasoning, proximity theory, and ontology. The common ground is what has become known as *mereotopology*, based, among others, on Whitehead's notion of connection [27], Leśniewski's mereology [12, 13], and "pointless geometry" which originates with the works of de Laguna [4], Nicod [16], and Tarski [25]. Historically, standard (models for) mereotopological structures were collections of regular closed (or regular open) sets of topological spaces  $\langle X, \tau \rangle$ . Following Whitehead [27], two regular closed sets are said to be *in contact*, if they have a non-empty intersection; the primary example is the collection of all regular closed sets of the Euclidean plane. Algebraizations of mereotopological structures have been considered for some time and in various ways, see e.g. [1, 3, 11, 20, 23]. In a sequence of papers, Pratt and Schoop have investigated the first order theory of Boolean algebras of polygons in the plane and have obtained some very satisfying results [17–19].

---

<sup>\*</sup>Ivo Düntsch gratefully acknowledges support from the National Sciences and Engineering Research Council of Canada.

<sup>†</sup>The order of authors is alphabetical and equal authorship is implied

From a different perspective, *proximity structures* have been investigated in a topological context since the 1950's; for an exhaustive treatment of proximity spaces we invite the reader to consult [15]. Proximity spaces are relational structures on families of sets that satisfy axioms which to some extent coincide with those for the connection structures mentioned earlier.

The search for representation theorems is motivated by a desire to relate general structures to – in some sense – more “familiar” ones, the most famous of such results being

- Each finite group is isomorphic to a group permutations [2].
- Each Boolean algebra is isomorphic to an algebra of sets [24].

Representation problems also play a part in the completeness of logical axioms: Having fixed a class of “standard models” of some theory, the non–existence of a standard model for a structure shows that the axiomatization is incomplete for this class.

Given the origin and original motivation for contact or proximity structures, one aimed at representing such structures as algebras of regular closed (or, equivalently, regular open) sets of some topological space where contact was defined in the Whiteheadian sense. Since contact algebras are first order structures, and the Boolean algebra  $\text{RegCl}(X)$  of all regular closed subsets of a topological space is complete [see 10], we need to be content with substructures of  $\text{RegCl}(X)$  if we want to stay in the realm of first order logic. Thus, a Boolean algebra  $B$  endowed with some contact relation  $C$  will be called *representable* if there are a topological space  $X$  and a mapping  $h : B \rightarrow \text{RegCl}(X)$  such that  $h$  is a Boolean embedding and  $xCy \iff h(x) \cap h(y) \neq \emptyset$ .

## 2 Topological spaces

For any notion not explained here, we invite the reader to consult [8]. We will denote topological spaces by  $\langle X, \tau \rangle$ , where  $\tau$  is the topology on  $X$ ; for  $x \subseteq X$ , we let  $\text{cl}_\tau(x)$  be the  $\tau$ -closure of  $x$ , and  $\text{int}_\tau(x)$  its  $\tau$ -interior. If  $\tau$  is understood, we will just speak of  $X$  as a topological space, and drop the subscripts from the operators. If  $x, y \in \tau$ , then  $x$  and  $y$  are called *separated*, if  $\text{cl}(x) \cap y = x \cap \text{cl}(y) = \emptyset$ . A non–empty open set  $x$  is called *connected* if it is not the union of two separated nonempty open sets. A set  $u \subseteq X$  is called *regular open* if  $u = \text{int}(\text{cl}(u))$ , and *regular closed*, if  $u = \text{cl}(\text{int}(u))$ . The set complement of a regular open set is regular closed and vice versa.  $\text{RegCl}(\langle X, \tau \rangle)$  is the collection of regular closed sets, and  $\text{RegOp}(\langle X, \tau \rangle)$  the collection of regular open sets; we will sometimes just write  $\text{RegCl}(X)$  or  $\text{RegCl}(\tau)$  (respectively,  $\text{RegOp}(X)$  or  $\text{RegOp}(\tau)$ ) if no confusion can arise. It is well known that  $\text{RegCl}(X)$  is a complete Boolean algebra with the operations  $a + b = a \cup b$ ,  $a \cdot b = \text{cl}(\text{int}(a \cap b))$ , and  $-a = \text{cl}(X \setminus a)$ . Note that we can have  $a \cdot b = \emptyset$ , while  $a \cap b \neq \emptyset$ . Similarly,  $\text{RegOp}(X)$  is a Boolean algebra with the operations  $a + b = \text{int}(\text{cl}(a \cup b))$ ,  $a \cdot b = a \cap b$ ,  $-a = \text{int}(X \setminus a)$ .

Let  $C_\tau$  be defined on  $\text{RegCl}(X)$  by  $aCb \iff a \cap b \neq \emptyset$ , and  $D_\tau$  be defined on  $\text{RegOp}(X)$  by  $aD_\tau b \iff \text{cl}(a) \cap \text{cl}(b) \neq \emptyset$ . These relations will be our standard contact relations, and the following known result shows that it is not structurally important whether we work with  $\text{RegCl}(X)$  or  $\text{RegOp}(X)$ :

**Lemma 2.1.**  *$\text{RegCl}(X)$  and  $\text{RegOp}(X)$  are isomorphic Boolean algebras, and the pairs  $\langle \text{RegCl}(X), C_\tau \rangle$  and  $\langle \text{RegOp}(X), D_\tau \rangle$  are isomorphic relational structures.*

*Proof.* It is easy to see that the assignment  $f : \text{RegOp}(X) \rightarrow \text{RegCl}(X)$  defined by  $u \mapsto \text{cl}(u)$  has the desired properties.  $\square$

A space is *regular* if a point  $x$  and a closed set not containing  $x$  have disjoint open neighborhoods, and *semiregular* if it has a basis of regular open sets; regularity implies semiregularity, but not vice versa.

Since representations of regions will be regular closed sets, we could restrict ourselves to semiregular spaces. The following result shows that we do not lose anything in this case. First, define the *semi-regularization* of  $\langle X, \tau \rangle$  as the topology  $r(\tau)$  on  $X$  whose open basis is  $\text{RegOp}(\tau)$ . Now,

**Proposition 2.2.** *Suppose that  $\langle X, \tau \rangle$  is a topological space. Then,  $\text{RegOp}(\tau) = \text{RegOp}(r(\tau))$ .*

*Proof.* [See also 8, p. 84] Let  $a \subseteq X$ . Then,

$$\text{cl}_{r(\tau)}(a) = -\bigcup\{m \in \text{RegOp}(\tau) : m \cap \text{cl}_\tau(a) = \emptyset\} \supseteq -\bigcup\{m \in \tau : m \cap \text{cl}_\tau(a) = \emptyset\} = \text{cl}_\tau(a).$$

Let  $a \in \text{RegOp}(\tau)$ . Then,

$$\begin{aligned} \text{int}_{r(\tau)} \text{cl}_{r(\tau)}(a) &= \text{int}_{r(\tau)} \left( -\bigcup\{m \in \text{RegOp}(\tau) : m \cap \text{cl}_\tau(a) = \emptyset\} \right), \\ &= \bigcup\{t \in \text{RegOp}(\tau) : t \cap m = \emptyset \text{ for all } m \in \text{RegOp}(\tau) \text{ with } m \cap \text{cl}_\tau(a) = \emptyset\}, \\ &= a, \end{aligned}$$

since  $a$  and  $t$  are regular open, and thus,  $t \subseteq \text{cl}_\tau(a)$  implies  $t \subseteq a$ .

Conversely, let  $a \in \text{RegOp}(r(\tau))$ . Then,

$$a = \text{int}_{r(\tau)} \text{cl}_{r(\tau)}(a) = \bigcup\{t \in \text{RegOp}(\tau) : t \subseteq \text{cl}_{r(\tau)}(a)\}.$$

Now,  $\text{int}_\tau \text{cl}_\tau(a) \in \text{RegOp}(\tau)$ , and thus,  $a \subseteq \text{int}_\tau \text{cl}_\tau(a) \subseteq \text{int}_{r(\tau)} \text{cl}_{r(\tau)}(a) = a$ .

If  $a \in \text{RegOp}(\tau)$ , then, by the preceding consideration,  $-\tau a = -_{r(\tau)} a$ , and thus,  $\text{cl}_\tau(a) = \text{cl}_{r(\tau)}(a)$ . This implies the claim.  $\square$

Together with Lemma 2.1, the following now is a straightforward consequence, and it shows that we can restrict our attention to semiregular spaces:

**Lemma 2.3.** *The enhanced Boolean algebras  $\langle \text{RegCl}(\tau), C_\tau \rangle$  and  $\langle \text{RegCl}(r(\tau)), C_{r(\tau)} \rangle$  are isomorphic.*

It is well known that  $X$  is regular, if and only if for each non-empty  $u \in \tau$  and each  $x \in u$  there is some  $v \in \tau$  such that  $x \in v \subseteq \text{cl}(v) \subseteq u$ . We call  $X$  *weakly regular* if it is semiregular and for each non-empty  $u \in \tau$  there is some non-empty  $v \in \tau$  such that  $\text{cl}(v) \subseteq u$ . Weak regularity may be called a “pointless version” of regularity, and each regular space is weakly regular.

A topological space  $X$  is called *normal*, if any two disjoint closed sets can be separated by disjoint open sets.  $X$  is called  $\kappa$ -*normal*, if any two disjoint regular closed sets can be separated by disjoint open sets. Then,

$$X \text{ is normal} \Rightarrow X \text{ is } \kappa\text{-normal} \Rightarrow X \text{ is regular} \Rightarrow X \text{ is weakly regular} \Rightarrow X \text{ is semiregular,}$$

and none of the implications can be reversed: Shchepin [21] gives an example of a  $\kappa$ -normal space which is not normal, and of a regular space which is not  $\kappa$ -normal. Example 5.3, together with the Representation Theorem 5.6 exhibits a weakly regular  $T_1$  space which is not  $T_2$ , and thus, it is not regular. Finally, here is an example of a connected, semiregular  $T_2$  space which is not weakly regular in a very strong way:

**Example 2.4.** The *relatively prime integer topology* is defined as follows [see 22, Example 60, for details]: Let  $X$  be the set of positive integers, and for all  $a, b \in X$  let  $U_a(b) = \{b + n \cdot a : n \in \mathbb{Z}\} \cap X$ . Consider the topology  $\tau$  generated by the basis

$$\mathfrak{B} = \{U_a(b) : a, b \in X, \text{gcd}(a, b) = 1\}.$$

This topology is  $T_2$  and has a basis of regular open sets. Furthermore, the closures of any two non-empty open sets intersect, and it follows that  $X$  is connected and not weakly regular.  $\square$

This also shows that weak regularity is independent of the  $T_2$  property.

### 3 Boolean contact algebras

In the sequel,  $\langle B, +, \cdot, -, 0, 1 \rangle$  will denote a Boolean algebra (BA); we will usual identify algebraic structures with their base set.  $B^+$  is the set of all non-zero elements of  $B$ . Furthermore, if  $a \in B$  and  $T \subseteq B$ , we let  $a \leq T$  if and only if  $a \leq b$  for all  $b \in T$ .

**Lemma 3.1.** 1. *For each ideal  $I$  and each filter  $F$  such that  $I \cap F = \emptyset$  there is some ultrafilter  $U$  of  $B$  such that  $F \subseteq U$  and  $U \cap I = \emptyset$ .*

2. *Suppose that  $M \subseteq B$  such that*

$$(a) \ 0 \notin M.$$

(b)  $y + z \in M \Rightarrow y \in M$  or  $z \in M$  for all  $y, z \in B$ .

(c) If  $x \in M$  and  $x \leq y$ , then  $y \in M$ .

Then, for each  $x \in M$  there is an ultrafilter  $F$  such that  $x \in F$  and  $F \subseteq M$ .

*Proof.* 1. is a well known consequence of the Boolean prime ideal theorem, see e.g. [10, p 33 ff]. 2. has previously been shown in the context of proximity structures [15, Lemma 5.7]; for completeness we provide a proof. Let  $x \in M$ , and suppose that  $F$  is a subset of  $M$  containing  $x$  which is maximal with respect to the property

$$(3.1) \quad y_0, \dots, y_k \in F \text{ implies } y_0 \cdot \dots \cdot y_k \in M.$$

The existence of such  $F$  is an easy consequence of Zorn's Lemma, and we show that  $F$  is an ultrafilter. Suppose that  $a, b \in F$ , and that  $y_0, \dots, y_k \in F$ . Then,  $y_0 \cdot \dots \cdot y_k \cdot a \cdot b$  is a finite product of elements of  $F$ , and thus it is an element of  $M$ ; the maximality of  $F$  now implies  $a \cdot b \in F$ . If  $a \in F$ ,  $a \leq b$ ,  $y_0, \dots, y_k \in F$ , then  $y_0 \cdot \dots \cdot y_k \cdot a \in F$ . By condition 2c above and  $a \leq b$ , we have  $y_0 \cdot \dots \cdot y_k \cdot a \in M$ , and the maximality of  $F$  implies  $b \in F$ . Finally, assume that there is some  $a \in B$  such that  $a, -a \notin F$ . Then, there are  $y, z \in F$  such that  $y \cdot a \notin M$  and  $z \cdot -a \notin M$ . By (3.1) and  $y, z \in F$  we have  $y \cdot z \in M$ . On the other hand,  $y \cdot a, z \cdot -a \notin M$  and condition 2c on  $M$  imply that  $y \cdot z \cdot a \notin M$  and  $y \cdot z \cdot -a \notin M$ . Since  $y \cdot z \cdot a + y \cdot z \cdot -a = y \cdot z \in M$ , this contradicts condition 2b.  $\square$

A binary relation  $C$  on  $B$  is called a *contact relation* if it satisfies

$$C0. \quad aCb \Rightarrow a, b \neq 0.$$

$$C1. \quad a \neq 0 \Rightarrow aCa.$$

$$C2. \quad C \text{ is symmetric.}$$

$$C3. \quad aCb \text{ and } b \leq c \Rightarrow aCc.$$

$$C4. \quad aC(b+c) \Rightarrow aCb \text{ or } aCc.$$

$$C5. \quad C(a) \subseteq C(b) \Rightarrow a \leq b.$$

As shown in [26], in the presence of the other axioms we can replace C5 by

$$C5'. \quad \text{If } a \not\leq b, \text{ there is some } c \in B \text{ such that } aCc \text{ and } c(-C)b.$$

If  $B$  is a Boolean algebra and  $C$  a contact relation on  $B$ , the pair  $\langle B, C \rangle$  will be called a *Boolean contact algebra* (BCA). We will consider the following additional properties of  $C$ :

C6.  $a(-C)b \Rightarrow (\exists c)[a(-C)c \text{ and } -c(-C)b]$  (The *strong axiom* of [15]).

C7.  $a, b \neq 0 \text{ and } a + b = 1 \Rightarrow aCb$  ( $C$  is *connecting*).

Let  $aOb \iff a \cdot b \neq 0$ . The following Lemma lists some easy properties of  $C$ . The proof is left to the reader:

**Lemma 3.2.** 1.  $O$  is a contact relation on  $B$  and  $O \subseteq C$ .

2.  $a(-C)b \Rightarrow a \cdot b = 0$ .

3. If  $a \notin \{0, 1\}$ , there is some  $b \neq 0$  such that  $a(-C)b$ .

4.  $aCb \text{ or } aCc \Rightarrow aC(b + c)$ .

5.  $a \leq b \Rightarrow C(a) \subseteq C(b)$ .

The *Region Connection Calculus* (RCC) of Randell et al. [20] has received some prominence as a structure for qualitative spatial reasoning. RCC models are BCAs which satisfy C7. Indeed, the primary motivation for this paper was to find a standard topological representation for each RCC model.

The presence of the extensionality axiom C5 causes that the interesting BCAs have a not too simple structure. Recall that a Boolean algebra is called *finite – cofinite* if every nonzero element is a finite sum of atoms or the complement of such an element. In particular, every finite BA is finite–cofinite.

**Lemma 3.3.** 1. If  $B$  is a finite–cofinite algebra, there is exactly one contact relation on  $B$ .

2. If  $C$  satisfies C7, then  $B$  is atomless.

*Proof.* 1. We show that  $O$  is the only contact relation on  $B$ . Assume that  $C$  is a contact relation on  $B$ , and that  $C \neq O$ . Since  $O$  is the smallest contact relation on  $B$  by Lemma 3.2(1), there are  $x, y \in B^+$  such that  $xCy$  and  $x \cdot y = 0$ . Since  $B$  is finite-cofinite, and the meet of two cofinite elements is always nonzero, we may assume by C4 that  $y$  is an atom; by C3 we may assume that  $x = -y$ , i.e. that  $x$  is the antiatom disjoint to  $y$ . But then,  $x \neq 1$  and  $x$  is connected to every region, contradicting C5'.

2. This was shown in [6]. □

This shows that BCAs are not the best choice for reasoning with finite or discrete structures: The RCC models assume a continuous interpretation of the world since by Lemma 3.3(2) every region is infinitely divisible, and Lemma 3.3(1) tells us that finite BCAs are quite trivial. A way of coping with this situation is to omit C5 as in [14] and also weaken the other axioms such as in [5], see also [9].

For each  $a \in B^+$  we let  $I_a = \{b : a(-C)b\}$ .

**Lemma 3.4.** Let  $a \neq 0, 1$ . Then  $I_a$  is a proper non-trivial ideal of  $B$  and  $\sup I_a = -a$ .

*Proof.* Since  $a \neq 1$ , there is some  $b \neq 0, 1$  such that  $a(-C)b$  by C5', and thus,  $I_a$  is non-trivial. Since  $a \neq 0$ , we have  $aCa$ , and hence,  $I_a$  is proper. If  $b, c \in I_a$ , then  $a(-C)b$  and  $a(-C)c$ , and C4 shows that  $a(-C)(b+c)$ . If  $a(-C)b$  and  $c \leq b$ , then  $a(-C)c$  by C3, and it follows that  $I_a$  is an ideal.

Observe that by Lemma 3.2(2),  $-a$  is an upper bound of  $I_a$ . Let  $b \leq s$  for all  $b \in I_a$ , and assume  $-a \not\leq s$ , i.e.  $a+s \neq 1$ . Since  $a+s \notin \{0, 1\}$ , there is some  $d \in B^+$  such that  $(a+s)(-C)d$  by C5'. By C4,  $a(-C)d$  and  $s(-C)d$ . The first condition implies  $d \in I_a$ , and hence  $d \leq s$ , and the second condition implies  $s \cdot d = 0$ , which together imply  $d = 0$ , contradicting our choice of  $d$ .  $\square$

If  $M \subseteq B$ , we let  $I_M$  be the ideal of  $B$  generated by  $\{I_a : a \in M\}$ . Thus,

$$(3.2) \quad x \in I_M \iff (\exists y_0, \dots, y_k \in M)(\exists p_0, \dots, p_k)[x = p_0 + \dots + p_k \text{ and } p_0(-C)y_0, \dots, p_k(-C)y_k].$$

We also let  $F_M = \{-x : x \in I_M\}$  be the filter containing the complements of the elements of  $I_M$ .

For later use we mention the following construction of a BCA:

**Example 3.5.** Let  $L$  be a linear order with smallest element  $0_L$ . Suppose that  $\infty$  is a symbol not in  $L$ , and set  $L^+ = L \cup \{\infty\}$  with  $x < \infty$  for all  $x \in L$ . An *interval of  $L$*  is a set of the form  $[s, t) = \{u \in L : s \leq u < t\}$ .  $Int(L)$  is the collection of all finite unions of intervals

$$(3.3) \quad [x_0^0, x_0^1) \cup [x_1^0, x_1^1) \cup \dots \cup [x_{t(x)}^0, x_{t(x)}^1),$$

along with the empty set. It is well known that  $Int(L)$  is a Boolean algebra [see 10, p.10], called *the interval algebra of  $L$* . Each nonzero  $x \in Int(L)$  can be written in the form (3.3) in such a way that  $x_j^i \in L^+$ ,  $x_j^0 < x_j^1 < x_{j+1}^0$ , and the intervals  $[x_j^0, x_j^1)$  are pairwise disjoint. The representation of  $x$  in this form is unique, and we call it the *standard representation*. In the sequel, we shall assume that all elements of  $IntAlg(L)$  are in standard representation. For each  $x \in IntAlg(L)^+$ , we let

$$rel(x) = \{x_j^0 : j \leq t(x)\} \cup \{x_j^1 : j \leq t(x)\}$$

be the set of *relevant points* of  $x$ .

Suppose that  $L$  is the rational unit interval  $[0, 1)$  and that  $B = IntAlg(L)$ . Set

$$(3.4) \quad xCy \stackrel{\text{def}}{\iff} x \cdot y \neq 0 \text{ or } rel(x) \cap rel(y) \neq \emptyset.$$

**Proposition 3.6.** [7]  $\langle B, C \rangle$  is a BCA which satisfies C6 and C7.

We also require the following construction:

**Proposition 3.7.** [7] Let  $B$  be atomless,  $F, G$  be distinct ultrafilters of  $B$ , and  $D = C \cup (F \times G) \cup (G \times F)$ . Then,  $D$  is a contact relation on  $B$ .

The common standard models of Boolean contact algebras are substructures of the regular closed algebra  $\text{RegCl}(X)$  for some semiregular topological space  $\langle X, \tau \rangle$  where the contact relation  $C_\tau$  is defined by  $aC_\tau b \iff a \cap b \neq \emptyset$ . The following result exhibits the connections between the BCA axioms and topological properties of  $\langle X, \tau \rangle$ :

**Proposition 3.8.** 1.  $C_\tau$  satisfies C0 – C4.

2.  $C_\tau$  satisfies C5 if and only if  $X$  is weakly regular.

3.  $C_\tau$  satisfies C6 if and only if  $X$  is  $\kappa$ -normal.

4.  $C_\tau$  satisfies C7 if and only if  $X$  is connected.

*Proof.* Clearly,  $C_\tau$  satisfies C0 – C4. For 2., suppose that  $C_\tau$  satisfies C5, and let  $u \neq \emptyset$  be open; since  $\tau$  is semiregular, we may assume that  $u$  is in fact regular open. Then,  $X \setminus u$  is regular closed and not equal to  $X$ . By C5' there is some non-empty regular closed  $v$  such that  $v(-C_\tau)(X \setminus u)$ , which implies  $v \cap (X \setminus u) = \emptyset$ , i.e.  $v \subseteq u$ .

Conversely, suppose that  $X$  is weakly regular, and that  $a, b \in \text{RegCl}(X)$  with  $a \not\subseteq b$ , i.e.  $a \cap (X \setminus b) \neq \emptyset$ . Since  $a$  is regular closed, we have in fact  $\text{int}(a) \cap (X \setminus b) \neq \emptyset$ , and  $\text{int}(a) \cap (X \setminus b)$  is regular open. By weak regularity, there is some non-empty regular closed  $c$  such that  $c \subseteq \text{int}(a) \cap (X \setminus b)$ , which shows that  $C_\tau(a) \not\subseteq C_\tau(b)$ .

For 3., suppose that  $X$  satisfies C6, and suppose that  $a, b$  are non-empty disjoint regular closed sets. By C6, there exists some regular closed  $c$  such that  $a(-C_\tau)c$  and  $-c(-C_\tau)b$ . Then,  $a \cap c = \text{cl}(X \setminus c) \cap b = \emptyset$ , and thus,  $a \subseteq \text{int}(-c)$  and  $b \subseteq \text{int}(c)$ , showing that  $a$  and  $b$  can be separated. Conversely, suppose that  $X$  is  $\kappa$ -normal, and that  $a, b \in \text{RC}(X)^+$ ,  $a(-C_\tau)b$ . Then,  $a \cap b = \emptyset$ , and by  $\kappa$ -normality there are disjoint open  $u, v$  such that  $a \subseteq u$  and  $b \subseteq v$ . Then, regularity of  $a$  implies  $a \cap \text{cl}(X \setminus u) = \emptyset$ , and  $\text{cl}(u) \cap b = \emptyset$ , i.e.  $a(-C_\tau)(-u)$  and  $u(-C_\tau)b$ .

Finally, suppose that  $X$  satisfies C7, and suppose that  $a, b$  are non-empty, disjoint open sets whose union is  $X$ . Then, in particular,  $a$  and  $b$  are regular closed, and  $a + b = X$ . C7 implies that  $aC_\tau b$ , i.e.  $a \cap b \neq \emptyset$ , a contradiction.

Conversely, suppose that  $C_\tau$  does not satisfy C7. Then, there are  $a, b \in \text{RegCl}(X)^+$  such that  $a \cup b = X$  and  $a \cap b = \emptyset$ . It follows that  $a$  and  $b$  are open and closed, showing that  $X$  is not connected.  $\square$

In the literature, standard representation spaces for contact algebras are usually assumed to be regular and  $T_1$ , but the previous result shows that this may not be a necessary condition. Indeed, Example 5.3 below will show that such spaces need not even be  $T_2$ .

## 4 Clusters

The points of Stone's topological representation space of a Boolean algebra  $B$  are the ultrafilters of  $B$ . The additional relation on  $B$  requires a different construction, which first was used in the theory of proximities, see [15]. A non-empty subset  $\Gamma$  of  $B$  is called a *clan* if for all  $x, y \in B$ ,

CL1. If  $x, y \in \Gamma$  then  $xCy$ .

CL2. If  $x + y \in \Gamma$  then  $x \in \Gamma$  or  $y \in \Gamma$ .

CL3. If  $x \in \Gamma$  and  $x \leq y$ , then  $y \in \Gamma$ .

**Lemma 4.1.** 1. If  $F, G$  are filters of  $B$  such that  $F \times G \subseteq C$ , then there is a clan containing  $F \cup G$ .

2. If  $\Gamma$  is a clan,  $U$  an ultrafilter, and  $U \times \Gamma \subseteq C$ , then  $U \cup \Gamma$  is a clan.

3. If  $\Gamma$  is a clan and  $x \in \Gamma$ , then there is some ultrafilter  $U$  such that  $x \in U \subseteq \Gamma$ .

*Proof.* 1. Since  $F \times G \subseteq C$ , we have  $G \cap I_F = \emptyset$ , and by 3.1(1), there is some ultrafilter  $U$  disjoint from  $I_F$  and containing  $G$ . Assume that  $x \in F \cap I_U$ . Then, there are  $y_0, \dots, y_k \in U$  and, for each  $i \leq k$  some  $p_i \in I_{y_i}$  such that  $x = p_0 + \dots + p_k$ . Since  $U$  is a filter we have  $y = y_0 \cdot \dots \cdot y_k \in U$ . By C3 and C4 we have  $(p_0 + \dots + p_k)(-C)y$ , i.e.  $x(-C)y$ . On the other hand,  $x \in F$  and  $y \in U$  imply  $xCy$ , since  $U \cap I_F = \emptyset$ . Again by 3.1(1) there is some ultrafilter  $V$  disjoint from  $I_U$  and containing  $F$ . Then,  $U \times V \subseteq C$ , and clearly,  $U \cup V$  is a clan.

Observe that this implies that

$$(4.1) \quad xCy \Rightarrow \text{There is a clan } \Gamma \text{ such that } x, y \in \Gamma.$$

2. This follows immediately from CL1 – CL3.

3. Since  $\Gamma$  satisfies the conditions of Lemma 3.1(2), there is some ultrafilter  $U$  such that  $x \in U \subseteq \Gamma$ .  $\square$

Clearly, the class of clans on  $B$  is closed under union of chains, and thus each clan is contained in a maximal element which we call *cluster*. The set of all clusters in  $B$  will be denoted by  $\text{Clust}(B)$ . A characterization of clusters is as follows:

**Proposition 4.2.** A clan  $\Gamma$  is a cluster if and only if

$$(4.2) \quad U \times \Gamma \subseteq C \Rightarrow U \subseteq \Gamma$$

for all ultrafilters  $U$ .

*Proof.* “ $\Rightarrow$ ”: If  $U \times \Gamma \subseteq C$ , then  $U \cup \Gamma$  is a clan by Lemma 4.1(2), and the maximality of  $\Gamma$  implies  $U \subseteq \Gamma$ .

“ $\Leftarrow$ ”: Suppose that  $U$  and  $\Gamma$  satisfy (4.2), and assume that  $\Delta$  is a clan such that  $\Gamma \subsetneq \Delta$ . Then, there is some  $z \in \Delta \setminus \Gamma$ , and, by 4.1(3) there is some ultrafilter  $U$  such that  $z \in U \subseteq \Delta$ . Since  $U \cup \Gamma \subseteq \Delta$  and  $\Delta \times \Delta \subseteq C$ , we have, in particular,  $U \times \Gamma \subseteq C$ . By our hypothesis, we have  $z \in U \subseteq \Gamma$ , a contradiction.  $\square$

**Lemma 4.3.** *Let  $\Gamma \in \text{Clust}(B)$ . Then,  $B \setminus \Gamma = I_\Gamma$ .*

*Proof.* Assume that  $x \in \Gamma \cap I_\Gamma$ . Then, there are  $y_0, \dots, y_k \in \Gamma, p_0, \dots, p_k \in B$  such that  $x = p_0 + \dots + p_k$  and  $p_i(-C)y_i$  for all  $i \leq k$ . Since  $x \in \Gamma$ , we have  $p_i \in \Gamma$  for some  $i \leq k$  by CL2. Now,  $p_i(-C)y_i$  contradicts CL1.

Next, assume that  $x \notin \Gamma \cup I_\Gamma$ . By Lemma 3.1(1), there is some ultrafilter  $U$  such that  $x \in U$  and  $U \cap I_\Gamma = \emptyset$ . It is easy to see that  $U \cap \Gamma$  satisfies CL1 – CL3, and thus,  $U \subseteq \Gamma$  by the maximality of  $\Gamma$ . This contradicts  $x \notin \Gamma$ .  $\square$

The representation theorem of Vakarelov et al. [26] uses a different definition of cluster; let us call  $\Gamma \subseteq B$  a *proximity cluster* (p-cluster), if it is a clan and furthermore satisfies

$$(P). \quad (\forall x)[\{x\} \times \Gamma \subseteq C \Rightarrow x \in \Gamma].$$

**Lemma 4.4.** *A clan  $\Gamma$  is a p-cluster if and only if*

$$(4.3) \quad F \times \Gamma \subseteq C \Rightarrow F \subseteq \Gamma$$

for all filters  $F$ .

*Proof.* “ $\Rightarrow$ ”: Suppose that  $F$  is a filter of  $B$  and  $F \times \Gamma \subseteq C$ . In particular,  $\{x\} \times \Gamma \subseteq C$  for all  $x \in F$ , and hence,  $F \subseteq \Gamma$  by (P).

“ $\Leftarrow$ ”: Suppose that  $\{x\} \times \Gamma \subseteq C$ , and let  $F$  be the principal filter generated by  $x$ . By C2 and C3 we have  $\{y\} \times \Gamma \subseteq C$  for all  $y \in F$ , and thus,  $F \subseteq \Gamma$ .  $\square$

Clearly, each p-cluster is a cluster, but the converse need not hold in general:

**Example 4.5.** Suppose that  $\langle B, C \rangle$  is the BCA of Example 3.5, and let  $0 \lesssim a \lesssim b \lesssim 1$ . Furthermore, let  $F_a$  be the ultrafilter of  $B$  of all sets containing  $a$ , and  $F_b$  be the ultrafilter of  $B$  of all sets containing  $b$ . By Proposition 3.7, the relation  $D = C \cup (F_a \times F_b) \cup (F_b \times F_a)$  is a contact relation on  $B$ . Let  $\Gamma = F_a \cup F_b$ ; clearly,  $\Gamma$  is a clan, and we shall show that it is a cluster. Suppose that  $U$  is an ultrafilter of  $B$  and that  $U \times \Gamma \subseteq D$ . By (4.2) it is sufficient to prove  $U \subseteq \Gamma$ . Thus, assume that  $x \in U$ ,  $x \notin \Gamma$ . Since  $U$  is an ultrafilter and no element below  $x$  can be in  $\Gamma$  by CL3, we can suppose that  $x = [s, t)$ . Since  $a, b \notin x$  and  $xCy$  for all  $y \in \Gamma = F_a \cup F_b$ , we must have  $x = [s, a) = [s, b)$ , contradicting  $a \neq b$ . Hence,  $U \subseteq \Gamma$ . If  $s \lesssim a \lesssim t \lesssim b$  and  $y = [s, a) \cup [t, b)$ , then  $\{y\} \times \Gamma \subseteq D$  and  $y \notin \Gamma$ .  $\square$

However, if C6 holds, then the two notions coincide:

**Proposition 4.6.** *Suppose that  $C$  satisfies C6. Then, each cluster is a  $p$ -cluster.*

*Proof.* Suppose  $\{x\} \times \Gamma \subseteq C$ ; by Lemma 4.3 it is enough to show that  $x \notin I_\Gamma$ . Assume otherwise; then, there are  $y_0, \dots, y_k \in \Gamma$  and for each  $i \leq n$  some  $p_i$  such that  $p_i(-C)y_i$ , and  $x \leq p_0 + \dots + p_n$ . With C6, choose for each  $1 \leq i \leq n$  some  $a_i$  such that  $p_i(-C) - a_i$  and  $a_i(-C)y_i$ , and let  $a = a_1 + \dots + a_n$ . Now, set  $z_0 = y_0 \cdot -a$ , and, for  $1 \leq i \leq n$  set  $z_i = y_0 \cdot a_i$ . Then,

$$z_0 + \dots + z_n = y_0 \cdot -a + y_0 \cdot (a_1 + \dots + a_n) = y_0 \cdot -a + y_0 \cdot a = y_0.$$

Our aim is to show that  $z_i \notin \Gamma$  for all  $i \leq n$  which contradicts  $y_0 \in \Gamma$ . First, we consider  $z_0$ . From  $z_0 = y_0 \cdot -a = y_0 \cdot -a_1 \cdot \dots \cdot -a_n$  we infer  $z_0 \leq y_0$  and  $z_0 \leq -a_i$  for  $1 \leq i \leq n$ . If  $z_0 C p_0$ , then  $z_0 \leq y_0$  implies  $p_0 C y_0$ , contradicting  $p_0 \in I_{y_0}$ . If  $z_0 C p_i$  for  $1 \leq i \leq n$ , then  $p_i C -a_i$ , contradicting the choice of  $a_i$ . Hence,  $z(-C)(p_0 + \dots + p_n)$ , and  $x \leq p_0 + \dots + p_n$  now implies  $x(-C)z$ . From  $\{x\} \times \Gamma \subseteq C$  we now obtain  $z_0 \notin \Gamma$ . If  $1 \leq i \leq n$ , then  $z_i \leq a_i$  and  $a_i(-C)y_i$  imply that  $z_i(-C)y_i$ , and hence,  $y_i \in \Gamma$  shows that  $z_i \notin \Gamma$ .  $\square$

## 5 The Representation Theorem

Let  $\langle B, C \rangle$  be a BCA,  $X = \text{Clust}(B)$  and  $h : B \rightarrow 2^X$  be defined by the Stone-like assignment  $h(x) = \{\Gamma \in \text{Clust}(B) : x \in \Gamma\}$ . Our first result shows that  $h$  is injective and preserves  $+$ :

**Lemma 5.1.** 1.  $x \leq y \iff h(x) \subseteq h(y)$ .

2.  $h(0) = \emptyset$ ,  $h(1) = X$ , and  $h(x) \cup h(y) = h(x+y)$  for all  $x, y \in B$ .

*Proof.* 1. “ $\Rightarrow$ ”: Let  $x \leq y$  and  $x \in \Gamma$ . By CL3 we have  $y \in \Gamma$ , and thus,  $\Gamma \in h(y)$ .

“ $\Leftarrow$ ”: Let  $x \not\leq y$ . C5 implies that there is some  $z \in B$  such that  $x C z$  and  $y(-C)z$ . By (4.1), there is some cluster  $\Gamma$  containing both  $x$  and  $z$ , and  $y(-C)z$  implies that  $y \notin \Gamma$ . Hence,  $h(x) \not\subseteq h(y)$ .

It may be worthy to note that, except for weak regularity, this is the only place in the proof of the Representation Theorem, where C5 is used.

2. Since 0 is not contained in any cluster by C0, and 1 is contained in every cluster by CL3, we have  $h(0) = \emptyset$  and  $h(1) = X$ . 1. above implies that  $h(x) \cup h(y) \subseteq h(x+y)$ . Conversely, if  $\Gamma \in h(x+y)$ , then  $x+y \in \Gamma$ , and it follows from CL2 that  $x \in \Gamma$  or  $y \in \Gamma$ , i.e.  $\Gamma \in h(x) \cup h(y)$ .  $\square$

Let  $\tau$  be the topology on  $X$  which has the family  $\mathfrak{A} = \{h(x) : x \in B\}$  as a basis for the closed sets. That  $\mathfrak{A}$  is indeed a basis follows from Lemma 5.1(2). Thus,  $\mathfrak{B} = \{X \setminus h(x) : x \in B\}$  is a basis for the open sets, and each open set  $U$  has the form

$$(5.1) \quad U = \bigcup \{X \setminus h(x) : x \in T\} = X \setminus \bigcap \{h(x) : x \in T\}$$

for some  $T \subseteq B$ .

The next Lemma will exhibit some properties of the topology  $\tau$  which will be useful for the proof of the Representation Theorem:

**Lemma 5.2.** 1.  $\langle X, \tau \rangle$  is a  $T_1$  space.

2. Let  $M \subseteq X$ . Then,  $\text{cl}(M) = \{\Gamma : \bigcap M \subseteq \Gamma\}$ .

3.  $A$  is closed iff there is some  $T \subseteq B$  such that  $A = \{\Gamma : T \subseteq \Gamma\}$ .

4. Let  $\emptyset \neq T \subseteq B$  and  $A = \{\Gamma : T \subseteq \Gamma\}$  be closed. Then,

(a)  $\text{int}(A) = \{\Gamma : (\exists y)[y \notin \Gamma \text{ and } -y \leq T]\}$ .

(b)  $\bigcap \text{int}(A) = \{x : (\forall y)[y \leq T \Rightarrow y \leq x]\}$ .

(c)  $A \subseteq \text{cl}(\text{int}(A)) \iff \bigcap \text{int}(A) \subseteq T$ .

(d) Let  $S = \{y : y^* \leq T\}$ . Then,  $\text{cl}(X \setminus A) = \{\Gamma : S \subseteq \Gamma\}$ .

*Proof.* 1. Suppose that  $\Gamma \in X$ . Clearly,  $\Gamma \in \bigcap \{h(y) : y \in \Gamma\}$ . Let  $\Delta \in \bigcap \{h(y) : y \in \Gamma\}$ . If  $y \in \Gamma$ , then  $\Delta \in h(y)$ , i.e.  $y \in \Delta$ . The maximality of  $\Gamma$  shows that  $\Gamma = \Delta$ , and it follows that  $\{\Gamma\} = \bigcap \{h(y) : y \in \Gamma\}$  is closed.

2. Since the sets  $h(x)$  are a basis for the closed sets of  $\tau$ , we have

$$\begin{aligned} \text{cl}(M) &= \bigcap \{h(y) : M \subseteq h(y)\} \\ &= \{\Gamma : (\forall y)[M \subseteq h(y) \Rightarrow y \in \Gamma]\} \\ &= \{\Gamma : y \in \bigcap M \Rightarrow y \in \Gamma\} \\ &= \{\Gamma : \bigcap M \subseteq \Gamma\}. \end{aligned}$$

3. Since the sets of the form  $h(x)$  are a basis for the closed sets of  $\tau$ ,  $A$  is closed if and only if there is some  $T \subseteq B$  such that  $A = \bigcap \{h(x) : x \in T\}$ . Now,

$$\Gamma \in \bigcap \{h(x) : x \in T\} \iff (\forall x \in T)\Gamma \in h(x) \iff (\forall x \in T)x \in \Gamma \iff T \subseteq \Gamma.$$

4a. By definition of  $\tau$  and  $A$ , we have

$$\begin{aligned}
\Gamma \in \text{int}(A) &\iff (\exists y)[\Gamma \in X \setminus h(y) \subseteq A] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall \Delta)(y \notin \Delta \Rightarrow \Delta \in A)] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall \Delta)(y \in \Delta \text{ or } T \subseteq \Delta)] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall \Delta)(\forall t \in T)(y+t \in \Delta)] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall t \in T)(y+t \in \bigcap \text{Clust}(B))] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } (\forall t \in T)(y+t = 1)] \\
&\iff (\exists y)[y \notin \Gamma \text{ and } -y \leq T].
\end{aligned}$$

Observe that this implies

$$(5.2) \quad \Gamma \in \text{int}(h(x)) \iff (\exists y)[y \notin \Gamma \text{ and } -y \leq x] \iff (\exists y)[y \notin \Gamma \text{ and } -x \leq y] \iff -x \notin \Gamma.$$

4b. Let  $x \in \bigcap \text{int}(A)$ ,  $y \leq T$ , and assume  $y \not\leq x$ , i.e.  $-y + x \neq 1$ . By C5' there is some  $z \notin \{0, 1\}$  such that  $z(-C)(-y+x)$ , and by C4 we have  $z(-C)-y$  and  $z(-C)x$ . Choose some  $\Gamma$  with  $z \in \Gamma$ . Then,  $-y \notin \Gamma$  and  $y \leq T$  shows that  $\Gamma \in \text{int}(A)$  by 4a. Since  $z(-C)x$  implies  $x \notin \Gamma$ , this contradicts  $x \in \bigcap \text{int}(A)$ .

Conversely, suppose that  $z \leq T$  implies  $z \leq x$ , and let  $\Gamma \in \text{int}(A)$ . Then, there is some  $y$  such that  $y \notin \Gamma$  and  $-y \leq T$ . Hence,  $-y \leq x$  and thus,  $x \in \Gamma$ .

4c. Consider the following:

$$A \subseteq \text{cl}(\text{int}(A)) \iff (\forall \Gamma)[\Gamma \in A \Rightarrow \bigcap \text{int}(A) \subseteq \Gamma] \iff (\forall \Gamma)[T \subseteq \Gamma \Rightarrow \bigcap \text{int}(A) \subseteq \Gamma] \iff \bigcap \text{int}(A) \subseteq T.$$

4d: Consider the following:

$$\begin{aligned}
\Gamma \in \text{cl}(X \setminus A) &\iff \Gamma \notin \text{int}(A) \\
&\iff (\forall y)[y \notin \Gamma \Rightarrow -y \not\leq T] && \text{by 4a} \\
&\iff (\forall y)[-y \leq T \Rightarrow y \in \Gamma] \\
&\iff (\forall y)[y \in S \Rightarrow y \in \Gamma] \\
&\iff S \subseteq \Gamma.
\end{aligned}$$

Note that  $S$  is a filter: If  $x, y \in S$ , then  $-x \leq T, -y \leq T$  implies  $-x + -y \leq T$ , and thus,  $x \cdot y \in S$ . If  $x \leq z$ , then  $-z \leq -x \leq T$ , and it follows that  $z \in S$ .  $\square$

The following example shows that  $\langle X, \tau \rangle$  need not be a  $T_2$  space:

**Example 5.3.** Let  $\langle B, C \rangle$  be the algebra of Example 3.5,  $0 \not\leq a \not\leq b \not\leq c \not\leq 1$  and  $F_a, F_b, F_c$  be the ultrafilters of  $B$  of all elements of  $B$  containing, respectively,  $a, b$  or  $c$ . Let  $D = C \cup (F_a \times F_b) \cup (F_b \times F_a) \cup (F_a \times F_c) \cup$

$(F_c \times F_a)$ . By Proposition 3.7,  $D$  is a contact relation. As in Example 4.5, one can show that  $\Gamma = F_a \cup F_b$  and  $\Delta = F_a \cup F_c$  are clusters. Incidentally, this shows that an ultrafilter can be contained in two different clusters which is not possible for  $p$ -clusters. Assume that there are open sets  $u, v$  such that  $\Gamma \in u, \Delta \in v$  and  $u \cap v = \emptyset$ . Since the sets  $h(x)$  are a basis for the closed sets, there are  $x, y$  in  $B$  such that  $\Gamma \notin h(x), \Delta \notin h(y)$  and  $h(x) + h(y) = X$ . Since  $h$  is an embedding, the latter implies  $x + y = 1$ . On the other hand,  $\Gamma \notin h(x)$  implies that  $a \notin x$  and  $b \notin x$ , and  $\Delta \notin h(y)$  implies that  $a \notin y$  and  $c \notin y$ . Together, we obtain  $a \notin x + y$ , contradicting  $x + y = 1$ .  $\square$

Our next result shows that  $T_2$  implies (P). As a preparation, we need the following

**Lemma 5.4.** *If  $\Gamma \in X$ , then  $\{h(y) : y \in F_\Gamma\}$  is a basis for the closed neighborhoods of  $\Gamma$ .*

*Proof.* The sets of the form  $\text{int}(h(y))$  are a basis for the open sets of  $X$  by 5.2(2), and, by 5.2(4a), we obtain

$$\Gamma \in \text{int}(h(y)) \iff y \in \Gamma \text{ and } -y \notin \Gamma,$$

i.e.  $y \in F_\Gamma$ .  $\square$

**Proposition 5.5.** *If  $\langle \text{Clust}(B), \tau \rangle$  is a  $T_2$  space, then  $B$  satisfies (P).*

*Proof.* Suppose that  $\Gamma \times \{x\} \subseteq C$ . Then, in particular,  $F_\Gamma \times \{x\} \subseteq C$ , and it follows from Lemma 4.1(1) that there is some  $\Delta \in X$  such that  $F_\Gamma \cup \{x\} \subseteq \Delta$ . Recall that a space is  $T_2$  iff each point is the intersection of all of its closed neighborhoods. Now, since  $X$  is  $T_2$ , and by 5.4 above we have  $\{\Gamma\} = \bigcap \{h(y) : y \in F_\Gamma\} = \{\Delta : F_\Gamma \subseteq \Delta\}$ . In other words,  $\Gamma$  is the only cluster containing  $F_\Gamma$ , and thus,  $x \in \Gamma$ .  $\square$

We will now prove the Representation Theorem:

**Proposition 5.6.** *1. Each BCA  $B$  is isomorphic to a dense substructure of some  $\langle \text{RegCl}(X), C_\tau \rangle$  where  $\tau$  is  $T_1$  and weakly regular.*

*2.  $B$  satisfies C7 if and only if  $X$  is connected.*

*Proof.* 1. Let  $X = \text{Clust}(B)$ . We show that  $h$  is a Boolean embedding into  $\text{RegCl}(X)$  such that  $h[B]$  is a dense subalgebra of  $\text{RegCl}(X)$ , and  $xCy \iff h(x) \cap h(y) \neq \emptyset$ . By Lemma 5.1,  $h$  is injective and preserves sums. From (4.1) and CL1 it is easily seen that  $xCy \iff h(x) \cap h(y) \neq \emptyset$ .

Next, we show that  $h(x) \in \text{RegCl}(X)$  and that  $h$  preserves complements. First, observe that for all  $x \in B$ ,

$$(5.3) \quad \text{cl}(X \setminus h(x)) = \bigcap \{h(y) : X \setminus h(x) \subseteq h(y)\} = \bigcap \{h(y) : h(x) \cup h(y) = X\} = \bigcap \{h(y) : x + y = 1\}.$$

If  $\Gamma \in h(-x)$ , then  $-x \in \Gamma$ , and, if  $x + y = 1$ , then  $-x \leq y$  which implies  $y \in \Gamma$ . Hence,  $\Gamma \in \text{cl}(X \setminus h(x))$ . Conversely, suppose that  $\Gamma \in h(y)$  whenever  $x + y = 1$ ; then, in particular,  $\Gamma \in h(-x)$ . It follows that  $h(x) = \text{cl}(X \setminus h(-x))$  for all  $x \in B$ . Now,

$$h(x) = \text{cl}(X \setminus h(-x)) = \text{cl}(X \setminus (\text{cl}(X \setminus h(x)))) = \text{cl}(\text{int}(h(x)))$$

shows that  $h(x)$  is regular closed and it follows from (5.3) that  $h$  preserves complements.

Suppose that  $\emptyset \neq A = \{\Gamma : T \subseteq \Gamma\}$  is regular closed. Then, there are some  $\Gamma \in \text{int}(A)$  and some  $y \in B$  such that  $y \notin \Gamma$  and  $-y \leq T$ . Since  $y \notin \Gamma$ , we have  $-y \in \Gamma$ , and  $-y \leq T$  implies  $T \subseteq \Gamma$ . It follows that  $\Gamma \in A$ , and hence,  $\emptyset \neq h(-y) \subseteq A$ . Therefore,  $h[B]$  is dense in  $\text{RegCl}(X)$ . Note that this implies  $h[B] = \text{RegCl}(X)$  just in case  $B$  is complete.

Since  $\tau$  is  $T_1$  by Lemma 5.2(1), it remains to show that  $\tau$  is weakly regular. Suppose that  $\emptyset \neq u \in \tau$ . Since the sets of the form  $X \setminus h(y)$  are regular open and a basis for the open sets, we can assume that  $X \setminus h(y) \subseteq u$  for some  $y \neq 1$ . By C5' there is some  $z \neq 0$  such that  $z(-C)y$ . Hence, by CL1 no cluster contains both  $y$  and  $z$ , and it follows that  $h(z) \cap h(y) = \emptyset$ , i.e.  $h(z) \subseteq X \setminus h(y)$ .

2. Suppose that  $M \neq \emptyset, X$  is clopen. By Lemma 5.2(4a) and (4d), there are filters  $F, G$  of  $B$  such that  $M = \{\Gamma : F \subseteq \Gamma\}$ ,  $X \setminus M = \{\Delta : G \subseteq \Delta\}$ , and  $G$  is the set of upper bounds of the set  $\{-x : x \in F\}$ . If  $x \in F$  and  $y \in G$ , then  $-x \leq y$  by definition of  $G$ , and thus  $x + y = 1$ . It follows from C7 that  $xCy$ . Hence,  $F \times G \subseteq C$ , and by Lemma 4.1(1) there is some cluster  $\Gamma$  such that  $F \cup G \subseteq \Gamma$ . On the other hand,

$$F \cup G \subseteq \Gamma \iff F \subseteq \Gamma \text{ and } G \subseteq \Gamma \iff \Gamma \in M \text{ and } \Gamma \in X \setminus M,$$

which is not possible. □

**Corollary 5.7.** *Each RCC model is isomorphic to a substructure of some  $\langle \text{RegCl}(X), C_\tau \rangle$  for a connected weakly regular  $T_1$  space  $X$ .*

## 6 Conclusion and outlook

We have provided a topological Representation Theorem for Boolean contact algebras which implies a representation of the Region Connection Calculus. As a consequence, one can observe that the common assumption of regularity and  $T_1$  of the representation space is too strong and not forced by the axioms, since our spaces need not be regular. We do not even need any point separation axioms: The  $T_1$  property of our representation space is a consequence of the fact that the clusters are pairwise incomparable under  $\subseteq$ . If we “double” the points of  $\text{Clust}(B)$  and choose the obvious topology, then the resulting space will still provide a representation for  $\langle B, C \rangle$ , but it will not even be  $T_0$ . Two interesting questions remain:

1. Is there a topological representation theorem for Boolean algebras where  $C$  only satisfies  $C_0 - C_4$ ? We are somewhat doubtful, since if  $C_5$  is not satisfied, then regions cannot be distinguished by  $C$  anymore, and the mapping  $h$  above need not be injective.
2. Are there BCAs which do not have a regular representation space? Even though our spaces are not necessarily regular, it may be that by another method such spaces could be obtained.

## References

- [1] Biacino, L. and Gerla, G. (1991). Connection structures. *Notre Dame Journal of Formal Logic*, 32:242–247.
- [2] Cayley, A. (1854). On the theory of groups as depending on the symbolic equation  $\theta^n = 1$ . *Philosophical Magazine*, 7:40–47.
- [3] Clarke, B. L. (1981). A calculus of individuals based on ‘connection’. *Notre Dame Journal of Formal Logic*, 22:204–218.
- [4] de Laguna, T. (1922). Point, line and surface as sets of solids. *The Journal of Philosophy*, 19:449–461.
- [5] Düntsch, I. and Vakarelov, D. (2003). Region-based theory of discrete spaces: A proximity approach. In Nadif, M., Napoli, A., SanJuan, E., and Sigayret, A., editors, *Proceedings of the Fourth International Conference Journées de l’informatique Messine*, pages 123–129, Metz, France.
- [6] Düntsch, I., Wang, H., and McCloskey, S. (2001). A relation algebraic approach to the Region Connection Calculus. *Theoretical Computer Science*, 255:63–83.
- [7] Düntsch, I. and Winter, M. (2003). Constructions of Boolean contact algebras. Manuscript.
- [8] Engelking, R. (1977). *General Topology*. PWN, Warszawa.
- [9] Galton, A. (1999). The mereotopology of discrete space. In Freksa, C. and Mark, D. M., editors, *Spatial Information Theory, Proceedings of the International Conference COSIT ’99*, Lecture Notes in Computer Science, pages 251–266. Springer–Verlag.
- [10] Koppelberg, S. (1989). *General Theory of Boolean Algebras*, volume 1 of *Handbook on Boolean Algebras*. North Holland.
- [11] Leonard, H. S. and Goodman, N. (1940). The calculus of individuals and its uses. *Journal of Symbolic Logic*, 5:45–55.
- [12] Leśniewski, S. (1928). O podstawach ogólnej teorii mnogości.I. *Przegląd Filozoficzny*, 31:261–291.

- [13] Leśniewski, S. (1983). On the foundation of mathematics. *Topoi*, 2:7–52.
- [14] Li, S. and Ying, M. (2003). Generalized region connection calculus. Preprint.
- [15] Naimpally, S. A. and Warrack, B. D. (1970). *Proximity Spaces*. Cambridge University Press, Cambridge.
- [16] Nicod, J. (1924). Geometry in a sensible world. Doctoral thesis, Sorbonne, Paris. English translation in *Geometry and Induction*, Routledge and Kegan Paul, 1969.
- [17] Pratt, I. and Schoop, D. (1998a). A complete axiom system for polygonal mereotopology of the real plane. *Journal of Philosophical Logic*, 27(6):621–658.
- [18] Pratt, I. and Schoop, D. (1998b). A complete axiom system for polygonal mereotopology of the real plane. *Journal of Philosophical Logic*, 27(6):621–661.
- [19] Pratt, I. and Schoop, D. (2000). Expressivity in polygonal, plane mereotopology. *Journal of Symbolic Logic*, 65:822–838.
- [20] Randell, D. A., Cohn, A. G., and Cui, Z. (1992). Computing transitivity tables: A challenge for automated theorem provers. In Kapur, D., editor, *Proceedings of the 11th International Conference on Automated Deduction (CADE-11)*, volume 607 of *LNAI*, pages 786–790, Saratoga Springs, NY. Springer.
- [21] Shchepin, E. (1972). Real-valued functions and spaces close to normal. *Siberian Mathematical Journal*, 13:820–830.
- [22] Steen, L. and Seebach jr., J. (1978). *Counterexamples in Topology*. Springer-Verlag, Berlin, Heidelberg, New York.
- [23] Stell, J. (2000). Boolean connection algebras: A new approach to the Region Connection Calculus. *Artificial Intelligence*, 122:111–136.
- [24] Stone, M. (1936). The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40:37–111.
- [25] Tarski, A. (1956). Foundation of the geometry of solids. In Woodger, J. H., editor, *Logic, Semantics, Metamathematics*, pages 24–29. Clarendon Press, Oxford. Translation of the summary of an address given by A. Tarski to the First Polish Mathematical Congress, Lwów, 1927.
- [26] Vakarelov, D., Dimov, G., Düntsch, I., and Bennett, B. (2002). A proximity approach to some region-based theories of space. *J. Appl. Non-Classical Logics*, 12:527–529.
- [27] Whitehead, A. N. (1929). *Process and reality*. MacMillan, New York.