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Improved lower bounds on the size of the smallest solution to a graph colouring problem, with an application to relation algebra

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Abstract. Let $G = (V, E)$ be a graph and let C be a finite set of colours. An *edge C-colouring* of G is a function $\lambda : E \to C$. If G is symmetric and $\lambda(x, y) = \lambda(y, x)$, for all $(x, y) \in E$, then we say that λ is a *symmetric* edge C-colouring.
Let n be a natural number and let $C_n = \{\mathsf{f}, \mathsf{c}_i : i < n\}$ be a set of $n+1$ colours. Using a probabilistic construction and an application of the *Local Lemma* we prove that there is a complete irreflexive graph G_n with at least two nodes and a symmetric edge C_n-colouring λ_n of G_n such that for any edge (x, y) of G_n and any $\beta, \gamma \in C_n$,

$$\mathsf{f} \in \{\lambda_n(x, y), \beta, \gamma\} \iff \exists z \in G_n\ (\beta = \lambda_n(x, z) \wedge \gamma = \lambda_n(y, z))$$

Moreover, such a graph exists of size $\binom{3k-4}{k}$ provided k is large enough so that

$$n^2\left(1 - \frac{1}{n^2}\right)^{(k-2)^2}\left(1 + \left(2\binom{2k-4}{k} + 2k\binom{2k-5}{k-1}\right)^2\right) \leq \frac{1}{e} \tag{1}$$

Equivalently, for k satisfying this inequality, the symmetric integral relation algebra with $n+1$ diversity atoms one of which is flexible but where all inflexible diversity triangles are forbidden has a representation over a base of size $\binom{3k-4}{k}$.
This significantly reduces the size of the smallest known edge-labelled graph satisfying these conditions.

1 Introduction and Preliminaries

We consider a certain combinatorial problem that arises in the study of representations of relation algebras but we present it here as a graph edge-colouring problem. Many of the familiar edge-colouring problems ask for graphs whose edges are coloured in such a way as to avoid specified structures, for example the Ramsay number $R(c_0, c_1, \ldots, c_{n-1})$ is one more than the size of the largest complete irreflexive graph where each undirected edge has one of n colours and there is no complete subgraph of size c_i whose edges all have the i'th colour (for $i < n$). We are asked to find a graph, as large as possible, satisfying some universally quantified constraints.

The problem we consider here is typical of another class of problems, where we seek graphs satisfying specified universal-existential conditions; if such graphs exist then we want to know

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whether a finite graph satisfying the conditions exists and if so we would like bounds on the smallest possible size. Model theorists have considered this kind of problem for some time now and have succeeded in proving, for example, that for finite m, a K_m free graph G and a sequence of partial isomorphisms of G may be extended to a finite K_m-free graph in which the partial isomorphisms are induced by automorphisms [5]. Applications of this include a proof that finite relation algebras have finite *relativized* representations [2], that various fragments of modal logic have the finite model property (e.g. [8]); however the finite models given by these results tend to be large. The techniques employed in this paper allow us to reduce the upper bound on the smallest solution, at least for one rather specific problem of this kind.

Consider the following graph problem: find the smallest triangle-free symmetric graph $G = (V, E)$ where for all $x, y \in V$ with $(x, y) \notin E$ there is $w \in V$ with $(x, w) \in E,\ (w, y) \notin E$ and there is $z \in V$ with $(x, z), (z, y) \in E$. A solution, given in [1], has vertices consisting of the ten three-element subsets of a set of size five and with an edge between such subsets iff their intersection has at most one element (see section 6 below). We will consider a family of generalisations of this problem to edge coloured graphs where for all vertices x, y and any consistent choice of colours, witness nodes, like w, z above, must exist with edges labelled by the chosen colours (see definition 2). Alm et. al. devized probabilistic constructions of edge coloured graphs to solve these edge colouring problems, but their solutions are rather large. Here we apply the *Local Lemma* to significantly reduce the upper bound on the size of the smallest solution.

Definition 1 (Labelled Graph). *A* labelled graph (G, λ) *consists of a graph G with vertices $V(G)$ and edges $E(G) \subseteq V(G) \times V(G)$, and a function $\lambda : E(G) \to C$, for some set C of colours. We say that (G, λ) is a* symmetric labelled graph *if G is symmetric and if $(x, y) \in E(G) \Rightarrow \lambda(x, y) = \lambda(y, x)$ (and we may say that the labelling λ is symmetric).*

Next we define our edge colouring problems. The graphs we use here are all complete (either reflexive or irreflexive), whereas the labelled graphs mentioned in the abstract were not assumed to be complete. The discrepancy is resolved later by introducing a special colour (f) that can be thought of as the colour of "non edges".

Definition 2 (Edge Colouring Problems). *We define three edge colouring problems: (i) symmetric labellings of irreflexive complete graphs, (ii) directed labellings of irreflexive complete graphs and (iii) directed labellings of reflexive complete graphs. An instance (C, T) of the first problem consists of a finite set C of colours and a set $T \subseteq C^3$ of consistent triangles closed under permutations, i.e. if $(c_1, c_2, c_3) \in T$ then $(c_2, c_1, c_3), (c_3, c_1, c_2) \in T$. A* solution (G, λ) *of (C, T) is a complete symmetric irreflexive labelled graph (G, λ) such that*

$$\{(\lambda(x, y), \lambda(y, z), \lambda(x, z)) : (x, y), (y, z), (x, z) \in E(G)\} = T \tag{2}$$

and for all $(x, y) \in E(G)$

$$(c_1, c_2, \lambda(x, y)) \in T \ \Rightarrow\ \exists z \in G\ (\lambda(x, z) = c_1 \wedge \lambda(z, y) = c_2). \tag{3}$$

The set T tells us which triangles are permitted *in a solution, the conditions above imply that permitted triangles are also* obligatory*, as Roger Maddux once put it. If property 3 holds for a certain edge (x, y) we say that (x, y) is* witnessed. *In model-theoretic terms,* (2) *says that (G, λ) is* universal *over T and* (3) *says that (G, λ) is* 3*-homogeneous, i.e. for any partial isomorphism p of (G, λ) of size strictly less than 3 and any $x \in G$ there is a partial isomorphism p^+ extending p and with x included in its domain.*

An instance $(C, \smile, T)$ *of the* directed irreflexive edge colouring problem *is defined similarly, but here* $\smile : C \to C$ *is any function such that* $(c^\smile)^\smile = c$ *(all* $c \in C$*) and* T *has to be closed under* Peircean Transforms, *i.e.* $(c_1, c_2, c_3) \in T \Rightarrow (c_2, c_3^\smile, c_1^\smile), (c_3, c_2^\smile, c_1) \in T$. *A solution is a complete irreflexive labelled graph* (G, λ) *(but the labelling* λ *is not necessarily symmetric) satisfying* (2), (3) *and in addition, for all* $(x, y) \in E(G)$ *we have*

$$\lambda(x, y) = (\lambda(y, x))^\smile. \tag{4}$$

Finally, an instance $(C, Id, \smile, T)$ *of the* reflexive directed edge colouring problem *has as an argument a specified subset* $Id \subseteq C$. *As before,* T *must be closed under Peircean Transforms, also for all* $c \in C$ *there is a unique* $st(c) \in Id$ *such that* $(st(c), c, c) \in T$ *and* $(c_1, c_2, c_3) \in T \Rightarrow st(c_1) = st(c_3)$. *A solution* (G, λ) *is a directed reflexive complete graph satisfying* (2), (3), (4) *and in addition, for all* $x, y \in G$ *we have* $\lambda(x, y) \in Id \iff x = y$.

For each of the three problems above, if there is a solution we may also wish to know the size of the smallest solution.

2 Equivalence with Representations of Relation Algebras

Solutions to these problems are related to representations of certain finite relation algebras, as we explain in outline next. The reader who is more interested in graph colouring problems than relation algebra might prefer to skip to the next section. Further references on relation algebras include [9, 7, 10].

Definition 3. *A* relation algebra $\mathcal{A} = (A, 0, 1, -, +, 1', \smile, ;)$ *consists of a set A, constants* $0, 1, 1' \in A$, *unary functions* $-, \smile$ *and binary functions* $+, ;$ *on A, satisfying certain equational axioms [7, definition 3.8], which state that* $(A, 0, 1, -, +)$ *is a boolean algebra* $(A, 1', \smile, ;)$ *is an involuted monoid, the operators* $\smile, ;$ *are normal and additive, and the algebra obeys the* Peircean law. *A* representation h *of* $\mathcal{A}$ *is a map* $h : A \to \wp(X \times X)$ *(some base set* X*) such that* $h(0) = \emptyset$, $h(-a) = h(1) \setminus h(a)$, $h(a + b) = h(a) \cup h(b)$, $h(1') = Id_X$, $h(a^\smile) = \{(x, y) : (y, x) \in h(a)\}$ *and* $h(a; b) = \{(x, y) : \exists z \in X,\ (x, z) \in h(a),\ (z, y) \in h(b)\}$, *for all* $a, b \in \mathcal{A}$. *The* representation problem *for finite relation algebras is to determine whether an arbitrary finite relation algebra is representable or not.*

An atom *of* $\mathcal{A}$ *is a minimal non-zero element with respect to the boolean ordering* $a \leq b \iff a + b = b$. *A relation algebra is* integral *if the identity* $1'$ *is itself an atom.* $\mathcal{A}$ *is* atomic *if every non-zero element of* $\mathcal{A}$ *is above some atom.*

Observe that every finite relation algebra is atomic.

Theorem 1. *The following decision problems are undecidable: the reflexive directed edge colouring problem and the irreflexive directed edge colouring problem.*

Proof. The representation problem for finite relation algebras is known to be undecidable [6, theorem 8]. We reduce that problem to the reflexive directed edge colouring problem. Given a finite relation algebra $\mathcal{A} = (A, 0, 1, -, +, 1', \smile, ;)$ let C be the set of atoms of $\mathcal{A}$, let Id be the set of atoms below the identity $1'$, let $\smile$ be obtained from the converse operator on $\mathcal{A}$ by restriction to C (it is easy to check that the converse of an atom is an atom and if an atom is below the identity then it is self-converse), and let $T = \{(a, b, c) : a, b, c \in C,\ a; b \geq c\}$ (it follows from the relation algebra axioms that T is closed under Peircean Transforms). The map that sends $\mathcal{A}$ to $(C, Id, \smile, T)$ can be shown to be a reduction, in fact every solution to $(C, Id, \smile, T)$ determines a representation of $\mathcal{A}$ and every representation of $\mathcal{A}$ determines a solution to $(C, Id, \smile, T)$. Hence the first decision problem in the theorem is undecidable.

There is a reduction of the reflexive directed edge colouring problem to the irreflexive one. An instance $(C, Id, \smile, T)$ of the reflexive directed edge colouring problem is mapped to $(C \setminus Id, \smile, T \cap (C \setminus Id)^3)$. Obviously if $(C, Id, \smile, T)$ is a yes instance, say (G, λ) is a solution, then by restricting λ to irreflexive edges of G we get a solution of $(C \setminus Id, \smile, T \cap (C \setminus Id)^3)$, so the latter is a yes-instance. Conversely, if $(C \setminus Id, \smile, T \cap (C \setminus Id)^3)$ has an irreflexive solution (G, λ) then let G^+ be the reflexive closure of G and extend λ to the labelling λ^+ of G^+ by letting $\lambda^+(x, x) = e \iff \exists y \in G\ e = st(\lambda(x, y))$ (the identity condition on instances of the reflexive problem ensures that e is uniquely determined). Now check that (G^+, λ^+) is a solution to $(C, Id, \smile, T)$.

It is not known whether the symmetric edge colouring problem is decidable, although there is a known finite symmetric relation algebra (due to Maddux) which is representable but has no representation over a finite base [7, § 11.4(2)]. It is also not known if the problem of determining whether a finite relation algebra has a representation over a finite base is decidable, hence if we modify our edge colouring problems so that solutions are additionally required to be finite, then we do not know whether the finite edge colouring problems are decidable, though it seems unlikely.

Although the representation problem for finite relation algebras is undecidable, there are certain cases where finite relation algebras are known to have representations. An example of such a case is where the integral relation algebra $\mathcal{A}$ has a *flexible atom* f, an atom that is consistent in any non-identity triangle — for all non-identity atoms a, b of $\mathcal{A}$ we have $a; b \geq \mathsf{f}$. Given an arbitrary finite integral relation algebra with a flexible atom f it is fairly easy to construct an infinite representation in a set-by-step manner, using f as the default label (see [7, exercise 11.4(1)]). Hence if (C, T) is an instance of the irreflexive symmetric edge colouring problem and there is a *flexible* colour $\mathsf{f} \in C$ such that for all $a, b \in C$ we have $(a, b, \mathsf{f}) \in T$, then (C, T) is a yes-instance (similarly if $(C, \smile, T)$ is an instance of the irreflexive directed edge colouring problem such that there is a flexible colour $\mathsf{f} \in C$ then $(C, \smile, T)$ is a yes-instance). However, it is not known if every finite integral relation algebra with a flexible atom has a representation over a finite base (see [7, problem 21(21)]), This problem remains unsolved, but it is hoped that the techniques used here may eventually be used to help solve it.

3 The colouring problem $\mathcal{M}_n$

We now focus on a special case of the symmetric edge colouring problem. Let $n \geq 1$, and let our colours be $C = \{\mathsf{f}, \mathsf{c}_0, \ldots, \mathsf{c}_{n-1}\}$. Let $\mathcal{M}_n = (C, T)$, where $T = C^3 \setminus \{(\mathsf{c}_i, \mathsf{c}_j, \mathsf{c}_k) : i, j, k < n\}$, i.e. T is the set of triangles in which at least one edge is f. A solution (G, λ) to $\mathcal{M}_n$ is a 3-homogeneous complete symmetric irreflexive labelled graph in which triangles involving f are allowed, but no other triangles.

Existing results

It was shown in [1] that finite solutions for $\mathcal{M}_n$ exist for all $n \geq 1$ using probabilistic methods and gave a bound for their size. In particular they proved the following theorem.

Theorem 2. *Given $n \geq 1$, if k is large enough such that*

$$\rho(n,k)\cdot H(k) < 1 \tag{5}$$

where

$$\rho(n,k) = n^2(1-\frac{1}{n^2})^{(k-2)^2} \tag{6}$$

$$H(k) = \frac{1}{2}\binom{3k-4}{k}^2 \tag{7}$$

then there exists a graph with $\binom{3k-4}{k}$ *vertices which is a solution for* $\mathcal{M}_n$.

The result of this paper is to provide a bound in the same form as above but replacing $H(k)$ with a function that takes much lower values for any given k, thus obtaining a lower upper bound on the minimum size of a solution for $\mathcal{M}_n$.

4 The main result

Theorem 3. *Given* $n \geq 1$, *if* k *is large enough such that*

$$\rho(n,k)\cdot L(k) < 1 \tag{8}$$

where

$$L(k) = e\left(1+\left(2\binom{2k-4}{k}+2k\binom{2k-5}{k-1}\right)^2\right) \tag{9}$$

(here e is the base of the natural logarithm) then there exists a graph with $\binom{3k-4}{k}$ *verticies which is a solution for* $\mathcal{M}_n$.

5 The local lemma

To prove this result we will follow a similar procedure to the proof given in [1] but employ a result from probabilistic graph theory, the local lemma, to reduce the bound.

The Local Lemma [4] lets us assert that given a set of events there is a positive probability that *none* of these occur if they each occur with low enough probability and are sufficiently independent.

Definition 4. *Given events* $\{A_1,\ldots,A_n\}$ *in a probability space, the* dependency graph *is the symmetric graph with vertices* $\{A_1,\ldots,A_n\}$ *and where there is an edge between* A_i, A_j *if* A_i *is dependent on* A_j.

Lemma 1 (The Local Lemma — Symmetric Version). *Let* $\{A_1,\ldots,A_n\}$ *be events in a probability space having a dependency graph with maximum degree d. Suppose that*

$$P(A_i) < \frac{1}{e(d+1)}$$

for all $i \in 1,\ldots,n$. *Then* $P(\overline{A_1}\cap\ldots\cap\overline{A_n}) > 0$ *(the probability that none of the events happen is non-zero).*

There are more general versions of the lemma, this one is the simplest to apply.

Proof. See [3, Theorem 13.14].

6 Sketch of proof of theorem 2

To prove theorem 3 we will first have to review the proof Alm et al. gave of theorem 2 in [1].

Let $n \geq 1$ and $k > 3$. Let $V_k = [3k-4]^k$ denote the set of k-subsets of $\{1, \dots, 3k-4\}$ and let G_k be the complete irreflexive graph with vertices V_k. Define a *random* edge colouring $\lambda_{k,n} : E(G) \to \{\mathsf{f}, \mathsf{c}_0, \dots, \mathsf{c}_{n-1}\}$ by

$$\lambda_{k,n}(X,Y) = \begin{cases} \text{uniform random choice from } \{\mathsf{c}_0, \dots, \mathsf{c}_{n-1}\} & \text{if } 0 \leq |X \cap Y| \leq 1 \\ \mathsf{f} & \text{otherwise} \end{cases}$$

for $X \neq Y \in V_k$. If $|X \cap Y| \leq 1$ we call (X,Y) a *randomly labelled edge.*

Alm et al. showed that $(G_k, \lambda_{k,1})$ is a solution for $\mathcal{M}_1$ ($n = 1$ makes the colouring non-random) when $k \geq 3$. For $n > 1$ it is easy to see that $(G_k, \lambda_{k,n})$ will not contain forbidden triangles since $\lambda_{k,n}$ defines the same set of flexible edges as $\lambda_{k,1}$. We must check that there is a non-zero probability that (2) and (3) hold. Since f is a flexible colour, for arbitrary colours c, d, the triangle (f, c, d) is consistent. If $(G_k, \lambda_{k,n})$ satisfies (3) and $\lambda_{k,n}(x,y) = \mathsf{f}$ then there must be z such that $\lambda_{k,n}(x,z) = c$ and $\lambda_{k,n}(z,y) = d$, hence (2) holds too. So it suffices to prove that there is a non-zero probability that (3) holds.

We say that an edge (X,Y) fails to be witnessed if (3) fails on (X,Y); failure occurs because there are two colours c, d such that $\mathsf{f} \in \{c, d, \lambda_{k,n}(X,Y)\}$ but there is no Z with $\lambda(X,Z) = c$ and $\lambda(Z,Y) = d$. Since the edges of G_k are mostly labelled by f it can be checked that if (X,Y) is a randomly labelled edge then it is very unlikely that (3) fails on such an edge. So we concentrate instead on the case where $\lambda_{k,n}(X,Y) = \mathsf{f}$ and there are $i, j < n$ but there is no Z such that $\lambda_{k,n}(X,Z) = \mathsf{c}_i$ and $\lambda_{k,n}(Z,Y) = \mathsf{c}_j$. Given X, Y, Z such that $|X \cap Z|, |Y \cap Z| \leq 1$ (so $|X \cap Y| > 1$), the probability that $\lambda_{k,n}(X,Z) = \mathsf{c}_i$ and $\lambda_{k,n}(Z,Y) = \mathsf{c}_j$ is $\frac{1}{n^2}$, so the probablity that Z does not witness $\mathsf{c}_i, \mathsf{c}_j$ is $1 - \frac{1}{n^2}$. The numbers of vertices Z such that $|X \cap Z|, |Y \cap Z| \leq 1$ is at least $(k-2)^2$, hence the probability that there is no Z such that $\lambda(X,Z) = \mathsf{c}_i$ and $\lambda(Y,Z) = \mathsf{c}_j$ is at most $(1 - \frac{1}{n^2})^{(k-2)^2}$. There are n inflexible colours, so the probability that there are $\mathsf{c}_i, \mathsf{c}_j$ such that there is no Z with $\lambda_{x,n}(X,Z) = \mathsf{c}_i$ and $\lambda_{x,n}(Y,Z) = \mathsf{c}_j$ is at most $n^2(1 - \frac{1}{n^2})^{(k-2)^2} = \rho(n,k)$. Thus, for any edge ϵ, the probability P_ϵ that (3) fails at ϵ is at most $\rho(n,k)$. Since $|V_k| = \binom{3k-4}{k}$, the number of edges of G_k is at most $\frac{1}{2}\binom{3k-4}{k}^2$. Using the union bound the probability that some edge of G_k fails to be witnessed is

$$P \leq \underbrace{\frac{1}{2}\binom{3k-4}{k}^2}_{=H(k)} . \rho(n,k) \tag{10}$$

For a fixed n they show we can find k big enough such that $P < 1$ therefore $\mathcal{M}_n$ has a solution of size $|V| = \binom{3k-4}{k}$.

7 Proof of theorem 3

Let $n \geq 1$ and $k \geq 3$. We will use the same labelled random graph, $(G_k, \lambda_{k,n})$, as in the proof of theorem 2 and demonstrate that it will be a representation for $\mathcal{M}_n$ given that k is large enough, but the bound on k will be lower than the one used in theorem 2. We will use the local lemma to show that there is a non zero probability that all edges in G_k are witnessed, where in the proof of theorem 2 the more coarse union bound was used. We will use the probabilities for edge witnessing shown in the sketch proof of theorem 2.

We will have to identify events and their dependencies. The events we wish to avoid are edges failing to be witnessed, which happens with probability $P_e \leq \rho(n,k)$. The dependency graph is a little trickier to work out (it is *not* G_k), if a randomly labelled edge is adjacent to ϵ and also adjacent to ϵ' then their may be a dependency between the witnessing of ϵ and the witnessing of ϵ', but if there is no randomly labelled edge adjacent to both ϵ and ϵ' then the two event must be independent.

Let $W_{x,y}$ denote the randomly labelled edges adjacent to x or y, that is

$$W_{x,y} = \{(x,z) : z \in V_k,\ \lambda_{k,n}(x,z) \neq \mathsf{f}\} \cup \{(z,y) : z \in V_k,\ \lambda_{k,n}(z,y) \neq \mathsf{f}\}. \tag{11}$$

The witnessing of edges (x,y), (u,v) are independent events if

$$W_{x,y} \cap W_{u,v} = \emptyset.$$

Each vertex is adjacent to no more than

$$\underbrace{\binom{2k-4}{k}}_{|X\cap Y|=0 \text{ edges}} + \underbrace{k\binom{2k-5}{k-1}}_{|X\cap Y|=1 \text{ edges}}$$

other vertices through randomly labelled edges. So each edge witnessing event is dependent on no more than

$$D = \left(2\binom{2k-4}{k} + 2k\binom{2k-5}{k-1}\right)^2$$

other edge witnessing events.

We can now apply the local lemma, where $P(A_i) \leq \rho(n,k)$ and maximum dependency degree $d \leq D$. The probability that no edges fail to be witnessed (no A_i's happen) is greater than 0 if

$$\rho(n,k) < \frac{1}{e(1+D)} \leq \frac{1}{e(1+d)} \tag{12}$$

So, there is a solution $(G_k, \lambda_{k,n})$ for $\mathcal{M}_n$ if

$$\underbrace{e \cdot \left(1 + \left(2\binom{2k-4}{k} + 2k\binom{2k-5}{k-1}\right)^2\right)}_{=L(K)} \cdot \rho(n,k) < 1$$

□

8 The new bound is significantly lower

Theorems 2 and 3 both give bounds upper bounds on the size of the smallest representation for $\mathcal{M}_n$. All that remains is to show that the bound of theorem 3 is significantly lower than that of theorem 2. One simply needs to check that $L(k) < H(k)$.

This is clearly true if you look asymptotically

$$e\left(1+\left(2\binom{2k-4}{k}+2k\binom{2k-5}{k-1}\right)^2\right)\in O\left(\binom{2k-4}{k}^2\right)$$

$$\binom{3k-4}{k}^2\notin O\left(\binom{2k-4}{k}^2\right)$$

So there exists some K such that for all $k \geq K$ we have $L(k) < H(k)$.

In fact it is clear from figure 1, that $L(k) < H(k)$ for all k and an inspection of the second graph in this figure shows that $\log_{10}(H(k) \approx 1.41 \times \log_{10}(L(k))$, hence $H(k) \approx L(k)^{1.41}$. Figure 2 shows the number of vertices in the smallest solution for $\mathcal{M}_n$ given by theorems 2 and 3. For each $n < \omega$, let $Alm(n)$ be the size of the smallest graph where theorem 2 proves a non-zero probability and let $M(n)$ be the size of the smallest graph where theorem 3 proves a non-zero probability. By inspection of figure 2, we have $\log_{10}(Alm(n)) \approx 1.3 \times \log_{10}(M(n))$, so $Alm(m) \approx M(n)^{1.3}$.

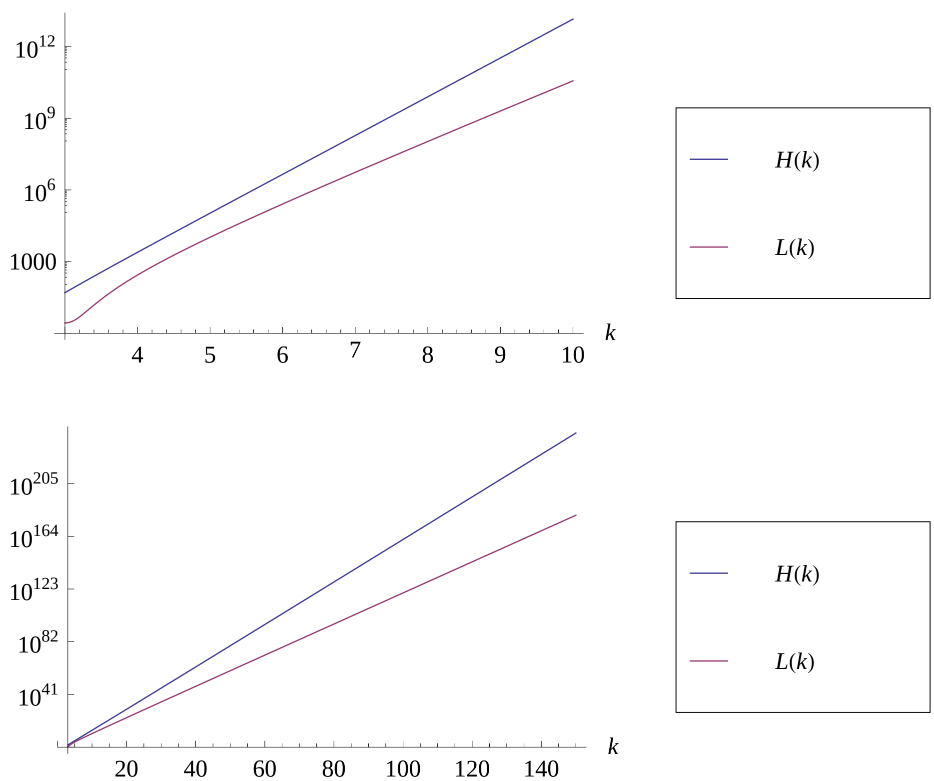

Fig. 1. Logarithmic plots of $L(k)$ and $H(k)$. It follows that the new upper bound on the smallest representation sizes obtained is asymptotically far lower than the previously attained results.

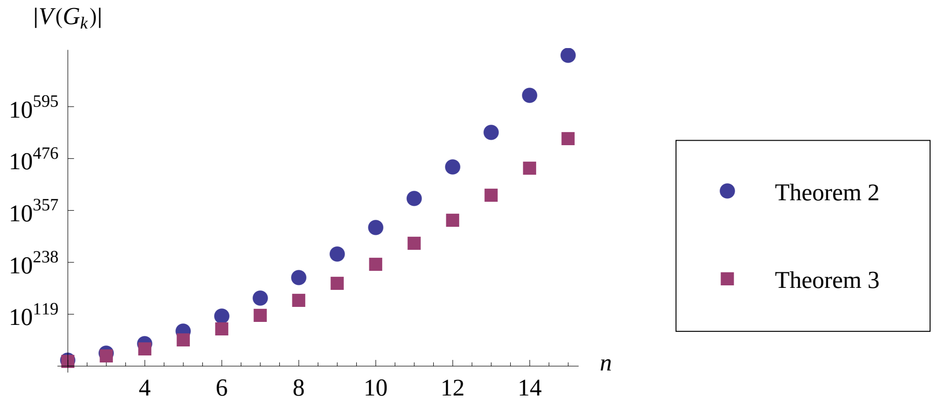

Fig. 2. Logarithmic plot of the number of vertices in the smallest graph $(G_k, \lambda_{k,n})$ that has a non-zero probability of being a solution for $\mathcal{M}_n$ according to theorems 2 and 3 for various values of n.

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