Hennessy-Milner Properties in Schröder Categories

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Abstract. Hennessy-Milner logic is a logic for labeled transition systems, which specifies behaviors of processes with logical formulae. Correspondence between semantical indistinguishability of processes in a sense of observational equivalence, bisimilarity, or experimental equivalence and logical indistinguishability of processes in the sense of satisfying the same formulae is called Hennessy-Milner property. This note presents a relational formalisation of Hennessy-Milner properties in Schröder categories.

Keywords: Process algebra, Labeled transition system, Relational calculus, Dedekind category, Schröder category.

1 Introduction

A modal logic for labeled transition systems due to Hennessey and Milner [4] (Hennessy–Milner logic, or HM logic for short) specifies behaviors of processes with logical formulae. This logic presents modality as occurrence of actions in transition systems and actions are classified into necessity and possibility. Labeled transition systems are used for operational semantics for parallel programs and concurrent processes which provide a notion of ‘observational equivalence’. Between such logical systems and semantic structures equivalent correspondences called Hennessy–Milner properties (HM properties, for short) [3] are shown in [4].

HM properties are classified into three levels with respect to semantics of labeled transition systems; observational level, simulational level and experimental
level. These correspond to HM logic with negation and conjunction, with con-
junction, and only with modality, respectively. The correspondence is that two
processes $p$ and $q$ are equivalent in a semantical level if and only if the sets of
logical formulae which $p$ enjoys and which $q$ enjoys are the same. One may see
that the equality of the two sets induces a binomial equivalence relation which
classifies processes enjoying all formulae and the others.

In this paper we present an alternative proof of HM properties in Schröder
categories which is a generalisation of the category of relations on sets. According
to our study it turns out that HM properties are also valid in abstract modal
logics, and consequently one may manipulate concurrent processes with behaviors
characterised by logical formulae, which are valid in values between true and false.

The paper is organised as follows: In Section 2 the definition of Dedekind
categories is reviewed and the basic properties of Dedekind categories are listed.
Then, a notion of binomial equivalence relations is given. Also, two operations
on relations defined by a given labeled transition system in a Dedekind category
are introduced and it is shown that the operations are anticontinuous. As the
well-known result of a classical fixed point theorem we can identify the observa-
tional and the simulation equivalences for labeled transition systems in Dedekind
categories. In Section 3 the definition of Schröder categories is reviewed and we
state a few fundamental properties of Schröder categories, needed in the later
sections. In Section 4 we verify HM properties in terms of relational calculus.
Readers may find out that methods of proof are algebraically unified in Schröder
categories. In Section 5 the experimental case is studied.

## 2 Dedekind Categories

In this section we recall the fundamentals on relation categories, which we will
call Dedekind categories.

Throughout this note, a morphism $\alpha$ from an object $X$ into an object $Y$ in
a Dedekind category (which will be defined below) will be denoted by a half
arrow $\alpha : X \rightarrow Y$, and the composite of a morphism $\alpha : X \rightarrow Y$ followed by a
morphism $\beta : Y \rightarrow Z$ will be written as $\alpha \beta : X \rightarrow Z$. Also we will denote the
identity morphism on $X$ as $\text{id}_X$.

**Definition 1.** A Dedekind category $\mathcal{D}$ is a category satisfying the following:

D1. [Complete Heyting Algebra] For all pairs of objects $X$ and $Y$ the hom-set
$\mathcal{D}(X,Y)$ consisting of all morphisms of $X$ into $Y$ is a complete Heyting alge-
bra with the least morphism $0_{XY}$ and the greatest morphism $\nabla_{XY}$. Its algebra
structure will be denoted by 
\[(\mathcal{D}(X,Y), \sqsubseteq, \sqcup, \sqcap, \Rightarrow, 0_{XY}, \nabla_{XY})\].

D2. [Converse] There is given a converse operation \(\sharp: \mathcal{D}(X,Y) \to \mathcal{D}(Y,X)\). That is, for all morphisms \(\alpha, \alpha' : X \to Y\), \(\beta : Y \to Z\), the following involutive laws hold:
(a) \((\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp\), (b) \((\alpha^\sharp)^\sharp = \alpha\), (c) If \(\alpha \sqsubseteq \alpha'\), then \(\alpha^\sharp \sqsubseteq \alpha'^\sharp\).

D3. [Dedekind Formula (DF, for short)] For all morphisms \(\alpha : X \to Y\), \(\beta : Y \to Z\) and \(\gamma : X \to Z\) the Dedekind formula \(\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)\) holds.

D4. [Residual Composition] For all morphisms \(\alpha : X \to Y\) and \(\beta : Y \to Z\) the residual composite \(\alpha \ominus \beta : X \to Z\) is a morphism such that \(\alpha^\sharp\delta \sqsubseteq \beta\) if and only if \(\delta \sqsubseteq \alpha \ominus \beta\) for all morphisms \(\delta : X \to Z\).

Since each hom-set \(\mathcal{D}(X,Y)\) in a Dedekind category \(\mathcal{D}\) is a complete Heyting algebra, a distributive law (DL)
\[\alpha \sqcap (\sqcup_{j \in J} \alpha_j) = \sqcup_{j \in J} (\alpha \sqcap \alpha_j)\]
holds for all \(\alpha, \alpha_j : X \to Y\) \((j \in J)\), and so we easily obtain the following extended distributive law (EDL)
\[\sqcap_{i=1}^k (\sqcup_{j \in J} \alpha_{ij}) = \sqcup_{j_i \in J, \ldots, j_k \in J} (\sqcap_{i=1}^k \alpha_{ij})\]
for all \(\alpha_{ij} : X \to Y\) \((i = 1, \ldots, k\) and \(j \in J)\).

An object \(I\) in a Dedekind category \(\mathcal{D}\) is called a strict unit if \(0_{II} \neq \text{id}_I = \nabla_{II}\) and \(\nabla_{XI} \nabla_{IX} = \nabla_{XX}\) for all objects \(X\). A morphism \(f : X \to Y\) such that \(f^\sharp f \sqsubseteq \text{id}_Y\) (univalent) is called a partial function and may be introduced as \(f : X \to Y\). A partial function \(f : X \to Y\) such that \(\text{id}_X \sqsubseteq ff^\sharp\) (total) is called a function.

Example 1. A category \text{Rel} with objects all sets and arrows \(\alpha: X \to Y\) all binary relations being subsets of \(X \times Y\) is a Dedekind category. For all morphisms \(\alpha : X \to Y\) and \(\beta : Y \to Z\) it holds that
\[\forall (x, z) \in (\alpha \sqcap \beta) \iff [\exists y \in Y. ((x, y) \in \alpha \Rightarrow (y, z) \in \beta)]\]
A singleton set is a strict unit.

In what follows the word relation is a synonym of morphisms in a Dedekind category. For the fundamental properties of relational categories it is referred to \([1,2,5,7]\).

In the rest of the section we assume that \(\mathcal{D}\) is a Dedekind category.
Proposition 1. Let $\alpha, \alpha', \alpha_j : X \to Y$, $\beta, \beta', \beta_j : Y \to Z$, $\gamma : Z \to W$ and $\delta : X \to Z$ ($j \in J$) be relations and $V$ an object in $\mathcal{D}$.

$$
X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} W \xleftarrow{\delta}
$$

Then the following hold:

(a) $\text{id}_Z^\sharp = \text{id}_X$, $0_X^\sharp = 0_Y$ and $\nabla_{XY}^\sharp = \nabla_{YX}^\sharp$.
(b) $(\bigcup_{j \in J} \beta_j)^\sharp = \bigcup_{j \in J} \beta_j^\sharp$ and $(\bigcap_{j \in J} \beta_j)^\sharp = \bigcap_{j \in J} \beta_j^\sharp$.
(c) If $\alpha \subseteq \alpha'$ and $\beta \subseteq \beta'$, then $\alpha \beta \subseteq \alpha' \beta'$.
(d) $0_{YX} \alpha = 0_{YV}$ and $\alpha 0_{YV} = 0_{XY}$.
(e) $\alpha(\bigcup_{j \in J} \beta_j) = \bigcup_{j \in J} \alpha \beta_j$ and $\alpha(\bigcap_{j \in J} \beta_j) \subseteq \bigcap_{j \in J} \alpha \beta_j$.
(f) If $J \neq \emptyset$ and $\alpha$ is univalent, then $\alpha(\bigcap_{j \in J} \beta_j) = \bigcap_{j \in J} \alpha \beta_j$.
(g) $\alpha^\sharp(\alpha \beta) \subseteq \beta$, $\delta \subseteq \alpha \alpha^\sharp \delta$ and $(\alpha \beta) \gamma \subseteq \alpha \beta \gamma$.
(h) $\alpha \beta \circ \gamma = (\alpha \beta) \circ \gamma$.
(i) If $\alpha' \subseteq \alpha$ and $\beta \subseteq \beta'$, then $\alpha \beta \subseteq \alpha' \beta'$.
(j) $(\bigcup_{j \in J} \alpha_j) \circ \beta = \bigcap_{j \in J} (\alpha_j \circ \beta)$ and $\alpha \circ (\bigcap_{j \in J} \beta_j) = \bigcap_{j \in J} (\alpha \circ \beta_j)$.

The next proposition shows some technical properties of Dedekind categories.

Proposition 2. Let $\alpha : X \to Y$, $\beta : Y \to Z$, $\rho, \rho_j : V \to Y$ and $\tau : V \to Z$ ($1 \leq j \leq k$) be relations in $\mathcal{D}$.

$$
X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\tau} V
$$

Then the following hold:

(a) If $\tau$ is univalent, then $\alpha \circ \beta \tau^\sharp = (\alpha \circ \beta \tau^\sharp) \cap \nabla_{XZ} \tau^\sharp$.
(b) If $\alpha$ is univalent, then $\alpha \circ \beta = \alpha \nabla_{YZ} \Rightarrow \alpha \beta$.
(c) $\bigcap_{j=1}^k (\rho_j^\sharp \circ \rho_j \alpha) \subseteq \bigcap_{j=1}^k (\rho_j^\sharp \circ \rho_j \alpha) \nabla_{VY} \Rightarrow (\bigcap_{j=1}^k \rho_j^\sharp \circ \rho_j \alpha) \Rightarrow (\bigcap_{j=1}^k \rho_j^\sharp \circ \rho_j \alpha)$.

Proof. (a) By 1(g) it is clear that $(\alpha \circ \beta) \tau^\sharp \subseteq \nabla_{XZ} \tau^\sharp \cap (\alpha \circ \beta \tau^\sharp)$. The converse inclusion follows from

$$
\nabla_{XZ} \tau^\sharp \cap (\alpha \circ \beta \tau^\sharp) \subseteq \{ \nabla_{XZ} \cap (\alpha \circ \beta \tau^\sharp) \tau^\sharp \} \cap \{ \text{DF} \}
$$

$$
= (\alpha \circ \beta \tau^\sharp) \tau^\sharp
$$

$$
\subseteq (\alpha \circ \beta \tau^\sharp) \tau^\sharp \quad \{ 1(g) \}
$$

$$
\subseteq (\alpha \circ \beta) \tau^\sharp \quad \{ \tau^\sharp \subseteq \text{id}_Z \}.
$$
(b) Let \( \alpha \) be univalent. We will show that \( \alpha^\sharp \delta \sqsubseteq \beta \) iff \( \delta \cap \alpha \nabla_{YZ} \sqsubseteq \alpha \beta \) for all relations \( \delta : X \to Z \). First assume \( \alpha^\sharp \delta \sqsubseteq \beta \). Then

\[
\delta \cap \alpha \nabla_{YZ} \sqsubseteq \alpha(\alpha^\sharp \delta \cap \nabla_{YZ}) \quad \{ \text{DF} \}
\]

\[
\sqsubseteq \alpha \beta. \quad \{ \alpha \beta \sqsubseteq \nabla_{XZ} \}
\]

Conversely assume \( \delta \cap \alpha \nabla_{YZ} \sqsubseteq \alpha \beta \). Then

\[
\alpha^\sharp \delta = \alpha^\sharp \delta \cap \nabla_{YZ} \quad \{ \alpha^\sharp \delta \sqsubseteq \nabla_{YZ} \}
\]

\[
\sqsubseteq \alpha^\sharp (\delta \cap \alpha \nabla_{YZ}) \quad \{ \text{DF} \}
\]

\[
\sqsubseteq \alpha^\sharp \alpha \beta \quad \{ \delta \cap \alpha \nabla_{YZ} \sqsubseteq \alpha \beta \}
\]

\[
\sqsubseteq \beta. \quad \{ \alpha^\sharp \alpha \sqsubseteq \text{id}_Y \}
\]

(c) The desired inclusion is immediate from

\[
\left\{ \cap_{j=1}^k (\rho_j^\sharp \cap \rho_j \alpha) \right\} \cap (\cup_{j=1}^k \rho_j^\sharp) \nabla_{VY}
\]

\[
= \left\{ \cap_{j=1}^k (\rho_j^\sharp \cap \rho_j \alpha) \right\} \cap (\cup_{j=1}^k \rho_j^\sharp) \nabla_{VY} \quad \{ 1(e) \}
\]

\[
\sqsubseteq \cup_{j=1}^k (\rho_j^\sharp \cap \rho_j \alpha) \nabla_{VY} \quad \{ \text{DL} \}
\]

\[
\sqsubseteq \cup_{j=1}^k \rho_j^\sharp (\rho_j^\sharp \cap \rho_j \alpha) \nabla_{VY} \quad \{ \text{DF} \}
\]

\[
\sqsubseteq \cup_{j=1}^k \rho_j^\sharp \rho_j \alpha \quad \{ 1(g) \}
\]

\[
\square
\]

**Proposition 3.** If \( \alpha : X \to Y \) is a finite join of partial functions and \( \beta_j : Y \to Z \)

\((j = 0, 1, \cdots)\) is a decreasing sequence of relations, that is, \( \beta_j \sqsupseteq \beta_{j+1} \) for all \( j \),

then the equality \( \alpha(\cap_{j \geq 0} \beta_j) = \cap_{j \geq 0} \alpha \beta_j \) holds.

Proof. First a semi-distributive law \( \alpha(\cap_{j \geq 0} \beta_j) \sqsubseteq \cap_{j \geq 0} \alpha \beta_j \) is trivial. Assume that

\( \alpha = \cup_{i=1}^k f_i \) for a finite number of partial functions \( f_i \). Then we have

\[
\alpha(\cap_{j \geq 0} \beta_j) = (\cup_{i=1}^k f_i)(\cap_{j \geq 0} \beta_j) \quad \{ \alpha = \cup_{i=1}^k f_i \}
\]

\[
= \cup_{i=1}^k f_i(\cap_{j \geq 0} \beta_j) \quad \{ 1(e) \}
\]

\[
= \cup_{i=1}^k (\cap_{j \geq 0} f_i \beta_j) \quad \{ 1(f) \}
\]

\[
= \cap_{j_0 \geq 0, \cdots, j_k \geq 0} (\cup_{i=1}^k f_i \beta_{j_i}) \quad \{ \text{EDL} \}
\]

Thus we have to see that \( \cap_{j \geq 0} \alpha \beta_j \sqsubseteq \cup_{i=1}^k f_i \beta_{j_i} \) for all \( j_1 \geq 0, \cdots, j_k \geq 0 \). Set

\( j_0 = \max\{j_1, \cdots, j_k\} \). Then \( \beta_{j_0} \sqsubseteq \beta_i \) for all \( i = 1, \cdots, k \), and hence

\[
\cap_{j \geq 0} \alpha \beta_j \sqsubseteq \alpha \beta_{j_0} = \cup_{i=1}^k f_i \beta_{j_0} \sqsubseteq \cup_{i=1}^k f_i \beta_{j_i},
\]

which completes the proof. \( \square \)
2.1 Binomial Equivalence Relations

Binomial equivalence relations generalise equivalence relations with at most two equivalence classes. It is easy to define them by suitable algorithms in theoretical computer science, and hence they are often used to generate general equivalence relations.

Definition 2. For a relation \( \rho : V \to X \) define two relations \( \zeta_\rho : X \to X \) and \( \xi_\rho : X \to X \) by

\[
\zeta_\rho = \rho^\sharp \circ \rho \quad \text{and} \quad \xi_\rho = \zeta_\rho \cap \zeta_\rho^\sharp,
\]
respectively. □

Example 2. Let \( X \) be a set, \( V = \{*\} \), and \( \rho \subseteq \{*\} \times X \). Then it holds that

\[
(x, x') \in \zeta_\rho \iff [x \in X_\rho \Rightarrow x' \in X_\rho],
\]
\[
(x, x') \in \xi_\rho \iff [x \in X_\rho \iff x' \in X_\rho],
\]
where \( X_\rho = \{x \in X \mid (*, x) \in \rho\} \). Therefore \( \xi_\rho \) is an equivalence relation with two equivalence classes \( X_\rho \) and complement of \( X_\rho \). □

The following is the basic property of the two relations defined above.

Lemma 1. Let \( \delta : X \to X \) and \( \rho, \tau : V \to X \) be relations in \( \mathcal{D} \). Then the following hold:

(a) \( \zeta_\rho : X \to X \) is reflexive and transitive,
(b) \( \xi_\rho : X \to X \) is an equivalence relation,
(c) \( \zeta_\rho \cap \zeta_\tau \subseteq \zeta_{\rho \cap \tau} \) and \( \xi_\rho \cap \xi_\tau \subseteq \xi_{\rho \cap \tau} \),
(d) \( \delta \circ \zeta_\rho \delta^\sharp \subseteq \zeta_{\delta \circ \rho \circ \delta^\sharp} \).

Proof. (a) First the reflexive law \( \text{id}_X \subseteq \zeta_\rho \) follows from \( \text{id}_X \subseteq \rho^\sharp \circ \rho \) by \( \rho \circ \text{id}_X \subseteq \rho \).

The transitive law \( \zeta_\rho \zeta_\rho \subseteq \zeta_\rho \) can be easily seen from \( \rho(\rho^\sharp \circ \rho)(\rho^\sharp \circ \rho) \subseteq \rho(\rho^\sharp \circ \rho) \subseteq \rho \) by using 1(g).

(b) First the reflexive law \( \text{id}_X \subseteq \xi_\rho \) follows from \( \text{id}_X \subseteq \zeta_\rho \cap \zeta_\rho^\sharp \) by \( \text{id}_X \subseteq \zeta_\rho \). The symmetric law \( \xi_\rho \xi_\rho \subseteq \xi_\rho \) is trivial by definition. The transitive law \( \xi_\rho \xi_\rho \subseteq \xi_\rho \) can be seen from \( (\zeta_\rho \cap \zeta_\rho^\sharp)(\zeta_\rho \cap \zeta_\rho^\sharp) \subseteq \zeta_\rho \zeta_\rho \cap \zeta_\rho^\sharp \zeta_\rho^\sharp \subseteq \zeta_\rho \cap \zeta_\rho^\sharp \).

(c) The first follows from

\[
(\rho^\sharp \circ \rho) \cap (\tau^\sharp \circ \tau) \subseteq \{(\rho^\sharp \cap \tau^\sharp) \circ \rho\} \cap \{(\rho^\sharp \cap \tau^\sharp) \circ \tau\} \{1(i)\}
\]
\[
= (\rho \cap \tau)^\sharp \circ (\rho \cap \tau) \{1(j)\}
\]
and the second is a simple corollary of the former.

(d) Using 1(g), (h) and (i) it follows at once from \( \delta \circ (\rho^\sharp \circ \rho) \delta^\sharp \subseteq \delta \circ (\rho^\sharp \circ \rho \delta^\sharp) = \delta \rho^\sharp \circ \rho \delta^\sharp \). □
2.2 Observational and Simulation Equivalences

Let $D$ be a Dedekind category and $A$ a set of labels. An $A$-labeled transition system in $D$ consists of an object $X$ of $D$ and an $A$-indexed set of transition relations $\delta_a : X \rightarrow X$ ($a \in A$) in $D$, which will be denoted by

$$(X, \delta_a : X \rightarrow X; a \in A).$$

In the rest of the section we will assume that an $A$-labeled transition system $(X, \delta_a : X \rightarrow X; a \in A)$ in $D$ is given.

Definition 3. For each relation $\theta : X \rightarrow X$ relations $\sigma(\theta) : X \rightarrow X$ and $\varepsilon(\theta) : X \rightarrow X$ are defined by

$$\sigma(\theta) = \bigsqcap_{a \in A}(\delta_a \circ \theta \delta_a^\sharp) \quad \text{and} \quad \varepsilon(\theta) = \sigma(\theta) \cap \sigma(\theta)^\sharp,$$

respectively.

The following proposition is the basic property of the two operations defined above.

Proposition 4. Let $\theta : X \rightarrow X$ be a relation in $D$.

(a) If $\theta$ is reflexive and transitive, then so is $\sigma(\theta)$.
(b) If $\theta$ is an equivalence relation, then so is $\varepsilon(\theta)$.

Proof. (a) First the reflexive $\text{id}_X \subseteq \sigma(\theta)$ follows from $\text{id}_X \subseteq \delta_a \circ \theta \delta_a^\sharp$ by $\delta_a^\sharp \text{id}_X \subseteq \theta \delta_a^\sharp$. The transitive law $\sigma(\theta) \sigma(\theta) \subseteq \sigma(\theta)$ can be seen from

$$\delta_a^\sharp(\delta_a \circ \theta \delta_a^\sharp)(\delta_a \circ \theta \delta_a^\sharp) \subseteq \theta \delta_a^\sharp(\delta_a \circ \theta \delta_a^\sharp) \subseteq \theta \theta \delta_a^\sharp \subseteq \theta \delta_a^\sharp.$$

(b) Since the symmetric law $\theta^\sharp = \theta$ implies $\varepsilon(\theta) = \sigma(\theta) \cap \sigma(\theta)^\sharp$, it is immediate from (a).

The operation $\sigma : D(X, X) \rightarrow D(X, X)$ is called anticontinuous if

$$\sigma(\bigsqcap_{n \geq 0} \theta_n) = \bigsqcap_{n \geq 0} \sigma(\theta_n)$$

for all decreasing sequences $\theta_n$ of relations in $D(X, X)$. In [4], transition relations are called image–finite if each process transits to at most finitely many processes. In the paper we postulate an image–finite relation to be a finite join of partial functions.

Proposition 5. If $\delta_a : X \rightarrow X$ is a finite join of partial functions for all $a \in A$, then the two operations $\sigma$ and $\varepsilon$ are anticontinuous.
Proof. Let $\theta_n : X \rightarrow X$ ($n \geq 0$) be a decreasing sequence of relations in $\mathcal{D}$.

(a) It simply follows from

$$\sigma(\bigcap_{n \geq 0} \theta_n) = \bigcap_{a \in A} \{ \delta_a \ominus (\bigcap_{n \geq 0} \theta_n) \delta_a^\sharp \} = \bigcap_{a \in A} \{ \delta_a \ominus (\bigcap_{n \geq 0} \theta_n) \delta_a^\sharp \} \{ 3 \} = \bigcap_{n \geq 0} \sigma(\theta_n).$$

(b) By making use of (a) above it is direct from

$$\varepsilon(\bigcap_{n \geq 0} \theta_n) = \sigma(\bigcap_{n \geq 0} \theta_n) \cap \sigma(\bigcap_{n \geq 0} \theta_n^\sharp) = \{ \bigcap_{n \geq 0} \sigma(\theta_n) \} \cap \{ \bigcap_{n \geq 0} \sigma(\theta_n^\sharp) \} \{ a \} = \bigcap_{n \geq 0} \{ \sigma(\theta_n) \cap \sigma(\theta_n^\sharp) \} = \bigcap_{n \geq 0} \varepsilon(\theta_n).$$

□

By the last proposition it follows from the classical fixed point theorem that $\sigma_\infty = \bigcap_{n \geq 0} \sigma^n(\nabla_{XX})$ and $\varepsilon_\infty = \bigcap_{n \geq 0} \varepsilon^n(\nabla_{XX})$ are the maximum solutions to $\sigma(\theta) = \theta$ and $\varepsilon(\theta) = \theta$, respectively. It is trivial that $\sigma_n = \sigma^n(\nabla_{XX})$ is reflexive and transitive, and $\varepsilon_n = \varepsilon^n(\nabla_{XX})$ is an equivalence relation. Also $\sigma_n \cap \sigma_n^\sharp \subseteq \varepsilon_n$ holds. Following Hennessy and Milner [4] the equivalence relations $\varepsilon_\infty$ and $\sigma_\infty \cap \sigma_\infty^\sharp$ are called the observational and the simulation equivalences, respectively.

## 3 Schröder Categories

**Definition 4.** A Schröder category $\mathcal{S}$ is a Dedekind category such that for all pairs of objects $X$ and $Y$ the hom-set $\mathcal{S}(X,Y)$ is a boolean algebra, in other words, every relation $\alpha : X \rightarrow Y$ has its complement $\alpha^{-} : X \rightarrow Y$.

In the rest of the section we assume that $\mathcal{S}$ is a Schröder category with a strict unit $I$.

**Proposition 6.** Let $\alpha, \alpha' : X \rightarrow Y$, $\beta : Y \rightarrow Z$ and $\rho, \tau : I \rightarrow X$ be relations in $\mathcal{S}$.

$$\begin{array}{c}
X \xrightarrow{\alpha} Y \\
\rho \uparrow \downarrow \tau
\end{array} \quad \begin{array}{c}
Y \xrightarrow{\beta} Z
\end{array}$$

Then the following hold:
(a) \( \alpha^{-z} = \alpha^z \).
(b) If \( \alpha \) is univalent, then \((\alpha\beta)^- = (\alpha \nabla_{YZ})^- \sqcup \alpha\beta^- \) holds. In particular, \((\nabla_{XI\rho})^- = \nabla_{XI\rho^-} \) is always valid.
(c) If \( \alpha \) is univalent, then \( \alpha \sqcap \beta = (\alpha \nabla_{YZ})^- \sqcup \alpha\beta \).
(d) \( \xi_\rho = \rho^\sharp \rho \sqcup \rho^- \rho^- \) and \( \xi_{\rho^-} = \xi_\rho \).
(e) \( (\rho \sqcap \tau)^\sharp \ominus (\rho \sqcap \tau)(\alpha \sqcup \alpha') \sqsubseteq (\rho^\sharp \rho \alpha) \sqcup (\tau^\sharp \rho \alpha') \).
(f) \( \rho^\sharp \rho \rho(\alpha \sqcup \alpha') \sqsubseteq (\rho^\sharp \rho \alpha) \sqcup \nabla_{XX\alpha} \).

Proof. (a) It is direct from \( \alpha^2 \sqcup \alpha^{-z} = (\alpha \sqcup \alpha^-)^2 = \nabla_{YY}^z = \nabla_{XY} \) and \( \alpha^2 \sqcup \alpha^{-z} = (\alpha \sqcup \alpha^-)^2 = 0^x_{XY} \) by 1(b).
(b) Assume that \( \alpha \) is univalent. The first equality simply follows from \( \alpha\beta^- \sqcup \alpha\beta = \alpha(\beta^- \sqcup \beta) = \alpha \nabla_{YZ} \) by 1(e) and \( \alpha\beta^- \sqcup \alpha\beta = \alpha(\beta^- \sqcup \beta) = 0_{XX} \) by 1(f). Moreover, as \( \nabla_{XI} \) is univalent, we have \((\nabla_{XI\rho})^- = (\nabla_{XI\rho^-})^- \sqcup \nabla_{XI\rho^-} = \nabla_{XI\rho^-} \).
(c) Recall that an equality \( a \Rightarrow b = a \sqcup b \) always holds in boolean algebras. Thus the assertion is just a corollary of 2(b).
(d) First note that \( \rho^\sharp \rho \sqcap \rho^\sharp \rho^- = 0_{XX} \). As \( \rho^\sharp : X \rightarrow I \) is univalent we have

\[
\xi_\rho = (\rho^\sharp \nabla_{IX})^- \sqcup \rho^\sharp \rho \quad \{ \text{(c)} \}
= \rho^\sharp \nabla_{IX} \sqcup \rho^\sharp \rho \quad \{ \text{by (a)} \}
= \rho^\sharp \rho \sqcup \rho^\sharp \rho^- \sqcup \rho^\sharp \rho, \{ \nabla_{IX} = \rho \sqcup \rho^- \}
\]
and so \( \xi_{\rho^-} = (\rho\sqcap \rho^\sharp \rho^-) \sqcup (\rho^\sharp \rho \rho^-) = \rho^\sharp \rho \rho \rho^- \).
(e) As \( \nabla_{XI}, \rho^\sharp, \tau^\sharp, (\rho \sqcap \tau)^\sharp : X \rightarrow I \) are univalent we have

\[
(\rho \sqcap \tau)^\sharp \ominus (\rho \sqcap \tau)(\alpha \sqcup \alpha') = (\rho \sqcap \tau)(\alpha \sqcup \alpha') \sqsubseteq (\rho^\sharp \nabla_{IX})^- \sqcup (\tau^\sharp \nabla_{IX})^- \sqcup \rho^\sharp \rho \alpha \sqcup \tau^\sharp \tau \alpha' \quad \{ \text{de Morgan} \}
\]
(f) Set \( \tau = \rho \). Then it directly follows from the proof of (e). \( \square \)

Let \( \rho_1, \cdots, \rho_k : I \rightarrow X \) be relations. For a subset \( J \subseteq \{1, \cdots, k\} \) we denote the relation \( \sqcap_{j \in J} \rho_j \) by \( \rho_J \). In particular \( \rho_\emptyset = \nabla_{IX} \).

**Proposition 7.** Let \( f_1, \cdots, f_k : X \rightarrow X \) and \( \rho_1, \cdots, \rho_k : I \rightarrow X \) \((k \geq 1)\) be relations in \( S \). If \( f_1, \cdots, f_k \) are univalent, then

\[\sqcap_{J \subseteq \{1, \cdots, k\}} \{ (\rho_J^\sharp \ominus \rho_J (\sqcup_{i=1}^k f_i)^z_\sharp) \} \sqsubseteq \sqcup_{i=1}^k \xi_{\rho_i} f_i^z.\]

Proof. First of all we set

\[ L_k = \sqcap_{J \subseteq \{1, \cdots, k\}} \{ (\rho_J^\sharp \ominus \rho_J (\sqcup_{i=1}^k f_i)^z_\sharp) \} \quad \text{and} \quad R_k = \sqcup_{i=1}^k \xi_{\rho_i} f_i^z. \]
We will prove the statement $L_k \subseteq R_k$ by induction on $k$. In the case of $k = 1$:

Set $f = f_1$ and $\rho = \rho_1$. Then we have

$$L_1 = (\nabla_{XI} \odot \nabla_{IX} f^2) \cap (\rho^2 \odot \rho f^2) \{ \rho_0 = \nabla_{IX} \}$$

$$= \nabla_{XX} f^2 \cap (\rho^2 \odot \rho f^2) \{ 6(c) \}$$

$$= \zeta_\rho f^2. \{ 2(a) \}$$

Next we will show the inductive step: Assume $L_k \subseteq R_k$ for $k \geq 1$ and set $\delta = \sqcup_{i=1}^{k} f_i$, $f = f_{k+1}$ and $\tau = \rho_{k+1}$. Now we decompose $L_{k+1}$ into two parts, that is, $L_{k+1} = L' \cap L''$, where

$$L' = \cap_{J \subseteq \{1, \ldots, k\}} \{ \rho_J^2 \odot \rho_J (\delta \sqcup f)^2 \}$$

and

$$L'' = \cap_{J \subseteq \{1, \ldots, k\}} \{ \rho_J^2 \odot \rho_J (\delta \sqcup f)^2 \} \cup \nabla_{XX} f^2 \{ \text{DL} \}$$

But we have

$$L' = \cap_{J \subseteq \{1, \ldots, k\}} \{ \rho_J^2 \odot \rho_J (\delta \sqcup f)^2 \} \cap \nabla_{XX} f^2 \{ 6(e) \}$$

$$= \{ \nabla_{XX} f^2 \} \{ \text{DL} \}$$

$$L'' = \cap_{J \subseteq \{1, \ldots, k\}} \{ \rho_J^2 \odot \rho_J (\delta \sqcup f)^2 \} \cup \nabla_{XX} f^2 \{ \text{DL} \}$$

Hence using the above results on $L'$ and $L''$ we have

$$L_{k+1} = L' \cap L''$$

$$= (L_k \sqcup \nabla_{XX} f^2) \cap \{ L_k \sqcup (\tau^2 \odot \tau f^2) \}$$

$$= L_k \sqcup \{ \nabla_{XX} f^2 \sqcup (\tau^2 \odot \tau f^2) \} \{ \text{DL} \}$$

$$= L_k \sqcup \zeta_f f^2 \{ 2(a) \}$$

$$\sqcup R_k \sqcup \zeta_f f^2 \{ \text{Induction hypothesis} \}$$

which completes the proof. \qed
4 Hennessy-Milner Properties

Hennessy and Milner [4] aimed at a logical characterisation for algebraic semantics of concurrency. We have prepared for our alternative proof, which will be given with unified methods in Schröder categories. In this section we state the characterisation theorems of the observational equivalences and the simulation equivalences.

In the HM logic a modality “\(a\)” denotes that “an action \(a\) will possibly occur in the next transition”. We define two languages \(\mathcal{L} = \bigcup_{n \geq 0} \mathcal{L}_n\) and \(\mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}_n\) for the observational case and the simulation case, respectively.

**Definition 5.** A sequence \(\mathcal{L}_0, \mathcal{L}_1, \cdots, \mathcal{L}_n, \cdots\) of sets of formulae is defined as follows:

(a) \(\mathcal{L}_0 = \{true\}\),
(b) If \(F \in \mathcal{L}_n\), then \(F \in \mathcal{L}_{n+1}\).
   - If \(F \in \mathcal{L}_n\) and \(a \in A\), then \(a.F \in \mathcal{L}_{n+1}\).
   - If \(F_1, F_2 \in \mathcal{L}_{n+1}\), then \(F_1 \land F_2 \in \mathcal{L}_{n+1}\).
   - If \(F \in \mathcal{L}_{n+1}\), then \(\neg F \in \mathcal{L}_{n+1}\).

**Definition 6.** A sequence \(\mathcal{M}_0, \mathcal{M}_1, \cdots, \mathcal{M}_n, \cdots\) of sets of formulae is defined as follows:

(a) \(\mathcal{M}_0 = \{true\}\),
(b) If \(F \in \mathcal{M}_n\), then \(F \in \mathcal{M}_{n+1}\).
   - If \(F \in \mathcal{M}_n\) and \(a \in A\), then \(a.F \in \mathcal{M}_{n+1}\).
   - If \(F_1, F_2 \in \mathcal{M}_{n+1}\), then \(F_1 \land F_2 \in \mathcal{M}_{n+1}\).

As is known that an interpretation of modal logic is given on sets of possible worlds, the semantics of HM logic is defined as a set of processes in labeled transition systems. Intuitively, \(\llbracket F \rrbracket\) stands for the set of processes enjoying a property \(F\) in \(\mathcal{L}\) or \(\mathcal{M}\).

**Definition 7.** For each formula \(F\) in \(\mathcal{L}\) or \(\mathcal{M}\) its interpretation \(\llbracket F \rrbracket : I \rightarrow X\) is defined as follows:

(i) \(\llbracket true \rrbracket = \nabla_{IX}\),
(ii) \(\llbracket a.F \rrbracket = \llbracket F \rrbracket \delta^a\),
(iii) \(\llbracket F_1 \land F_2 \rrbracket = \llbracket F_1 \rrbracket \cap \llbracket F_2 \rrbracket\),
(iv) \(\llbracket \neg F \rrbracket = \llbracket F \rrbracket^c\).
For each formula $F$ in $L$ or $M$ we then attach a binomial equivalence relation
$\xi[\![F]\!] : X \rightarrow X$. In $\text{Rel}$, as the case of Example 2, $\xi[\![F]\!]$ classifies into indistinguishable processes and the others in the sense of satisfying $F$.

**Definition 8.** For each natural number $n$ a reflexive and transitive relation $\zeta_n : X \rightarrow X$ and an equivalence relation $\xi_n : X \rightarrow X$ is defined as follows:

$$
\zeta_n = \cap_{F \in M_n} \zeta[\![F]\!] \\
\xi_n = \cap_{F \in L_n} \xi[\![F]\!].
$$

Moreover we set $\zeta_\infty = \cap_{n \geq 0} \zeta_n$ and $\xi_\infty = \cap_{n \geq 0} \xi_n$. □

The purpose of the paper is to prove the following theorem, the so-called Hennessy-Milner (HM) properties, in Schröder categories.

**Theorem 1.** If transition relations are image-finite, then the observational equivalence $\varepsilon_\infty$ coincides with the logical equivalence $\xi_\infty$ and also the simulation equivalence $\sigma_\infty \cap \sigma_\infty^\sharp$ coincides with $\zeta_\infty \cap \zeta_\infty^\sharp$. More precisely the following identities hold for all natural numbers $n$.

(a) $\sigma_n = \zeta_n$,
(b) $\varepsilon_n = \xi_n$. □

To prove the above characterisation theorem it is sufficient to show the following propositions 8 and 9. The former is straightforward, but the latter needs an assumption that transition relations are image-finite, postulated to be finite joins of partial functions.

We now show the simpler part of the HM properties in Schröder categories.

**Proposition 8.** The following inclusions hold for all natural numbers $n$.

(a) $\sigma_n \subseteq \zeta_n$,
(b) $\varepsilon_n \subseteq \xi_n$.

Proof. The statements will be proved by structural induction on formulae. In the case of $n = 0$ the inclusions are trivial.

(a) Recall $\sigma_n = \sigma^n(\nabla_{XX})$ and assume that $\sigma_n \subseteq \zeta[\![F]\!]$ for all $F \in M_n$. We will show that $\sigma_{n+1} \subseteq \zeta[\![G]\!]$ for all $G \in M_{n+1}$. (i) If $G \in M_n$, then $\sigma_{n+1} \subseteq \sigma_n \subseteq \zeta[\![G]\!]$.

(ii) Assume that $G = a.F$ for $F \in M_n$ and $a \in A$. Then from the induction hypothesis $\sigma_n \subseteq \zeta[\![F]\!]$ we have

$$
\sigma_{n+1} \subseteq \delta_a \circ \sigma_n \delta_a^\sharp \{ \sigma_{n+1} = \sigma(\sigma_n) \} \\
\subseteq \delta_a \circ \zeta[\![F]\!] \delta_a^\sharp \{ \sigma_n \subseteq \zeta[\![F]\!] \} \\
\subseteq \zeta[G] \{ 1(d), 7(ii) \} \\
= \zeta[\![G]\!].
$$
If \( G = G_1 \land G_2 \) for \( G_1, G_2 \in \mathcal{M}_{n+1} \) with \( \sigma_{n+1} \subseteq \zeta_{[G_1]} \) and \( \sigma_{n+1} \subseteq \zeta_{[G_2]} \), then
\[
\sigma_{n+1} \subseteq \zeta_{[G_1]} \cap \zeta_{[G_2]}
\subseteq \zeta_{[G_1 \land G_2]} \quad \{ \text{1(c), 7(iii)} \}
= \zeta_{[G]}.
\]

(b) Recall \( \varepsilon_n = \varepsilon^n(\nabla_{XX}) \) and assume that \( \varepsilon_n \subseteq \xi_{[F]} \) for all \( F \in \mathcal{L}_n \). We will show that \( \varepsilon_{n+1} \subseteq \xi_{[G]} \) for all \( G \in \mathcal{L}_{n+1} \) by the structural induction. (i) If \( G \in \mathcal{L}_n \), then \( \varepsilon_{n+1} \subseteq \varepsilon_n \subseteq \xi_{[G]} \). (ii) Assume that \( G = a.F \) for \( F \in \mathcal{L}_n \) and \( a \in A \). Then from \( \varepsilon_n \subseteq \xi_{[F]} \subseteq \zeta_{[F]} \) we have \( \delta_a \sqcap \varepsilon_n \delta^a \sqsubseteq \delta_a \sqcap \zeta_{[F]} \delta^a \sqsubseteq \zeta_{[a.F]} = \zeta_{[G]} \) (Cf. the proof of (a)(ii)) and hence
\[
\varepsilon_{n+1} \sqsubseteq (\delta_a \sqcap \varepsilon_n \delta^a) \sqcap (\delta_a \sqcap \varepsilon_n \delta^a)^2 \quad \{ \varepsilon_{n+1} = \sigma(\varepsilon_n) \sqcap \sigma(\varepsilon_n)^2 \}
\subseteq \xi_{[G]} \sqcap \zeta_{[G]}^2 \quad \{ \delta_a \sqcap \varepsilon_n \delta^a \sqsubseteq \xi_{[G]} \}
= \xi_{[G]}.
\]

(iii) If \( G = \neg G' \) for \( G' \in \mathcal{L}_{n+1} \) with \( \varepsilon_{n+1} \subseteq \xi_{[G']} \), then \( \varepsilon_{n+1} \subseteq \xi_{[G']} = \xi_{[\neg G']} = \xi_{[G]} \) by 6(d) and 7(iv). (iv) If \( G = G_1 \land G_2 \) for \( G_1, G_2 \in \mathcal{L}_{n+1} \) with \( \varepsilon_{n+1} \subseteq \xi_{[G_1]} \) and \( \varepsilon_{n+1} \subseteq \xi_{[G_2]} \), then \( \varepsilon_{n+1} \subseteq \xi_{[G_1]} \cap \xi_{[G_2]} \subseteq \xi_{[G_1 \land G_2]} \) by 1(c) and 7(iii). \( \square \)

The essential part of HM properties in Schröder categories will be shown in the following proposition.

**Proposition 9.** If each transition relation \( \delta_a \) is a finite join of partial functions, then the following inclusions hold for all natural numbers \( n \).

(a) \( \zeta_n \subseteq \sigma_n \).
(b) \( \xi_n \subseteq \varepsilon_n \).

**Proof.** (a) We will see that \( \zeta_{n+1} \subseteq \sigma_{n+1} \) can be derived from the induction hypothesis \( \zeta_n \subseteq \sigma_n \). Assume that \( \zeta_n \subseteq \sigma_n \). If \( \zeta_{n+1} \subseteq \sigma(\zeta_n) \), then \( \zeta_{n+1} \subseteq \sigma(\zeta_n) \subseteq \sigma(\sigma_n) = \sigma_{n+1} \). So it suffices to prove \( \zeta_{n+1} \subseteq \sigma(\zeta_n) \) and so \( \zeta_{n+1} \subseteq \delta_a \cap \zeta_n \delta^a \) for all \( a \in A \). But this fact follows from the following:
\[
\zeta_{n+1} \subseteq \cap_{F \in \mathcal{M}_n} \zeta_{[a.F]}
= \cap_{F \in \mathcal{M}_n} (\delta_a [F]^2 \cap [F] \delta^a)
= \cap_{F \in \mathcal{M}_n} \{ \delta_a \cap ([F] \cap [F] \delta^a) \}
\subseteq \delta_a \cap \zeta_n \delta^a, \quad \{ \text{2(a)} \}
\]

(b) We will see that \( \xi_{n+1} \subseteq \varepsilon_{n+1} \) can be derived from the induction hypothesis \( \xi_n \subseteq \varepsilon_n \). Assume that \( \xi_n \subseteq \varepsilon_n \). If \( \xi_{n+1} \subseteq \varepsilon(\xi_n) \), then \( \xi_{n+1} \subseteq \varepsilon(\xi_n) \subseteq \varepsilon(\varepsilon_n) = \varepsilon_{n+1} \).
So it suffices to prove $\xi_{n+1} \sqsubseteq \varepsilon(\xi_n)$, which is equivalent to the condition that $\xi_{n+1} \sqsubseteq \delta_a \ominus \xi_n \delta_a^2$ for all $a \in A$. But this condition follows from

$$\begin{align*}
\xi_{n+1} & \sqsubseteq \bigcap_{F \in \mathcal{L}_n} \zeta_{[a,F]} \\
& = \bigcap_{F \in \mathcal{L}_n} (\delta_a [[F]^2] \odot [[F]] \delta_a^2) \\
& = \delta_a \ominus \{ \bigcap_{F \in \mathcal{L}_n} ([[F]^2] \odot [[F]] \delta_a^2) \} \\
& \sqsubseteq \delta_a \ominus \xi_n \delta_a^2. & \{ \text{2(b)} \} 
\end{align*}$$

We close the section by showing the Lemma 2 which has already been used in the above proof.

**Lemma 2.** If a transition relation $\delta_a$ is a finite join of partial functions, then the following inclusions hold for all natural numbers $n$.

(a) $\bigcap_{F \in \mathcal{M}_n} ([[F]^2] \odot [[F]] \delta_a^2) \sqsubseteq \zeta_n \delta_a^2$.

(b) $\bigcap_{F \in \mathcal{L}_n} ([[F]^2] \odot [[F]] \delta_a^2) \sqsubseteq \xi_n \delta_a^2$.

**Proof.** By the assumption we can set $\delta_a = \bigcup_{i=1}^k f_i$, where $f_i$ is a partial function for $i = 1, \cdots, k$. Then it follows that

$$\begin{align*}
\zeta_n \delta_a^2 &= \zeta_n (\bigcup_{i=1}^k f_i)^2 \\
& = \bigcup_{i=1}^k \zeta_n f_i^2 \\
& = \bigcup_{i=1}^k \{ (\bigcap_{F \in \mathcal{M}_n} \zeta_{[F]} f_i^2) \} \\
& = \bigcup_{i=1}^k \{ (\bigcap_{F \in \mathcal{L}_n} \zeta_{[F]} f_i^2) \} & \{ \text{DL} \} \\
& = \bigcap_{F_1, \cdots, F_k \in \mathcal{M}_n} (\bigcup_{i=1}^k \zeta_{[F]} f_i^2) & \{ \text{EDL} \}
\end{align*}$$

and in the similar way we have $\xi_n \delta_a^2 = \bigcap_{F_1, \cdots, F_k \in \mathcal{L}_n} (\bigcup_{i=1}^k \zeta_{[F]} f_i^2)$.

(a) Let $F_1, \cdots, F_k$ be formulae in $\mathcal{M}_n$. It suffices to prove

$$\bigcap_{F \in \mathcal{M}_n} ([[F]^2] \odot [[F]] \delta_a^2) \sqsubseteq \bigcup_{i=1}^k \zeta_{[F]} f_i^2.$$

For each subset $J \subseteq \{1, \cdots, k\}$ we define a formula $F_J$ by

$$F_0 = \text{true}, \quad \text{and} \quad F_J = F_{j_1} \land \cdots \land F_{j_t} \quad \text{if} \quad J = \{j_1, \cdots, j_t\}.$$

Then by the virtue of Proposition 7 we obtain

$$\bigcap_{J \subseteq \{1, \cdots, k\}} ([[F_J]^2] \odot [[F_J]] \delta_a^2) \sqsubseteq \bigcup_{i=1}^k \zeta_{[F]} f_i^2.$$

By the way all formulae $F_J$ are in $\mathcal{M}_n$, since all $F_1, \cdots, F_k$ are in $\mathcal{M}_n$. Hence it is trivial that

$$\bigcap_{F \in \mathcal{M}_n} ([[F]^2] \odot [[F]] \delta_a^2) \sqsubseteq \bigcap_{J \subseteq \{1, \cdots, k\}} ([[F_J]^2] \odot [[F_J]] \delta_a^2).$$
which completes the proof.

(b) Let \( F_1, \cdots, F_k \) be formulae in \( \mathcal{L}_n \). It suffices to see that
\[
\bigcap_{F \in \mathcal{L}_n} ([F] \circ [F]) \subseteq \bigcup_{i=1}^k \xi_{[F_i]} f_i^\sharp.
\]
Now consider the set \( \mathcal{F} \) of all formulae \( F_1' \land \cdots \land F_k' \), where \( F_i' = F_i \) or \( F_i' = \neg F_i \) for \( i = 1, \cdots, k \). (In other words \( F_i' \) is a literal of \( F_i \).) That is,
\[
\mathcal{F} = \{ \neg^{\epsilon_i} F_1 \land \cdots \land \neg^{\epsilon_k} F_k \mid \epsilon_i = 0 \text{ or } 1 \ (i = 1, \cdots, k) \},
\]
where \( \neg^0 F_i = F_i \) and \( \neg^1 F_i = \neg F_i \). It is readily seen that
\[
\bigcup_{F \in \mathcal{F}} [F] = \nabla_{IX} \quad \text{and} \quad \bigcup_{F \in \mathcal{F}} [F] \subseteq \bigcap_{j=1}^k \xi_{[F_j]},
\]
because for all \( F \in \mathcal{F} \) and all \( j = 1, \cdots, k \) just one of two inclusions \([F] \subseteq [F_j]\) and \([F] \subseteq \neg [F_j]\) holds and so
\[
[F] \subseteq [F_j] \cup \neg [F_j] \quad \text{with} \quad \{ \neg [F_j] = [F_j] \}.
\]
Thus we have
\[
\bigcap_{F \in \mathcal{F}} ([F] \circ [F]) \subseteq \bigcup_{F \in \mathcal{F}} [F] \subseteq \bigcap_{j=1}^k \xi_{[F_j]} f_i^\sharp.
\]
By the way it is trivial that \( \mathcal{F} \) is a subset of \( \mathcal{L}_n \), since all \( F_1, \cdots, F_k \) are in \( \mathcal{L}_n \). Hence it is trivial that
\[
\bigcap_{F \in \mathcal{L}_n} ([F] \circ [F]) \subseteq \bigcup_{F \in \mathcal{F}} ([F] \circ [F]) \subseteq \bigcap_{F \in \mathcal{L}_n} ([F] \circ [F])
\]
which completes the proof. \( \Box \)

## 5 Experimental Equivalences

Experimental equivalences are regarded as equivalences of action–strings, that is, traces. Therefore the proof of this case is straightforward.
Let \((X, \delta : X \rightarrow X; a \in A)\) be an \(A\)-labeled transition system. For each finite string \(w \in A^*\) we define a relation \(\delta_w : X \rightarrow X\) by

\[
\delta_\varepsilon = \text{id}_X \quad \text{and} \quad \delta_aw = \delta_a\delta_w \quad \text{for } a \in A \text{ and } w \in A^* .
\]

**Definition 9.** Define a relation \(\varepsilon_n : X \rightarrow X\) by

\[
\varepsilon_n = \bigsqcap_{w \in A^{(n)}} \{(\delta_w \nabla_{XI} \circ \nabla_{IX} \delta_w^2) \sqcap (\delta_w \nabla_{XI} \circ \nabla_{IX} \delta_w^2)^2\}
\]

where \(A^{(n)}\) is the set of all finite strings \(w \in A^*\) with length \(|w| \leq n\).

It is trivial that \(\varepsilon_n\) is an equivalence relation on \(X\) such that \(\varepsilon_{n+1} \subseteq \varepsilon_n\). We set \(\varepsilon_\infty = \bigsqcap_{n \geq 0} \varepsilon_n\). Considering the case of experimental equivalences we require the language to describe actions sequentially.

**Definition 10.** A sequence \(N_0, N_1, \cdots, N_n, \cdots\) of sets of formulae will be defined as follows:

(a) \(N_0 = \{\text{true}\}\),

(b) If \(F \in N_n\), then \(F \in N_{n+1}\),

\[\text{If } F \in N_n \text{ and } a \in A, \text{ then } a.F \in N_{n+1}.\]

We now define a formula \(w.\text{true}\) for each finite string \(w \in A^*\) by

\[
\varepsilon.\text{true} = \text{true} \quad \text{and} \quad aw.\text{true} = a.(w.\text{true}) \quad \text{for } a \in A \text{ and } w \in A^* .
\]

It is trivial that \(N_n = \{w.\text{true} \mid w \in A^{(n)}\}\) for all natural numbers \(n\) and \([w.\text{true}] = \nabla_{IX} \delta_w^2\) for all finite strings \(w \in A^*\). Hence we have the following proposition.

**Proposition 10.** For each natural number \(n\) an equality \(\varepsilon_n = \bigsqcap_{F \in N_n} \xi[F]\) holds, and so the experimental equivalence \(\bigsqcap_{n \geq 0} \varepsilon_n\) coincides with \(\bigsqcap_{F \in N} \xi[F]\).

Proof. It is direct from the following computation.

\[
\varepsilon_n = \bigsqcap_{w \in A^{(n)}} \xi_{\nabla_{IX} \delta_w^2}
\]

\[
= \bigsqcap_{w \in A^{(n)}} \xi[w.\text{true}]
\]

\[
= \bigsqcap_{F \in N_n} \xi[F].
\]

\[\square\]
6 Conclusion

We presented an alternative proof of HM properties in a Schröder category which is a generalisation of the category of relations and sets. Considering behavior of communicating concurrent processes, one cannot determine their logical semantics with either true or false. It is worth studying logical semantics of concurrency in general Boolean algebras and lattice–valued logical semantics. Results of this paper show a possibility to extend HM properties to this direction.

References