Journal on Relational Methods in Computer Science, Vol. 1, 2004, pp. 132 - 160

# Relational Representation Theorems for Some Lattice-Based Structures

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## 1 Introduction

The major elements of the method of proving relational representation theorems presented in this paper are, on the one hand, Urquhart representation theorem for lattices [17] and Allwein and Dunn developments on Kripke semantics for linear logic (see [1] and also [4]), and on the other hand, a generalisation of Jonsson and Tarski ideas [9] which inspired the developments in [5], [6] and [13]. We present an extension of the Urquhart-Allwein-Dunn (UAD) methodology to lattice-based modal algebras, lattice-based relation algebras and algebras of substructural logics. Our generalisation of the UAD framework requires the following steps:

*Step 1.* A class of algebras is given. It consists of a signature and/or axiomatic extensions of not necessarily distributive lattices.

*Step 2.* We define a class of relational structures (frames) such that it provides a Kripke-style semantics for the logic whose algebraic semantics is determined by the class of algebras in question.

*Step 3.* For any algebra $W$ of the given class we define its canonical frame. The universe $X(W)$ of this frame consists of the pairs $(x_1, x_2)$ such that $x_1$

Received by the editors March 30, 2004, and, in revised form, October 06, 2004.
Published on December 10, 2004.

is a filter and $x_2$ is an ideal of the lattice reduct of $W$ and, moreover, they satisfy some additional conditions. It follows that $X(W)$ is a binary relation on $2^W$. The relations of the canonical frame correspond in an appropriate way to the operations of the algebra.

*Step 4.* For any frame $X$ of the class in question we define its complex algebra. The universe of the complex algebra is a family $L(X)$ of some special subsets of $X$ referred to as $l$–stable sets.

*Step 5.* We prove a representation theorem saying that every algebra from the class in question is isomorphic to a subalgebra of the complex algebra of its canonical frame. It follows that the universe of the representation algebra is the set $L(X(W))$ consisting of subrelations of relation $X(W)$.

The framework described above serves, on the one hand, as a tool for investigation of classes of lattices with operators and, on the other hand, as a means for developing Kripke-style semantics for the logics whose algebraic semantics is given. Then representation theorems play an essential role in proving completeness of the logics with respect to a Kripke semantics determined by a class of frames associated with a given class of algebras. In this paper we deal only with the algebraic aspects.

The framework presented above has been suggested in [16] in the context of lattice-based modal logics. The methodology elaborated there has a broad applicability. It has been applied to lattice-based relation algebras in [7], [8] and to substructural logics in [15]. The present paper provides a survey of those results. In Section 2 we recall the Urquhart representation theorem for lattices. Sections 3 and 4 are concerned with representation theory for lattices with a necessity operator and negative possibility operator, respectively. Section 5 presents lattice based relation algebras and their representability in the sense of our framework. In Section 6 our methodology is applied to the algebras of substructural logics.

## 2 Relational Representation of Lattices

In this section we recall Urquhart representation theorem for lattices [17] in the formulation of Orłowska and Vakarelov [16]. Throughout the paper we often use the same symbol for denoting an algebra or a relational system and their corresponding universes.

By a *doubly ordered set* we mean a relational system $(X, \leq_1, \leq_2)$, where $X$ is a non-empty set, $\leq_1$ and $\leq_2$ are quasi orders on $X$ such that for all $x, y \in X$, if $x \leq_1 y$ and $x \leq_2 y$, then $x = y$.

Given a doubly ordered set, the mappings $l : 2^X \to 2^X$ and $r : 2^X \to 2^X$ are defined as follows. For any $A \subseteq X$:

$l(A) = \{x \in X : \text{for every } y \in X, \text{ if } x \leq_1 y, \text{ then } y \notin A\}$,

$r(A) = \{x \in X : \text{for every } y \in X, \text{ if } x \leq_2 y, \text{ then } y \notin A\}$.

A set $A \subseteq X$ is said to be *$l$–stable* (resp. *$r$–stable*) whenever $lr(A) = A$ (resp. $rl(A) = A$). We say that $A$ is a *$\leq_i$–increasing* set, $i = 1, 2$, whenever for all $x, y \in X$, if $x \in A$ and $x \leq_i y$, then $y \in A$.

**Lemma 1.** *For every doubly ordered set $(X, \leq_1, \leq_2)$ and for all $A, B \subseteq X$ the following conditions are satisfied:*

(I) *$l(A)$ is a $\leq_1$–increasing set,*
(II) *$r(A)$ is a $\leq_2$–increasing set,*
(III) *$l(A), r(A) \subseteq -A$,*
(IV) *if $A \subseteq B$, then $l(B) \subseteq l(A)$ and $r(B) \subseteq r(A)$.*

□

**Lemma 2.** *The family of $\leq_i$–increasing sets, $i = 1, 2$, forms a distributive lattice, where join and meet are union and intersection of sets.* □

Observe that mapping $l$ treated as an operation of the lattice of $\leq_1$–increasing sets is an intuitionistic negation, and so is $r$ treated as an operation of the lattice of $\leq_2$–increasing sets. Consequently, if $A$ is $\leq_1$–increasing, then $l(A) = [\leq_1](-A)$ and if $A$ is $\leq_2$–increasing, then $r(A) = [\leq_2](-A)$. We recall that given a relational system $(X, R)$ such that $R$ is a binary relation on a set $X$, then for any $A \subseteq X$ the classical necessity operator $[R]$ determined by relation $R$ is defined as $[R]A = \{x \in X : \text{for every } y \in X \text{ if } R(x, y), \text{ then } y \in A\}$. Its well known properties are as follows.

**Lemma 3.** *Let $R$ and $S$ be binary relations on a set $X$. Then for all $A, B \subseteq X$ the following assertions hold:*

(I) *if $A \subseteq B$, then $[R]A \subseteq [R]B$,*
(II) *if $R \subseteq S$, then $[S]A \subseteq [R]A$,*
(III) *$[R; S]A = [R][S]A$, where ; is the composition of binary relations,*
(IV) *$[R](A \cap B) = [R]A \cap [R]B$.*

□

**Lemma 4.** [17]. *For every doubly ordered set $(X, \leq_1, \leq_2)$ the mappings $l$ and $r$ form a Galois connection between the lattice of $\leq_1$–increasing subsets of $X$ and the lattice of $\leq_2$–increasing subsets of $X$.* □

This lemma amounts to saying that $A \subseteq l(B)$ iff $B \subseteq r(A)$ for all $A$ and $B$ such that $A$ is a $\leq_1$–increasing set and $B$ is a $\leq_2$–increasing set.

**Lemma 5.** *For every doubly ordered set $(X, \leq_1, \leq_2)$ and for all $A, B \in X$ the following conditions are satisfied:*

(I) *if $A$ is $\leq_2$–increasing, then $l(A)$ is an $l$–stable set,*
(II) *if $A$ is $\leq_1$–increasing, then $r(A)$ is an $r$–stable set,*
(III) *$lr(A)$ is an $l$–stable set,*
(IV) *$rl(A)$ is an $r$–stable set,*
(V) *if $A$ and $B$ are $l$–stable sets, then $r(A) \cap r(B)$ is an r-stable set,*
(VI) *if $A$ is $l$–stable, then $r(A)$ is $r$–stable and if $A$ is $r$–stable, then $l(A)$ is $l$–stable,*
(VII) *if $A$ is $\leq_1$–increasing, then $A \subseteq lr(A)$,*
(VIII) *if $A$ is $\leq_2$–increasing, then $A \subseteq rl(A)$.* □

Given a doubly ordered set $(X, \leq_1, \leq_2)$, by $L(X)$ we denote the family of $l$–stable subsets of $X$. We define the following operations and constants on $L(X)$. For all $A, B \in L(X)$:
$A \wedge^c B = A \cap B$,
$A \vee^c B = l(r(A) \cap r(B))$,
$0^c = \emptyset$,
$1^c = X$.

**Lemma 6.** *For every doubly ordered set $(X, \leq_1, \leq_2)$ and for all $A, B \in L(X)$, $A \wedge^c B$, $A \vee^c B$, $X$, and $\emptyset$ are $l$–stable sets.* □

The system $(L(X), \vee^c, \wedge^c\ 1^c, 0^c)$ with the operations and constants defined above is referred to as the *complex algebra of a doubly ordered set* $(X, \leq_1, \leq_2)$.

**Lemma 7.** *For every doubly ordered set $(X, \leq_1, \leq_2)$, its complex algebra $(L(X), \vee^c, \wedge^c, 1^c, 0^c)$ is a lattice.* □

Complex algebras are not necessarily distributive.

Let $(W, \vee, \wedge, 1, 0)$ be a bounded lattice. By a *filter-ideal pair* of $W$ we mean a pair $(x_1, x_2)$ such that $x_1$ is a filter of $W$, $x_2$ is an ideal of $W$ and $x_1 \cap x_2 = \emptyset$. We define an ordering $\leq$ on the family of filter-ideal pairs of a lattice $W$:

$$(x_1, x_2) \leq (y_1, y_2) \quad \text{iff} \quad x_1 \subseteq y_1 \ \text{ and } \ x_2 \subseteq y_2\,.$$

A filter-ideal pair $(x_1, x_2)$ is said to be maximal whenever it is maximal with respect to the partial order $\leq$. Let $X(W)$ be the family of the maximal filter-ideal pairs of a lattice $(W, \vee, \wedge, 1, 0)$. We define relations $\subseteq_1$ and $\subseteq_2$ on $X(W)$ as follows. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of $X(W)$, then:

$$x \subseteq_1 y \quad \text{iff} \quad x_1 \subseteq y_1 \,,$$

$$x \subseteq_2 y \quad \text{iff} \quad x_2 \subseteq y_2 \,.$$

The system $(X(W), \subseteq_1, \subseteq_2)$ is referred to as a *canonical frame of a lattice* $(W, \vee, \wedge, 1, 0)$. Observe that $X(W)$ is a binary relation on $2^W$.

**Lemma 8.** *For every lattice* $(W, \vee, \wedge, 1, 0)$*, its canonical frame* $(X(W), \subseteq_1, \subseteq_2)$ *is a doubly ordered set.* □

The condition saying that for all $x, y \in X(W)$, if $x \subseteq_1 y$ and $x \subseteq_2 y$, then $x = y$, follows from the maximality of the pairs $x$ and $y$.

Observe that if a lattice $(W, \vee, \wedge, 1, 0)$ is distributive and $(x_1, x_2) \in X(W)$, then $x_1$ and $x_2$ are a prime filter and a prime ideal of $W$, respectively, $x_1 \cap x_2 = \emptyset$, and $x_1 \cup x_2 = W$. It follows that $\subseteq_2 = \subseteq_1^{-1}$.

Consider the complex algebra $L(X(W))$ of the doubly ordered set $(X(W), \subseteq_1, \subseteq_2)$ defined above. Clearly, $L(X(W))$ is a family of subrelations of $X(W)$. Define the mapping $h$: $W \to 2^{X(W)}$ as follows. For every $a \in W$:

$$h(a) = \{x = (x_1, x_2) \in X(W) : a \in x_1\} \,.$$

**Theorem 1.** [17]. *For every lattice* $(W, \vee, \wedge, 1, 0)$ *and for every* $a \in W$ *the following assertions hold:*

(I) $rh(a) = \{x = (x_1, x_2) \in X(W) : a \in x_2\}$,

(II) $h(a)$ *is an* $l$*-stable set,*

(III) $h$ *is a lattice embedding.*

□

As a corollary we obtain the following weak form of the Urquhart representation theorem.

**Theorem 2. (Representation theorem for lattices)**. *Every bounded lattice is isomorphic to a subalgebra of the complex algebra of its canonical frame.* □

# 3 Relational Representation of Lattice-Based Necessity Algebras

The (not necessarily distributive) lattice-based modal algebras and a generalised UAD framework for them have been introduced in [16]. The necessity algebras and their representation theorem have been presented there without proofs. In this section we describe all the technical details leading to those results.

By a *necessity algebra* we mean an algebra $N = (W, \vee, \wedge, 1, 0, \Box)$, where $(W, \vee, \wedge, 1, 0)$ is a bounded lattice and $\Box$ is a unary operation on $W$ satisfying for all $a, b \in W$:

(N1) $\Box(a \wedge b) = \Box a \wedge \Box b$ multiplicative
(N2) $\Box 1 = 1$ dual normal

Observe that any necessity operator is isotone.
For every $A \subseteq W$ we define:

$\Box A = \{a \in W : \Box a \in A\}$,
$<>^{-1} A = \{a \in W : \text{there is } b \in A \text{ such that } a \leq \Box b\}$,

where $\leq$ is the natural ordering of the lattice $W$.

By a filter (resp. ideal) of a necessity algebra $(W, \vee, \wedge, 1, 0, \Box)$ we mean a filter (resp. ideal) of the underlying lattice $(W, \vee, \wedge, 1, 0)$.

**Lemma 9.** *For every necessity algebra $N = (W, \vee, \wedge, 1, 0, \Box)$ and for all $A, B \subseteq W$ the following conditions are satisfied:*

(I) *if $A$ is a filter of $N$, then so is $\Box A$,*
(II) *if $A \subseteq B$, then $\Box A \subseteq \Box B$,*
(III) *if $A$ is an ideal of $N$, then so is $<>^{-1} A$,*
(IV) *$<>^{-1} A \subseteq B$ implies $A \subseteq \Box B$; if $B$ is an ideal of $N$, then $A \subseteq \Box B$ implies $<>^{-1} A \subseteq B$.*

$\Box$

A *necessity frame* is a relational system $(X, \leq_1, \leq_2, R_\Box, S_\Box)$ such that $(X, \leq_1, \leq_2)$ is a doubly ordered set, and $R_\Box$, $S_\Box$ are binary relations on $X$ satisfying the following conditions for all $x, y, x', y' \in X$:

($M\ R_\Box$) if $x' \leq_1 x, xR_\Box y$, and $y \leq_1 y'$, then $x' R_\Box y'$,

($M\ S_\Box$) if $x \leq_2 x', xS_\Box y$, and $y' \leq_2 y$, then $x' S_\Box y'$,

($S\ R_\Box S_\Box$) if $xR_\Box y$, then there is $x' \in X$ such that $x \leq_1 x'$ and $x' S_\Box y$,

($S\ S_\Box R_\Box$) if $xS_\Box y$, then there is $y' \in X$ such that $y \leq_2 y'$ and $xR_\Box y'$.

The conditions ($M\ R_\Box$) and ($M\ S_\Box$) are referred to as *necessity monotonicity conditions*, and conditions ($S\ R_\Box S\Box$) and ($S\ S_\Box R_\Box$) as *necessity stability conditions*.

We define unary operators $[R_\Box]$ and $<S_\Box>$ on $2^X$ as follows. For every $A \subseteq X$:

$[R_\Box]A = \{x \in X : \text{for every } y \in X \text{ if } xR_\Box y, \text{ then } y \in A\}$,
$<S_\Box>A = \{x \in X : \text{there is } y \in X \text{ such that } xS_\Box y \text{ and } y \in A\}$.

It is clear that these operators are the usual necessity and possibility operators determined by relations $R_\Box$ and $S_{<>}$, respectively.

**Lemma 10.** *For every necessity frame* $(X, \leq_1, \leq_2, R_\Box, S_\Box)$ *and for every* $A \subseteq X$ *the following conditions are satisfied:*

(I) $[R_\Box]A$ *is a* $\leq_1$*–increasing set,*

(II) $<S_\Box>A$ *is a* $\leq_2$*–increasing set.*

*Proof.* (I) Assume that (i) $x \in [R_\Box]A$ and (ii) $x \leq_1 y$. Suppose that there is $t \in X$ such that (iii) $yR_\Box t$, and $t \notin A$. From (ii), (iii) and (Mono $R_\Box$) we obtain (iv) $xRt$. Now (i) and (iv) yield $t \in A$, a contradiction which completes the proof of (I).

The proof of (II) is similar. $\Box$

A *complex algebra of a necessity frame* $(X, \leq_1, \leq_2, R_\Box, S_\Box)$ is an algebra $(L(X), \vee^c, \wedge^c, 1^c, 0^c, \Box^c)$, where $(L(X), \vee^c, \wedge^c, 1^c, 0^c)$ is the complex algebra of the doubly ordered set $(X, \leq_1, \leq_2)$, and $\Box^c$ is defined as:

$$\Box^c A = [R_\Box]A\,.$$

In the following lemma we present an equivalent formulation of the operation $\Box^c$ which uses the relation $S_\Box$.

**Lemma 11.** *For every necessity frame* $(X, \leq_1, \leq_2, R_\Box, S_\Box)$ *and for every* $l$*–stable set* $A \subseteq X$ *the following conditions are satisfied:*

(I) $\Box^c A = l{<}S_\Box{>}r(A)$,

(II) $\Box^c A$ *is* $l$*–stable.*

*Proof.* (I) ($\subseteq$) Suppose that for some $x \in X$ the conditions (i) $x \in \Box^c A$ and (ii) $x \notin l{<}S_\Box{>}r(A)$ are satisfied. From (ii) there is $y \in X$ such that (iii) $x \leq_1 y$ and (iv) $y \in {<}S_\Box{>}r(A)$. From (iv) there is $z \in X$ such that (v) $yS_\Box z$ and (vi) $z \in r(A)$. From (v) and $(S\,S_\Box R_\Box)$ there is $z' \in X$ such that (vii) $z \leq_2 z'$ and (viii) $yR_\Box z'$. (vi) and (vii) yield (ix) $z' \notin A$. From (iii), (viii), and $(M\,R_\Box)$ we get (x) $xR_\Box z'$. From (i) and (x) we conclude that $z' \in A$, which contradicts (ix).

($\supseteq$) Suppose that for some $x \in X$ the conditions (i) $x \in l{<}S_\Box{>}r(A)$ and (ii) $x \notin \Box^c A$ are satisfied. (ii) means that there is $y \in X$ such that (iii) $xR_\Box y$ and (iv) $y \notin A$. Since $A$ is an $l$–stable set, from (iv) we get (v) $y \notin lr(A)$. This means that there is $z \in X$ such that (vi) $y \leq_1 z$ and (vii) $z \in r(A)$. From (iii) and (vi) we have (viii) $xR_{\Box}z$. From (viii) and $(S\,R_{\Box}S_{\Box})$ there is $x' \in X$ such that (ix) $x \leq_1 x'$ and (x) $x'S_{\Box}z$. From (i) and (ix) we get $x' \notin <S_{\Box}>r(A)$. But from (vii) and (x) we obtain $x' \in <S_{\Box}>r(A)$, a contradiction.

(II) By Lemma 10(II) $<S_{\Box}>r(A)$ is a $\leq_2$–increasing set. From Lemma 5(I) we get $l<S_{\Box}>r(A)$ is $l$–stable. By (I), $\Box^c A$ is $l$–stable. □

**Theorem 3.** *For every necessity frame $(X, \leq_1, \leq_2, R_{\Box}, S_{\Box})$ the complex algebra $(L(X), \vee^c, \wedge^c, 1^c, 0^c, \Box^c)$ is a necessity algebra.*

*Proof.* In view of Lemma 7 we only have to show that $\Box^c$ satisfies the axioms (N1) and (N2). This follows easily from the facts that $[R_{\Box}]$ distributes over $\cap$ and preserves 1. □

Let $N = (W, \vee, \wedge, 1, 0, \Box)$ be a necessity algebra. By a filter-ideal pair of $N$ we mean a filter-ideal pair of the underlying lattice $(W, \vee, \wedge, 1, 0, )$. Let $X(N)$ be the family of all the maximal filter-ideal pairs of $N$ and let $(X(N), \subseteq_1, \subseteq_2)$ be a canonical frame of the lattice reduct of $N$ as defined in Section 2.

We define binary relations $R^c_{\Box}$ and $S^c_{\Box}$ on $X(N)$ as follows. Let $x = (x_1,\, x_2)$ and $y = (y_1,\, y_2)$ be elements of $X(N)$, then:

$$xR^c_{\Box}y \quad \text{iff} \quad \Box x_1 \subseteq y_1\,,$$

$$xS^c_{\Box}y \quad \text{iff} \quad y_2 \subseteq \Box x_2\,.$$

A *canonical frame of a necessity algebra* is a system $(X(N), \subseteq_1, \subseteq_2, R^c_{\Box}, S^c_{\Box})$.

**Lemma 12.** *For every necessity algebra $N = (W, \vee, \wedge, 1, 0, \Box)$, for every $x = (x_1,\, x_2) \in X(N)$, and for every $a \in W$ the following conditions are equivalent:*

(I) $\Box a \in x_1$,
(II) *for every $y = (y_1,\, y_2) \in X(N)$, if $xR^c_{\Box}y$ then $a \in y_1$.*

*Proof.* $(\rightarrow)$ Assume that $\Box a \in x_1$ and $xR^c_{\Box}y$. From the corresponding definitions we get $a \in \Box x_1$ and $\Box x_1 \subseteq y_1$. Hence, $a \in y_1$.

$(\leftarrow)$ Suppose that $\Box a \notin x_1$ which means (i) $a \notin \Box x_1$. Let $(a] = \{b \in W : b \leq a\}$ be the ideal of $N$ generated by $a$, where $\leq$ is the natural ordering of the lattice $W$. Since by Lemma 9(I) $\Box x_1$ is a filter, it follows from (i) that $\Box x_1 \cap (a] \neq \emptyset$, and hence $(\Box x_1, (a])$ is a filter-ideal pair. We extend it to a maximal filter-ideal pair, say $y = (y_1,\, y_2)$. So we have $\Box x_1 \subseteq y_1$ which means that (ii) $xR^c_{\Box}y$. We also have $(a] \subseteq y_2$, so $a \in y_2$. Since $y_1 \cap y_2 = \emptyset$, we have (iii) $a \notin y_1$. From (ii) and (iii) we conclude that (II) does not hold, which completes the proof. □

**Lemma 13.** *For every necessity algebra $N = (W, \vee, \wedge, 1, 0, \Box)$ the relations of its canonical frame satisfy the necessity monotonicity conditions.*

*Proof.* By way of example we prove the monotonicity condition for relation $R^c{}_\Box$. We have to show that for all $x, y, x', y' \in X(N)$, if $x' \leq_1 x, xR^c_\Box y$, and $y \leq_1 y'$, then $x'R^c_\Box y'$. Assume that (i) $x'_1 \subseteq x_1$, (ii) $\Box x_1 \subseteq y_1$, and (iii) $y_1 \subseteq y'_1$. From (i) and Lemma 9(II) we obtain (iv) $\Box x'_1 \subseteq \Box x_1$. From (ii) and (iv) we get (v) $\Box x'_1 \subseteq \Box y_1$. Using (iii) and (v) we conclude that $\Box x'_1 \subseteq y'_1$.

The proof of the monotonicity condition for $S^c_\Box$ is similar. $\Box$

**Lemma 14.** *For every necessity algebra $N = (W, \vee, \wedge, 1, 0, \Box)$ the relations in its canonical frame satisfy necessity stability conditions.*

*Proof.* We show $(S\, R_\Box S\Box)$. We have to prove that for all $x,\, y \in X(N)$, if $xR^c_\Box y$, then there is $x' \in X(N)$ such that $x \leq_1 x'$ and $x'S^c_\Box y$. Assume that $\Box x_1 \subseteq y_1$. We will show that $x_1 \cap \Diamond^{-1} y_2 = \emptyset$. For suppose conversely, then there is $a \in W$ such that (i) $a \in x_1$ and (ii) $a \in \Diamond^{-1} y_2$. By (ii) there is $b \in W$ such that (iii) $b \in y_2$ and (iv) $a \leq \Box b$. Since $x_1$ is a filter, from (i) and (iv) we obtain (v) $\Box b \in x_1$. Using (v) and the assumption we get that (vi) $b \in y_1$. (iii) and (vi) lead to $y_1 \cap y_2 \neq \emptyset$ – a contradiction. By Lemma 9(III) $\Diamond^{-1} y_2$ is an ideal, so $(x_1, \Diamond^{-1} y_2)$ is a filter-ideal pair. Let $x' = (x'_1,\, x'_2)$ be its extension to a maximal filter-ideal pair. It follows that $x_1 \subseteq x'_1$, so $x \leq_1 x'$. Similarly, $\Diamond^{-1} y_2 \subseteq x'_2$, so by Lemma 9(IV) we obtain $y_2 \subseteq \Box x'_2$, and hence $x'S^c_\Box y$.

Now we prove $(S\, S_\Box R_\Box)$, that is for all $x,\, y \in X(N)$, if $xS^c_\Box y$, then there is $y' \in X(N)$ such that $y \leq_2 y'$ and $xR^c_\Box y'$. Assume that $y_2 \subseteq \Box x_2$. We show that $\Box x_1 \cap y_2 = \emptyset$. For suppose conversely, then there is $a \in W$ such that (i) $a \in \Box x_1$ and (ii) $a \in y_2$. From (i) we get (iii) $\Box a \in x_1$. The assumption and (ii) yield (iv) $\Box a \in x_2$. From (iii) and (iv) we have $x_1 \cap x_2 \neq \emptyset$, a contradiction. Since by Lemma 9(I) $\Box x_1$ is a filter, $(\Box x_1,\, y_2)$ is a filter-ideal pair of $N$. Let $y' = (y'_1,\, y'_2)$ be its extension to a maximal filter-ideal pair. It follows that $\Box x_1 \subseteq y'_1$, and hence $xR^c_\Box y'$. Similarly, since $y_2 \subseteq y'_2$, we get $y \leq_2 y'$, which completes the proof. $\Box$

We conclude that the following is true.

**Theorem 4.** *For every necessity algebra, its canonical frame is a necessity frame.*

*Proof.* Follows from Lemma 8 and Lemma 14. $\Box$

Let $N = (W, \vee, \wedge, 1, 0, \Box)$ be a necessity algebra and let $h : W \to 2^{X(N)}$ be the mapping defined as in Section 2: $h(a) = \{x \in X(N) :\ a \in x_1\}$.

**Lemma 15.** *For every necessity algebra $N = (W, \vee, \wedge, 1, 0, \Box)$ and for every $a \in W$, $h(\Box a) = \Box^c h(a)$.*

*Proof.* Follows from the equivalence of the following statements: $x \in h(\Box a)$ iff $\Box a \in x_1$ iff for every $y \in X(N)$, $xR^c_\Box y$ implies $a \in y_1$ (by Lemma 12) iff for every $y \in X(N)$, $xR^c_\Box y$ implies $y \in h(a)$ iff $x\Box^c h(a)$. □

**Theorem 5. (Representation theorem for necessity algebras)**. *Every necessity algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

*Proof.* Follows from Theorem 2 and Lemma 15. □

## 4 Relational Representation of Negative Possibility Algebras

In [16] negative possibility algebras (or equivalently dual sufficiency algebras, according to the terminology in [3] and [6]) and a generalised UAD framework for them have been introduced without proofs. In the present section we describe all the technical details.

By a *negative possibility (dual sufficiency) algebra* we mean an algebra $P^\circ = (W, \vee, \wedge, 1, 0, \Diamond^\circ)$, where $(W, \vee, \wedge, 1, 0)$ is a bounded lattice and $\Diamond^\circ$ is a unary operation on $W$ satisfying for all $a, b \in W$:

$(P^\circ 1)\Diamond^\circ(a \wedge b) = \Diamond^\circ a \vee \Diamond^\circ b$ comultiplicative

$(P^\circ 2)\Diamond^\circ 1 = 0$ codual normal

Observe that any negative possibility operator is antitone.

For every $A \subseteq W$ we define:

$\Diamond^\circ A = \{a \in W : \Diamond^\circ a \in A\}$,

$\Box^{\circ -1} A = \{a \in W : \text{there is } b \in A \text{ such that } \Diamond^\circ b \leq a\}$,

where $\leq$ is the natural ordering of the lattice $W$.

As previously, by a filter (resp. ideal) of a negative possibility algebra $(W, \vee, \wedge, 1, 0, \Diamond^\circ)$ we mean a filter (resp. ideal) of the underlying lattice $(W, \vee, \wedge, 1, 0)$.

**Lemma 16.** *For every negative possibility algebra $P^\circ = (W, \vee, \wedge, 1, 0, \Diamond^\circ)$ and for all $A$, $B \in W$ the following conditions are satisfied:*

(I) *if $A$ is an ideal of $P^{\circ}$, then $\Diamond^{\circ}A$ is a filter of $P^{\circ}$,*

(II) *if $A \subseteq B$, then $\Diamond^{\circ}A \subseteq \Diamond^{\circ}B$,*

(III) *if $A$ is an ideal of $P^{\circ}$, then $\Box^{\circ-1}A$ is a filter of $P^{\circ}$,*

(IV) *$\Box^{\circ-1}A \subseteq B$ implies $A \subseteq \Diamond^{\circ}B$; if $B$ is a filter of $P^{\circ}$, then $A \subseteq \Diamond^{\circ}B$ implies $\Box^{\circ-1}A \subseteq B$.*

*Proof.* By straightforward verification.

A *negative possibility frame* is a system $(X, \leq_1, \leq_2, R_{\Diamond\circ}, S_{\Diamond\circ})$, where the relations $R_{\Diamond\circ}$ and $S_{\Diamond\circ}$ satisfy the following conditions:

($M\,R_{\Diamond\circ}$) if $x \leq_1 x'$, $xR_{\Diamond\circ}y$, and $y' \leq_2 y$, then $x'R_{\Diamond\circ}y'$,

($M\,S_{\Diamond\circ}$) if $x' \leq_2 x$, $x\,S_{\Diamond\circ}y$, and $y \leq_1 y'$, then $x'S_{\Diamond\circ}y'$,

($S\,R_{\Diamond\circ}S_{\Diamond\circ}$) if $xR_{\Diamond\circ}y$, then there is $y' \in X$ such that $y \leq_2 y'$ and $x\,S_{\Diamond\circ}y'$,

($S\,S_{\Diamond\circ}R_{\Diamond\circ}$) if $xS_{\Diamond\circ}y$, then there is $x' \in X$ such that $x \leq_2 x'$ and $x'R_{\Diamond\circ}y$.

In analogy to the respective conditions for the relations in necessity frames, the conditions ($M\,R_{\Diamond\circ}$) and ($M\,S_{\Diamond\circ}$) are referred to as *negative possibility monotonicity conditions*, and conditions ($S\,R_{\Diamond\circ}S_{\Diamond\circ}$) and ($S\,S_{\Diamond\circ}R_{\Diamond\circ}$) as *negative possibility stability conditions*.

We define unary operators $[S_{\Diamond\circ}]$ and $<R_{\Diamond\circ}>$ on $2^X$ as follows. For every $A \subseteq X$:

$$[S_{\Diamond\circ}]A = \{x \in X : \text{ for every } \ y \in X \ \text{ if } \ xS_{\Diamond\circ}y, \ \text{ then } \ y \in A\},$$
$$<R_{\Diamond\circ}>A = \{x \in X : \text{ there is } \ y \in X \ \text{ such that } \ xR_{\Diamond\circ}y \ \text{ and } \ y \in A\}. \quad \Box$$

**Lemma 17.** *For every negative possibility frame $(X, \leq_1, \leq_2, R_{\Diamond\circ}, S_{\Diamond\circ})$ and for every $A \subseteq X$, the following conditions are satisfied:*

(I) *$[S_{\Diamond\circ}]A$ is a $\leq_2$–increasing set,*

(II) *$<R_{\Diamond\circ}>A$ is a $\leq_1$–increasing set.*

*Proof.* (II) Assume that (i) $x \in <R_{\Diamond\circ}>A$, (ii) $x \leq_1 y$, and suppose that (iii) $y \notin <R_{\Diamond\circ}>A$. It follows from (i) that there is $t \in X$ such that (iv) $xR_{\Diamond\circ}t$ and (v) $t \in A$. From (ii), (iv) and (Mono $R_{\Diamond\circ}$) we get (v) $yR_{\Diamond\circ}t$. It follows from (iii) that for every $z \in X$, (vi) if $yR_{\Diamond\circ}z$, then $z \notin A$. From (vi) and (v) we obtain $t \notin A$, a contradiction.

The proof of (I) is similar and uses ($M\,S_{\Diamond\circ}$). $\Box$

A *complex algebra* of a negative possibility frame $(X, \leq_1, \leq_2, R_{\Diamond\circ}, S_{\Diamond\circ})$ is an algebra $(L(X), \vee^c, \wedge^c, 1^c, 0^c, \Diamond^{\circ c})$, where $(L(X), \vee^c, \wedge^c, 1^c, 0^c)$ is the complex algebra of the doubly ordered set $(X, \leq_1, \leq_2)$ and $\Diamond^{\circ c}$ is defined as:

$$\Diamond^{\circ c} A = l[S_{\Diamond\circ}](A)\,.$$

**Lemma 18.** *For every negative possibility frame* $(X, \leq_1, \leq_2, R_{\Diamond^\circ}, S_{\Diamond^\circ})$ *and for every l–stable set* $A \subseteq X$*, the following conditions are satisfied:*

(I) $[S_{\Diamond^\circ}]A = r<R_{\Diamond^\circ}>r(A)$,

(II) $\Diamond^{\circ c} A$ *is an l–stable set.*

*Proof.* (I) $(\subseteq)$ Assume that $x \in [S_{\Diamond^\circ}]lr(A)$. This means that (i) for every $y \in X$, if $x\, S_{\Diamond^\circ} y$, then for every $z$, if $y \leq_1 z$, then $z \notin r(A)$. Suppose that $x \in r<R_{\Diamond^\circ}>r(A)$. It follows that there is $u \in X$ such that (ii) $x \leq_2 u$ and there is $v \in X$ such that (iii) $uR_{\Diamond^\circ}v$ and (iv) $v \in r(A)$. From (iii) and $(S\, R_{\Diamond^\circ} S_{\Diamond^\circ})$ we get there is $v' \in X$ such that (v) $v \leq_2 v'$ and (vi) $uS_{\Diamond^\circ}v'$. From (ii), (vi) and $(M\, S_{\Diamond^\circ})$ we have (vii) $xS_{\Diamond^\circ}v'$. The assumption (i) and (vii) yield for every $z$, if $y \leq_1 z$, then $z \notin r(A)$. In particular, (viii) $v' \notin r(A)$. Since by Lemma 1(II) $r(A)$ is $\leq_2$–increasing, from (v), (iv), and (viii) we obtain a contradiction.

The proof of $(\supseteq)$ is similar.

(II) This follows from Lemma 17(I) and Lemma 5(I). □

**Theorem 6.** *For every negative possibility frame* $(X, \leq_1, \leq_2, R_{\Diamond^\circ}, S_{\Diamond^\circ})$ *the complex algebra* $(L(X), \vee^c, \wedge^c, 1^c, 0^c, \Diamond^{\circ c})$ *is a negative possibility algebra.*

*Proof.* The proof is by showing that $\Diamond^{\circ c}$ satisfies the axioms $(P^\circ 1)$ and $(P^\circ 2)$. Let $A, B \in L(X)$. We have $\Diamond^{\circ c}(A \wedge B) = l[S_{\Diamond^\circ}](A \cap B) = l([S_{\Diamond^\circ}]A \cap [S_{\Diamond^\circ}]B) = l(r<R_{\Diamond^\circ}>r(A) \cap r<R_{\Diamond^\circ}>r(B))$ (by Lemma 18(I)) $= l(rlr<R_{\Diamond^\circ}>r(A) \cap rlr<R_{\Diamond^\circ}>r(B))$ (by Lemma 17(II) and Lemma 5(II)) $= lr<R_{\Diamond^\circ}>r(A) \vee^c lr<R_{\Diamond^\circ}>r(B)$ (by definition of $\vee^c$) $= l[S_{\Diamond^\circ}]A \vee^c l[S_{\Diamond^\circ}]B$ (by Lemma 18(I)) $= \Diamond^\circ A \vee^c \Diamond^\circ B$.

We also have $\Diamond^\circ 1 = l[S_{\Diamond^\circ}]X = lX = \emptyset = 0$. For the lattice axioms we rely on Lemma 7. □

Let $P^\circ = (W, \vee, \wedge, 1, 0, \Diamond^\circ)$ be a negative possibility algebra. By a filter-ideal pair of $P^\circ$ we mean a filter-ideal pair of the underlying lattice $(W, \vee, \wedge, 1, 0)$. Let $X(P^\circ)$ be the family of all the maximal filter-ideal pairs of $P^\circ$ and let $(X(P^\circ), \subseteq_1, \subseteq_2)$ be a canonical frame of the lattice reduct of $P^\circ$ as defined in Section 2.

We define binary relations $R^c_{\Diamond^\circ}$ and $S^c_{\Diamond^\circ}$ on $X(P^\circ)$ as follows. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of $X(P^\circ)$, then:

$xR^c_{\Diamond^\circ}y$ iff $y_2 \subseteq \Diamond^\circ x_1$,

$xS^c_{\Diamond^\circ}y$ iff $\Diamond^\circ x_2 \subseteq y_1$.

By the *canonical frame of the negative possibility algebra* $P^\circ$ we mean a system $(X(P^\circ), \subseteq_1, \subseteq_2, R^c_{\Diamond^\circ}, S^c_{\Diamond^\circ})$.

**Lemma 19.** *For every negative possibility algebra* $P^\circ = (W, \vee, \wedge, 1, 0, \Diamond^\circ)$*, for every* $x \in X(P^\circ)$*, and for every* $a \in W$ *the following conditions are equivalent:*

(I) $\Diamond^\circ a \in x_2$,
(II) *for every* $y \in X(P^\circ)$*, if* $x S^c_{\Diamond^\circ} y$*, then* $a \in y_1$.

*Proof.* ($\rightarrow$) Assume that $\Diamond^\circ a \in x_2$. Hence (i) $a \in \Diamond^\circ x_2$. Suppose that there is $y \in X(P^\circ)$ such that (ii) $\Diamond^\circ x_2 \subseteq y_1$ and (iii) $a \notin y_1$. From (i) and (ii) we get $a \in y_1$, which contradicts (iii).

($\leftarrow$) Now assume that $\Diamond^\circ a \in x_2$ which means (i) $a \notin \Diamond^\circ x_2$. Let $(a] = \{b \in W : b \leq a\}$ be the ideal of $P^\circ$ generated by $a$, where $\leq$ is the natural ordering of the lattice $W$. By Lemma 16(I) $\Diamond^\circ x_2$ is an ideal. It follows from (i) that $\Diamond^\circ x_2 \cap (a] = \emptyset$. Hence $(\Diamond^\circ x_2, (a])$ is a filter-ideal pair. We extend it to a maximal filter-ideal pair, say $y = (y_1, y_2)$. So we have (ii) $(a] \subseteq y_2$ and (iii) $\Diamond^\circ x_2 \subseteq y_1$. From (ii) we have (iv) $a \in y_2$. Since $y_1 \cap y_2 = \emptyset$, from (iv) we obtain $a \notin y_1$. Moreover, (iii) means that $x S^c_{\Diamond^\circ} y$. It follows that (II) does not hold, which completes the proof. $\square$

**Lemma 20.** *For every negative possibility algebra* $P^\circ = (W, \vee, \wedge, 1, 0, \Diamond^\circ)$ *the relations of its canonical frame satisfy the negative possibility monotonicity conditions.*

*Proof.* By way of example we prove the monotonicity condition for relation $R^c_{\Diamond^\circ}$. We have to show that for all $x$, $y$, $x'$, $y' \in X(P^\circ)$, if $x \leq_1 x'$, $x R^c_{\Diamond^\circ} y$, and $y' \leq_2 y$, then $x' R^c_{\Diamond^\circ} y'$. Assume that (i) $x_1 \subseteq x'_1$, (ii) $y_2 \subseteq \Diamond^\circ x_1$, and (iii) $y'_2 \subset y_2$. From (i) and Lemma 16(II) we obtain (iv) $\Diamond^\circ x_1 \subseteq \Diamond^\circ x'_1$. From (iv), (ii), and (iii) we get $y'_2 \subseteq \Diamond^\circ x'_1$.

The proof of the monotonicity condition for $S^c_{\Diamond^\circ}$ is similar. $\square$

**Lemma 21.** *For every negative possibility algebra* $P^\circ = (W, \vee, \wedge, 1, 0, \Diamond^\circ)$ *the relations of its canonical frame satisfy the negative possibility stability conditions.*

*Proof.* We show $(S\, R_{\Diamond^\circ} S_{\Diamond^\circ})$. We have to prove that for all $x$, $y \in X(P^\circ)$ if $x R^c_{\Diamond^\circ} y$, then there is $y' \in X(P^\circ)$ such that $y \leq_2 y'$ and $x S^c_{\Diamond^\circ} y'$. Assume that $y_2 \subseteq \Diamond^\circ x_1$. We will show that $\Diamond^\circ x_2 \cap y_2 = \emptyset$. For suppose conversely, then there is $a \in W$ such that (i) $\Diamond^\circ a \in x_2$ and (ii) $a \in y_2$. By the assumption and (ii) we have $a \in \Diamond^\circ x_1$, so $\Diamond^\circ a \in x_1$, a contradiction since $x_1 \cap x_2 = \emptyset$. By Lemma 16(I) $\Diamond^\circ x_2$ is a filter of $P^\circ$. So $(\Diamond^\circ x_2, y_2)$ is a filter-ideal pair. Let $y' = (y'_1, y'_2)$ be its extension to a maximal pair. It is easy to see that $y'$ satisfies the required conditions.

Now we prove $(S\, S_{\Diamond^\circ} R_{\Diamond^\circ})$, that is for all $x$, $y \in X(P^\circ)$, if $x S^c_{\Diamond^\circ} y$, then there is $x' \in X(P^\circ)$ such that $x \leq_2 x'$ and $x' R^c_{\Diamond^\circ} y$. Assume that $\Diamond^\circ x_2 \subseteq y_1$.

We show that $\Box^{\circ -1} y_2 \cap x_2 = \emptyset$. For suppose conversely, then there is $a \in W$ such that (i) $a \in \Box^{\circ -1} y_2$ and (ii) $a \in x_2$. From (i) we get there is $b \in W$ such that (iii) $b \in y_2$ and (iv) $\Diamond^\circ\, b \leq a$, where $\leq$ is the lattice ordering of $W$. Since $x_2$ is an ideal, (ii) and (iv) yield $\Diamond^\circ\, b \in x_2$. Therefore by the assumption we obtain $b \in y_1$, which is in contradiction with (iii). It follows that $(\Box^{\circ -1} y_2, x_2)$ is a filter-ideal pair. Let $x' = (x'_1, x'_2)$ be its extension to a maximal pair. So we have $\Box^{\circ -1} y_2 \subseteq x'_1$, which by Lemma 16(IV) yields $y_2 \subseteq \Diamond^\circ\, x'_1$, and $x_2 \subseteq x'_2$ as required. $\Box$

We conclude with the following theorem.

**Theorem 7.** *For every negative possibility algebra, its canonical frame is a negative possibility frame.*

*Proof.* Follows from Lemma 8, Lemma 20, and Lemma 21. $\Box$

Let $P^\circ = (W, \vee, \wedge, 1, 0, \Diamond^\circ)$ be a negative possibility algebra and let $h$ be the mapping defined as in Section 3: $h(a) = \{x \in X(P^\circ) : a \in x_1\}$.

**Lemma 22.** *For every negative possibility algebra $P^\circ = (W, \vee, \wedge, 1, 0, \Diamond^\circ)$ and for every $a \in W$, $h(\Diamond^\circ\, a) = \Diamond^{\circ c}\, h(a)$.*

*Proof.* Observe that $x \in \Diamond^{\circ c}\, h(a)$ iff $x \in l[S^c_{\Diamond^\circ}]h(a)$ iff for every $y \in X(P^\circ)$, if $x \leq_1 y$, then $y \notin [S^c_{\Diamond^\circ}]h(a)$ iff for every $y \in X(P^\circ)$, if $x \leq_1 y$, then there is $z \in X(P^\circ)$ such that $y S^c_{\Diamond^\circ} z$ and $a \notin z_1$. We will show that $x \in h(\Diamond^\circ\, a)$ iff for every $y \in X(P^\circ)$, if $x \leq_1 y$, then there is $z \in X(P^\circ)$ such that $y S^c_{\Diamond^\circ} z$ and $a \notin z_1$.

($\rightarrow$) Assume that $\Diamond^\circ\, a \in x_1$ and $x_1 \subseteq y_1$. It follows that $\Diamond^\circ\, a \in y_1$, so $\Diamond^\circ\, a \notin y_2$. By Lemma 19 the required condition follows.

($\leftarrow$) Assume that $\Diamond^\circ\, a \notin x_1$. We show that there is $y \in X(P^\circ)$ such that $x \leq_1 y$ and for every $z \in X(P^\circ)$, if $y S^c_{\Diamond^\circ} z$, then $a \in z_1$. Let $y = (y_1, y_2)$ be an extension of a filter-ideal pair $(x_1, (\Diamond^\circ\, a])$ to a maximal pair. Then, we have $x_1 \subseteq y_1$ and $(\Diamond^\circ\, a] \subseteq y_2$. It follows that $x \leq_1 y$ and $\Diamond^\circ\, a \in y_2$. From Lemma 19 we get the required condition. $\Box$

We conclude that the following representation theorem holds for negative possibility algebras.

**Theorem 8. (Representation theorem for negative possibility algebras).** *Every negative possibility algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

*Proof.* Follows from Theorems 1 and 6, and Lemma 22.

One can observe that there is some similarity between the definitions of the notions involved in the developments of Sections 3 and 4 and, as a consequence, a similarity between the constructions used in the proofs of the theorems. However, an abstract principle of this similarity which would enable us to obtain the definitions and theorems for one class of algebras once the respective notions and theorems are known for the other class is by no means obvious. It is an interesting open question which should be solved for all the four classes of operators: possibility, necessity, negative possibility, and negative necessity.

## 5 Relational Representation of Lattice-Based Relation Algebras

In this section we present a representation theorem for the class LCP of relation algebras introduced in [7]. The abbreviation LCP is intended to indicate that the algebras are lattices with the operations analogous to the converse and the product of binary relations in the classical relation algebras. However, the representation theorem presented here differs from the one given in that paper. Namely, in the frames corresponding to our algebras we assume a binary relation $C$ which provides a means of defining the converse operator in a complex algebra, and it is not necessarily a function, as it is the case in [7]. This treatment of converse has an advantage of providing a Kripke style semantics for an underlying logic without any operations on states (possible words). The converse operation is interpreted in terms of a relation in the usual Kripke style. This representation of converse is based on Allwein and MacCaull [2]. However, our presentation and proofs are in most cases different. The presentation follows the general UAD framework described in Section 1.

An *LCP algebra* is an algebra $(W, \vee, \wedge, 1, 0, \otimes, 1', ^{\smile})$ such that $(W, \vee, \wedge, 1, 0)$ is a bounded lattice, $(W, \otimes, 1', ^{\smile})$ is an involutive monoid, $\otimes$ and $^{\smile}$ distribute over join $\vee$. It follows that the following conditions are satisfied:

(LCP1) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$,

(LCP2) $a \otimes 1' = a = 1' \otimes a$,

(LCP3) $a^{\smile\smile} = a$,

(LCP4) $(a \otimes b)^{\smile} = b^{\smile} \otimes a^{\smile}$,

(LCP5) $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$,

(LCP6) $(a \vee b) \otimes c = (a \otimes c) \vee (b \otimes c)$,

(LCP7) $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$.

**Lemma 23.** *For all $a$, $b \in W$ the following assertions hold:*

(I) *if $a \leq b$, then $a^{\smile} \leq b^{\smile}$,*
(II) $(a \wedge b)^{\smile} = a^{\smile} \wedge b^{\smile}$,
(III) $0^{\smile} = 0$, $1^{\smile} = 1$, $1'^{\smile} = 1'$.

For any $A$, $B \subseteq W$ we define:

$A^{\smile} = \{a^{\smile} : a \in A\}$,
$A \otimes B = \{a \otimes b : a \in A \text{ and } b \in B\}$. □

**Lemma 24.** *Let $(W, \vee, \wedge, 1, 0, \otimes, 1', ^{\smile})$ be an LCP algebra. Then for all $A, B \subseteq W$ the following assertions hold:*

(I) $A^{\smile} = \{a \in W : a^{\smile} \in A\}$,
(II) $A^{\smile\smile} = A$,
(III) $A \subseteq B$ iff $A^{\smile} \subseteq B^{\smile}$,
(IV) $(A \cup B)^{\smile} = A^{\smile} \cup B^{\smile}$,
(V) $(A \cap B)^{\smile} = A^{\smile} \cap B^{\smile}$.

□

**Lemma 25.** *Let $(W, \vee, \wedge, 1, 0, \otimes, 1', ^{\smile})$ be an LCP algebra and let $F$ and $I$ be a filter and an ideal of $W$, respectively. Then:*

(I) *$F^{\smile}$ is a filter of $W$,*
(II) *$I^{\smile}$ is an ideal of $W$,*
(III) *$\{a \in W : (F \otimes \{a\}) \cap I \neq \emptyset\}$ is an ideal of $W$,*
(IV) *$\{a \in W : ((a) \otimes F) \cap I \neq \emptyset\}$ is an ideal of $W$.*

*Proof.* (I) and (II) can be proved by an easy verification. The proofs of (III) and (IV) can be found in [7].

We recall that $[a)$ is a filter of $W$ generated by $a$, that is $[a) = \{b \in W : a \leq b\}$. Similarly, $(a] = \{b \in W : b \leq a\}$ is an ideal of $W$ generated by $a$. □

**Lemma 26.** *Let $(W, \vee, \wedge, 1, 0, \otimes, 1', ^{\smile})$ be an LCP algebra and let $a \in W$. Then:*

(I) $[a)^{\smile} = [a^{\smile})$,
(II) $(a]^{\smile} = (a^{\smile}]$.

*Proof.* (I) We have $b \in [a)^{\smile}$ iff $b^{\smile} \in [a)$ iff $a \leq b^{\smile}$ iff $a^{\smile} \leq b^{\smile\smile} = b$ iff $b \in [a^{\smile})$.

The proof of (II) is similar. □

Now our aim is to define a class of frames associated with LCP algebras so that the theorems needed in Steps 3, 4, and 5 mentioned in the introduction will hold. Intuitively, the relations in the frames are intended to be relational counterparts to the operations of the respective algebras. If an operation is $n$-ary, $n \geq 0$, then there are $n$ corresponding $n+1$-ary relations each of which should describe how the result of applying the operation depends on one of the arguments of the operation. Consequently, an LCP frame should contain one unary relation (I) corresponding to constant $1'$ (0-ary operation), two binary relations ($C$ and $C'$) corresponding to the unary operation $\smile$, and three ternary relations $(R, S, Q)$ associated with ;. A precise formal principles of the relationship between an operation and its corresponding relations are still to be discovered.

An *LCP frame* is a relational system $(X, \leq_1, \leq_2, R, S, Q, I, C, C')$ such that $(X, \leq_1, \leq_2)$ is a doubly ordered set, $R$, $S$, and $Q$ are ternary relations on $X$, $I$ is a subset (unary relation) of $X$, $C$ and $C'$ are binary relations on $X$, and for all $x$, $x'$, $y$, $y'$, $z$, $z' \in X$ the following conditions are satisfied:

(M1) $R(x,y,z)$ & $x' \leq_1 x$ & $y' \leq_1 y$ & $z \leq_1 z'$ imply $R(x',y',z')$,
(M2) $S(x,y,z)$ & $x \leq_2 x'$ & $y' \leq_1 y$ & $z' \leq_2 z$ imply $S(x',y',z')$,
(M3) $Q(x,y,z)$ & $x' \leq_1 x$ & $y \leq_2 y'$ & $z' \leq_2 z$ imply $Q(x',y',z')$,
(M4) $I(x)$ & $x \leq_1 x'$ imply $I(x')$,
(M5) $C(x,y)$ & $x \leq_2 x'$ & $y' \leq_2 y$ imply $C(x',y')$,
(M6) $C'(x,y)$ & $x' \leq_1 x$ & $y \leq_1 y'$ imply $C'(x',y')$.

(S1) $R(x,y,z)$ implies $\exists x' \in X\ (x \leq_1 x'$ & $S(x',y,z))$,
(S2) $R(x,y,z)$ implies $\exists y' \in X(y \leq_1 y'$ & $Q(x,y',z))$,
(S3) $S(x,y,z)$ implies $\exists z' \in X(z \leq_2 z'$ & $R(x,y,z'))$,
(S4) $Q(x,y,z)$ implies $\exists z' \in X(z \leq_2 z'$ & $R(x,y,z'))$,
(S5) $\exists u \in X\ (R(x,y,u)$ & $Q(x',u,z))$ implies $\exists w \in X\ (R(x',x,w)$ & $S(w,y,z))$,
(S6) $\exists u \in X\ (R(x,y,u)$ & $S(u,z,z'))$ implies $\exists w \in X\ (R(y,z,w)$ & $Q(x,w,z'))$,
(S7) $I(x)$ & $(R(x,y,z)$ or $R(y,x,z))$ imply $y \leq_1 z$,
(S8) $\exists u \in X\ (I(u)$ & $S(u,x,x))$,
(S9) $\exists u \in X(I(u)$ & $Q(x,u,x))$,
(S10) $C(x,y)$ implies $\exists y' \in X\ (y \leq_2 y'$ & $C'(x,y'))$,
(S11) $C'(x,y)$ implies $\exists x' \in X\ (x \leq_1 x'$ & $C(x',y))$,
(S12) $\exists w \in X\ (C(w,x)$ & $C(x,w))$,
(S13) $C(w,x)$ & $C(y,w)$ imply $x \leq_2 y$,
(S14) $S(u,v,z')$ & $C(z',z)$ imply $\exists x,y \in X\ (Q(x,y,z)$ & $C'(v,x)$ & $C(u,y))$,

(S15) $Q(x,y,z)$ & $C'(v,x)$ & $C(u,y)$ imply $\exists z' \in X\,(S(u,v,z')$ & $C(z',z))$.

The conditions (M1), ..., (M6) and (S1), ..., (S15) are referred to as the *LCP monotonicity conditions and LCP stability conditions*, respectively.

In terms of the relations of an LCP frame we define the following operations:

$$A \otimes_Q B = \{z \in X : \forall x,y \in X\,(Q(x,y,z) \ \ \& \ \ x \in A \ \text{ imply } \ y \in r(B))\},$$
$$A \otimes_S B = \{z \in X : \forall x,y \in X\,(S(x,y,z) \ \ \& \ \ y \in B \ \text{ imply } \ x \in r(A))\},$$
$$f_C(A) = \{z \in X : \forall x \in X\,(C(x,z) \ \text{ implies } \ x \in A)\}.$$

Observe that the operator $f_C$ can be expressed in modal terms as $f_C(A) = [C^{-1}]A$, where $[C^{-1}]$ is the classical necessity operator determined by converse of relation $C$. From this observation several properties of $f_C$ can be easily deduced due to the well known properties of the necessity operator listed in Lemma 3.

**Lemma 27.** *For every LCP frame $(X, \leq_1, \leq_2, R, S, Q, I, C, C')$ and for all $A, B \subseteq X$ the following assertions hold:*

(I) *$A \otimes_Q B$, $A \otimes_S B$, and $f_C(A)$ are $\leq_2$–increasing,*

(II) *if $A$ and $B$ are $l$–stable, then $A \otimes_Q B$, $A \otimes_S B$, and $f_C(r(A))$ are $r$–stable.*

*Proof.* We prove the conditions (I) and (II) for $f_C$; for $\otimes_Q$ and $\otimes_S$ the proofs can be found in [7].

(I) Let (i) $x \leq_2 y$ and (ii) $x \in f_C(A)$. Take $y'$ such that (iii) $C(y'y)$. From (M5), (iii), and (i) we get (iv) $C(y',x)$. Then (ii) and (iv) yield $y \in f_C(A)$.

(II) We show that $rlf_Cr(A) = f_Cr(A)$.

($\subseteq$) Let $y \notin f_Cr(A)$. Hence, we have (i) $C(x,y)$ and there is $z$ such that (ii) $x \leq_2 z$ and (iii) $z \in A$. From (i), (ii) and (M5) we get (iv) $C(z,y)$. (iv) and (S10) yield that there is $y'$ such that (v) $y \leq_2 y'$ and (vi) $C'(z,y')$. We show that $y \notin rlf_Cr(A)$. Take $y''$ such that (vii) $y' \leq_1 y''$. Then we have (viii) $C'(z,y'')$ because of (M6) and (vi). From (viii) and (S11) there is $z'$ such that (ix) $z \leq_1 z'$ and (x) $C(z',y'')$. Since $A$ is $l$–stable, from (iii) and (ix) we get (xi) $z' \in A$. It follows that (xii) $z' \notin r(A)$. So (x) and (xii) enable us to conclude that (xiii) $y'' \notin f_Cr(A)$. From (vii) and (xiii) we have $y' \in lf_Cr(A)$, and hence $y \notin rlf_Cr(A)$.

($\supseteq$) This part follows immediately from (I) and Lemma 5(VIII). $\square$

It is known that if $A$ and $B$ are $l$–stable sets, then $A \otimes_Q B = A \otimes_S B$. The proofs can be found in [1] or in [7].

*A complex algebra of an LCP frame* $(X, \leq_1, \leq_2, R, S, Q, I, C, C')$ is an algebra $(L(X), \vee^c, \wedge^c, 1^c, 0^c, \otimes^c, 1'^c, {}^{\smile c})$ such that $(X(W), \vee^c, \wedge^c, 1^c, 0^c)$ is a complex algebra of the doubly ordered set $(X, \leq_1, \leq_2)$ and the operations $\otimes^c$, ${}^{\smile c}$ and constant $1'^c$ are defined as:

$A \otimes^c B = l(A \otimes_S B)$,
$A^{\smile c} = l f_C r(A)$,
$1'^c = lr(I)$.

**Lemma 28.** *If $A$ and $B$ are $l$–stable, then so are $A \otimes^c B$, $A^{\smile c}$, and $1'^c$.*

*Proof.* The required property of $A^{\smile c}$ follows from Lemma 27(II) and Lemma 5(VI). The proofs for $\otimes^c$ and $1'^c$ can be found in [7]. □

Now we verify that complex algebras of the LCP frames are LCP algebras. The proofs that $\otimes^c$ and $1'^c$ satisfy the axioms (LCP1), (LCP2), (LCP5), and (LCP6) are presented in [7]. Here we prove the axioms that involve the operation ${}^{\smile c}$.

**Lemma 29.** *Let $(L(X), \vee^c, \wedge^c, 1^c, 0^c, \otimes^c, 1'^c, {}^{\smile c})$ be a complex algebra of an LCP frame. Then for all $l$–stable sets $A$ and $B$ the following conditions are satisfied:*

(I) $A^{\smile c \smile c} = A$,
(II) $(A \otimes^c B)^{\smile c} = B^{\smile c} \otimes^c A^{\smile c}$,
(III) $(A \vee^c B)^{\smile c} = A^{\smile c} \vee^c B^{\smile c}$.

*Proof.* In some parts of this proof we will make use of the modal representation of the operator $f_C$.

(I) ($\subseteq$) First, we show that $r(A) \subseteq f_C f_C r(A)$. So assume that (i) for every $y$, if $x \leq_2 y$, then $y \notin A$ and suppose that $x \notin f_C f_C r(A)$. It follows that there are $z$, $t$, and $u$ such that (ii) $C(z, x)$, (iii) $C(t, z)$, (iv) $t \leq_2 u$, and (v) $u \in A$. Applying (S13) to (ii) and (iii) we have (vi) $x \leq_2 t$. Since $\leq_2$ is transitive, from (vi) and (iv) we get $x \leq_2 u$. Then from (i) we obtain $u \notin A$, a contradiction with (v).

($\supseteq$) Stability condition (S12) amounts to saying that $Id \subseteq C^{-1}; C^{-1}$, where $Id$ is the identity relation on $X$. From the properties of the classical necessity operator (Lemma 3(II),(III)) we deduce (i) $[C^{-1}][C^{-1}]r(A) \subseteq r(A)$. Hence,

| | |
|---|---|
| $A = lr(A) \subseteq l[C^{-1}][C^{-1}]r(A)$ | $A$ is $l$–stable, (i), and Lemma 1(IV) |
| $= l f_C r l f_C r(A)$ | definition of $f_C$, $f_C r(A)$ is $r$–stable (Lemma 27(II)) |
| $= A^{\smile c \smile c}$ | definition of ${}^{\smile c}$. |

(II) ($\subseteq$) First, we will show that if $z \notin f_C(A \otimes_S B)$, then $z \notin B^{\smile c} \otimes_Q A^{\smile c}$. The assumption says that there is $z'$ such that (i) $C(z'z)$ and (ii) $z' \notin (A \otimes_S B)$. (ii) amounts to saying that there are $u$ and $v$ such that (iii) $S(u, v, z')$, (iv) $v \in B$, and (v) $u \notin r(A)$. From (i), (iii) and (S14) there are $x$ and $y$ such that (vi) $Q(x, y, z)$, (vii) $C'(v, x)$, and (viii) $C(u, y)$.

Now we show that $x \in B^{\smile c} = lf_C r(B)$. Take $x'$ such that (ix) $x \leq_1 x'$. We have to show that $x' \notin f_C r(B)$. For suppose conversely, then (x) for every $p$, $C(p, x')$ implies $p \in r(B)$. From (ix) and (vii) we get by (M6) that (xi) $C'(v, x')$ holds. (S11) applied to (xi) gives that there is $v'$ such that (xii) $v \leq_1 v'$ and (xiii) $C(v', x')$. From (x), taking $v'$ for $p$ we obtain (xiv) $v' \in r(B)$. From (iv) and (xii) we get (xv) $v' \in B$, since $B$ is $l$–stable, and hence $\leq_1$–increasing. But (xiv) and (xv) are in contradiction.

Now we show that $y \notin r(A^{\smile c})$. From (v) and (viii) we get by Lemma 27(II) that $y \notin f_C r(A) = rlf_C r(A) = r(A^{\smile c})$.

So we have $Q(x, y, z)$ from (vi), $x \in B^{\smile c}$, and $y \notin r(A^{\smile c})$, which means $z \notin B^{\smile c} \otimes_Q A^{\smile c}$. It follows that (xvi) $B^{\smile c} \otimes_Q A^{\smile c} \subseteq f_C(A \otimes_S B)$.

In conclusion we have

$$
\begin{array}{ll}
(A \otimes^c B)^{\smile c} = & \\
\quad = lf_C r(A \otimes^c B) & \text{definition of } {}^{\smile c} \\
\quad = lf_C rl(A \otimes_S B) & \text{definition of } \otimes^c \\
\quad = lf_C(A \otimes_S B) & A \otimes_S B \text{ is } r\text{–stable (Lemma 27(II))} \\
\quad \subseteq l(B^{\smile c} \otimes_Q A^{\smile c}) & \text{(xvi) and Lemma 1(IV)} \\
\quad = B^{\smile c} \otimes^c A^{\smile c} & \text{definition of } \otimes^c,\ A^{\smile c} \text{ and } B^{\smile c} \text{ are } l\text{–stable} \\
 & \text{so } \otimes_S = \otimes_Q.
\end{array}
$$

($\supseteq$) First, we show that if $z \notin B^{\smile c} \otimes_Q A^{\smile c}$, then $z \notin f_C(A \otimes_S B)$. The assumption means that there are $x$ and $y$ such that (i) $S(x, y, z)$, (ii) $x \notin r(B^{\smile c})$ and (iii) $y \in A^{\smile c}$. From (ii) we obtain $x \notin rlf_C r(B)$ which means (iv) $x \notin f_C r(B)$ because $f_C r(B)$ is $r$–stable. It follows that there is $w$ such that (v) $C(w, x)$ and there is $v$ such that (vi) $w \leq_2 v$ and (vii) $v \in B$. (M5), (v), and (vi) yield (viii) $C(v, x)$. Applying (S10) to (viii) we get an $x'$ such that (ix) $x \leq_2 x'$ and (x) $C'(v, x')$. From (i) and (ix) we get (xi) $S(x', y, z)$ by applying (M2). Applying (S3) to (xi) we get a $z'$ such that (xii) $z \leq_2 z'$ and (xiii) $R(x', y, z')$. From (S2) applied to (xiii) we obtain a $y'$ such that (xiv) $y \leq_1 y'$ and (xv) $Q(x', y', z')$. From (iii), (xiv), and since $A^{\smile c}$ is $l$–stable, we get $y' \in A^{\smile c}$. It follows that $y' \notin r(A^{\smile c}) = rlf_C r(A) = f_C r(A)$. This means that there is $u$ such that (xvi) $C(u, y')$ and (xvii) $u \notin r(A)$. From (xv), (x), and (xvi) we obtain by application of (S15) that there is $t$ such that (xviii) $S(u, v, t)$ and (xix) $C(t, z)$. (xviii), (xvii), and (vii) yield (xx) $t \notin A \otimes_S B$. Now (xix) and (xx) lead to $z \notin f_C(A \otimes_S B)$. We conclude that (xxi) $f_C(A \otimes_S B) \subseteq B^{\smile c} \otimes_Q A^{\smile c}$. Now we have:

$$\begin{array}{ll}
B^{\smile c} \otimes^c A^{\smile c} = & \\
= l(B^{\smile c} \otimes_Q A^{\smile c}) & \text{definition of } \otimes^c \\
\subseteq l f_C(A \otimes_S B) & \text{(xxi) and Lemma 5(V)} \\
= l f_C r l(A \otimes_S B) & A \otimes_S B \text{ is } r\text{–stable} \\
= l f_C r(A \otimes^c B) & \text{definition of } \otimes^c \\
= (A \otimes^c B)^{\smile c} & \text{definition of } {}^{\smile c}.
\end{array}$$

(III) $(A \vee^c B)^{\smile c} =$

$$\begin{array}{ll}
= l f_C r l(r(A) \cap r(B)) & \text{definitions of } \vee^c \text{ and } {}^{\smile c} \\
= l f_C(r(A) \cap r(B)) & \text{Lemma 5(V)} \\
= l(f_C r(A) \cap f_C r(B)) & f_C \text{ distributes over } \cap \text{ (Lemma 3(IV))} \\
= l(r l f_C r(A) \cap r l f_C r(B)) & f_C r(A) \text{ is } r\text{–stable (Lemma 27(II))} \\
= l f_C r(A) \vee^c l f_C r(B) & \text{definition of } \vee^c \\
= A^{\smile c} \vee^c B^{\smile c} & \text{definition of } {}^{\smile c}.
\end{array}$$

□

We conclude that the following theorem holds.

**Theorem 9.** *The complex algebra of an LCP frame is an LCP algebra.*

*Proof.* The theorem follows from Lemma 7 and Lemma 29. □

*A canonical frame of an LCP algebra* $(W, \vee, \wedge, 1, 0, \otimes, 1', {}^{\smile})$ is a structure $(X(W), \subseteq_1, \subseteq_2, R^c, S^c, Q^c, I^c, C^c, C'^c)$ such that $(X(W), \subseteq_1, \subseteq_2)$ is a canonical frame of the lattice reduct of $W$ and the relations $R^c, S^c, Q^c, I^c, C^c, C'^c$ are defined as follows:

$$\begin{array}{l}
R^c(x,y,z) \quad \text{iff} \quad \forall a, b \in W\, (a \in x_1 \;\;\&\;\; b \in y_1 \quad \text{imply} \quad a \otimes b \in z_1), \\
Q^c(x,y,z) \quad \text{iff} \quad \forall a, b \in W\, (a \otimes b \in z_2 \;\;\&\;\; a \in x_1 \quad \text{imply} \quad b \in y_2), \\
S^c(x,y,z) \quad \text{iff} \quad \forall a, b \in W\, (a \otimes b \in z_2 \;\;\&\;\; b \in y_1 \quad \text{imply} \quad a \in x_2), \\
C^c(x,y) \quad \text{iff} \quad \forall a \in W\, (a^{\smile} \in y_2 \quad \text{implies} \quad a \in x_2), \\
C'^c(x,y) \quad \text{iff} \quad \forall a \in W\, (a \in x_1 \quad \text{implies} \quad a^{\smile} \in y_1), \\
I^c = \{x \in X(W) :\, 1' \in x_1\}.
\end{array}$$

In order to show that the canonical frame of an LCP algebra is an LCP frame, we have to verify that the relations $\subseteq_1, \subseteq_2, R^c, S^c, Q^c, I^c, C^c$, and $C'^c$ satisfy all the monotonicity and stability conditions. We will present the proofs of conditions involving $C^c$ and $C'^c$, the proofs of the remaining conditions can be found in [1].

**Lemma 30.** *For every LCP algebra* $(W, \vee, \wedge, 1, 0, \otimes, 1', {}^{\smile})$ *and for all* $x_1, x_2 \subseteq W$, $(x_1, x_2)$ *is a filter-ideal pair of* $W$ *iff so is* $(x_1^{\smile}, x_2^{\smile})$.

*Proof.* The proof is the same as in [7]. □

**Lemma 31.** *Let $(X(W), \subseteq_1, \subseteq_2, R^c, S^c, Q^c, I^c, C^c, C'^c)$ be a canonical frame of an LCP algebra. Then the relations $\subseteq_1, \subseteq_2, R^c, S^c, Q^c, I^c, C^c$, and $C'^c$ satisfy the monotonicity conditions (M1), ..., (M6).*

*Proof.* By way of example we show condition (M5). Assume that (i) $\forall a \in W$ ($a^{\smile} \in y_2$ implies $a \in x_2$), (ii) $x_2 \subseteq x_2'$, (iii) $y_2' \subseteq y_2$ and let (iv) $b^{\smile} \in y_2'$. From (iv) and (iii) we get (v) $b^{\smile} y_2$. From (v) and (i) we obtain $b \in x_2$ which together with (ii) yields $b \in x_2'$. Hence, $C(x', y')$ holds. □

**Lemma 32.** *Let $(X(W), \subseteq_1, \subseteq_2, R^c, S^c, Q^c, I^c, C^c, C'^c)$ be a canonical frame of an LCP algebra. Then the relations $\subseteq_1, \subseteq_2, R^c, S^c, Q^c, I^c, C^c$, and $C'^c$ satisfy the stability conditions (S1), ..., (S15).*

*Proof.* (S10) Assume that (i) for every $b \in W$, if $b^{\smile} \in y_2$, then $b \in x_2$. Take $a \in x_1$. We show that $[a^{\smile}) \cap y_2 = \emptyset$. For suppose conversely, then there is $b \in y_2$ such that $a^{\smile} \leq b$. Since $y_2$ is an ideal, we have $a^{\smile} \in y_2$. From (i) we get $a \in x_2$, a contradiction. Hence, $([a^{\smile}), y_2)$ is a filter-ideal pair. We claim that $y'$ defined as an extension of $([a^{\smile}), y_2)$ to a maximal pair satisfies the required condition. Indeed, since $[a^{\smile}) \subseteq y_1'$, we have $a^{\smile} \in y_1'$. We also have $y_2 \subseteq y_2'$.

(S11) Assume that (i) for every $b \in W$, if $b \in x_1$, then $b^{\smile} \in y_1$. Let $a$ be such that (ii) $a^{\smile} \in y_2$. We show that $x_1 \cap (a] = \emptyset$. For suppose conversely, then there is $b$ such that (iii) $b \in x_1$ and (iv) $b \leq a$. From (i) and (iii) we get (v) $b^{\smile} \in y_1$, and (iv) yields (vi) $b^{\smile} \leq a^{\smile}$. Since $y_1$ is a filter, from (v) and (vi) we get $a^{\smile} \in y_1$, which contradicts (ii). Define $x'$ as an extension of $(x_1, (a])$ to a maximal filter-ideal pair. So we have $x_1 \subseteq x_1'$ and $(a] \subseteq x_2'$ which yields $a \in x_2'$ as required.

(S12) Let $x = (x_1, x_2)$ be a maximal filter-ideal pair. Define $w$ as an extension of $(x_1^{\smile}, x_2^{\smile})$ to a maximal filter-ideal pair. If $b^{\smile} \in x_2$, then $b = b^{\smile\smile} \in w_2$, and hence $C^c(w, x)$ holds. Similarly, if $b^{\smile} \in w_2$, then $b = b^{\smile\smile} \in x_2$, so $C^c(x, w)$ holds.

(S13) Assume that (i) for every $b \in W$, if $b^{\smile} x_2$, then $b \in w_2$ and (ii) for every $b \in W$, if $b^{\smile} \in w_2$, then $b \in y_2$. Let $a \in x_2$. Since $a^{\smile\smile} \leq a$ and $x_2$ is an ideal, we have $a^{\smile\smile} \in x_2$. By (i) $a^{\smile} \in w_2$ and by (ii) $a \in y_2$. Hence, $x_2 \subseteq y_2$ as required.

(S14) Assume that $S^c(u, v, z')$ and $C^c(z', z)$ hold. Define $x_1 = v_1^{\smile}$, $y_1 = u_1^{\smile}$, $x_2 = \{a \in W : (\{a\} \otimes y_1) \cap z_2 \neq \emptyset\}$, $y_2 = \{a \in W : (x_1 \otimes \{a\}) \cap z_2 \neq \emptyset\}$.

We show that $x_1 \cap x_2 = \emptyset$. For suppose conversely, then there are $b \in v_1$ and $c \in u_1$ such that $b^{\smile} \otimes c^{\smile} = (c \otimes b)^{\smile} \in z_2$. Since $C^c(z', z)$ holds, we have $c \otimes b \in z_2'$. Since $S^c(u, v, z')$ holds, we have $c \in u_2$, a contradiction. Hence, $(x_1, x_2)$ is a filter-ideal pair. In a similar way we can show that $(y_1, y_2)$ is a filter-ideal pair. Let $x$ and $y$ be their extensions to maximal filter-ideal pairs, respectively.

To show that $Q^c(x,y,z)$ holds, take $a,b \in W$ such that $a \otimes b \in z_2$ and $a \in v_1^{\smile}$. Hence, there is $c \in v_1$ such that $c^{\smile} \otimes b \in z_2$. So according to the definition of $y_2$ we get $b \in y_2$.

To show that $C'^c(v,x)$ holds, take $a \in v_1$. Then $a^{\smile} \in v_1^{\smile} = x_1$.

To show that $C^c(u,y)$ holds, take $a \in W$ such that $a^{\smile} \in y_2$. It follows that there is $c \in v_1$ such that $c^{\smile} \otimes a^{\smile} = (a \otimes c)^{\smile} \in z_2$. Since $C^c(z',z)$ holds, we get $a \otimes c \in z_2'$. Since $S^c(u,v,z')$ holds, we obtain $a \in u_2$ as needed.

(S15) Assume that $Q^c(x,y,z)$, $C'^c(v,x)$, and $C^c(u,y)$ hold. From Lemma 30 we know that $(z_1^{\smile}, z_2^{\smile})$ is a filter-ideal pair. Let $z'$ be its extension to a maximal filter-ideal pair.

To show that $S^c(u,v,z')$ holds, take $a,b \in W$ such that (i) $a \otimes b \in z_2'$ and (ii) $b \in v_1$. From (i) and the definition of $z_2'$ we get (iii) $(a \otimes b)^{\smile} = b^{\smile} \otimes a^{\smile} \in z_2$. From (ii) and $C'^c(v,x)$ we obtain (iv) $b^{\smile} \in x_1$. (iii), (iv) and $Q^c(x,y,z)$ yield $a^{\smile} \in y_2$. Since $C^c(u,y)$ holds, we get $a \in u_2$.

To show that $C^c(z',z)$ holds, take $a \in W$ such that $a^{\smile} \in z_2$. It follows that $a \in z_2^{\smile} = z_2'$, which completes the proof. □

**Theorem 10.** *For every LCP algebra its canonical frame is an LCP frame.*

*Proof.* Follows from Lemma 8, Lemma 31, and Lemma 32. □

Let $(W, \vee, \wedge, 1, 0, \otimes, 1', {}^{\smile})$ be an LCP algebra. Define mapping $h\colon W \to 2^{X(W)}$ as

$$h(a) = \{x \in X(W) : a \in x_1\}.$$

**Lemma 33.** *For every LCP algebra* $(W, \vee, \wedge, 1, 0, \otimes, 1', {}^{\smile})$ *and for every* $a \in W$ *the following conditions are satisfied:*

(I) $h(a^{\smile}) = h(a)^{\smile c}$,

(II) $h(a \otimes b) = h(a) \otimes^c h(b)$,

(III) $h(1') = 1'^c$.

*Proof.* We prove (I), the proofs of the remaining conditions can be found in [7].

First, we show that (i) $f_C^c rh(a) = rh(a^{\smile})$.

($\subseteq$) Let $y \in f_C^c rh(a)$. This means that (ii) for any $x \in X(W)$ if $C^c(x,y)$, then $x \in rh(a) = \{z \in X(W) : a \in z_2\}$ (by Theorem 8(I)). Suppose that $y \notin rh(a^{\smile})$. This means that $a^{\smile} \notin y_2$, and hence (iii) $a \notin y_2^{\smile}$. Observe that due to (iii) $[a) \cap y_2^{\smile} = \emptyset$, so $([a), y_2^{\smile})$ is a filter-ideal pair. Let $x$ be its extension to a maximal pair. Then $C^c(x,y)$ holds. It follows that $a \in x_2 = y_2^{\smile}$, which contradicts (iii).

($\supseteq$) Let $y \in rh(a^{\smile})$. By Theorem 8(I) this means that (iv) $a^{\smile} \in y_2$. Take $z \in X(W)$ such that $C^c(z,y)$ holds. This means that (v) for every $a \in W$, if $a^{\smile} \in y_2$, then $a \in z_2$. (iv) and (v) yield $a \in z_2$. Hence $z \in rh(a)$ and as a consequence $y \in f_C^c rh(a)$.

Now, from (i) we get $lf_C^c rh(a) = lrh(a^{\smile})$. Since $h(a^{\smile})$ is $l$–stable, condition (I) follows. □

**Theorem 11. (Representation theorem for LCP algebras)**. *Every LCP algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

*Proof.* The theorem follows from Theorem 9 and Lemma 33. □

## 6 Relational Representation of the Algebras of Substructural Logics

In this section we apply the generalised UAD framework to basic substructural logics. A survey of substructural logics and their algebras can be found in [11] and [12]. The content of the present section is based on [15]. All the proofs can be found there.

An *FL algebra* is an algebra $(W, \vee, \wedge, \otimes, \rightarrow, \leftarrow, 1', d)$ such that $(W, \vee, \wedge, \otimes, \rightarrow, \leftarrow, 1')$ is a residuated lattice and $d$ is an arbitrary fixed element of $W$. It follows that in any FL algebra the following conditions hold:

(FL1) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$,

(FL2) $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$,

(FL3) $(a \vee b) \otimes c = (a \otimes c) \vee (b \otimes c)$,

(FL4) $a \otimes b \leq c$ iff $b \leq a \rightarrow c$,

(FL5) $a \otimes b \leq c$ iff $a \leq c \leftarrow b$,

(FL6) $a \otimes 1' = a = 1' \otimes a$.

FL algebras provide an algebraic semantics for the Full Lambek Calculus.

An *$FL_e$ algebra* is an FL algebra such that the product $\otimes$ is commutative:

($\mathrm{FL_e}$) $a \otimes b = b \otimes a$.

An *$FL_c$ algebra* is an FL algebra such that product $\otimes$ satisfies the following condition:

($\mathrm{FL_c}$) $a \leq a \otimes a$.

An *$FL_w$ algebra* is an FL algebra satisfying the following conditions:

($\mathrm{FL_w}$1) $a \otimes b \leq a,\ a \otimes b \leq b$,

($\mathrm{FL_w}$2) $d$ is the least element of the lattice $(W, \vee, \wedge)$.

$\mathrm{FL_e}$, $\mathrm{FL_c}$, and $\mathrm{FL_w}$ algebras provide an algebraic semantics of the substructural logics with exchange, contraction, and weakening rules, respectively. The other substructural logics $\mathrm{FL_{ec}}$, $\mathrm{FL_{ew}}$, and $\mathrm{FL_{ecw}}$ are obtained by joining the corresponding axioms.

An *FL frame* is a relational system $(X, \leq_1, \leq_2, R, S, Q, I, D)$ such that $(X, \leq_1, \leq_2)$ is a doubly ordered set, $R$, $S$, and $Q$ are ternary relations on $X$, $I$ and $D$ are subsets (unary relations) of $X$, the conditions (M1), ..., (M4) and (S1), ..., (S9) of Section 5 are satisfied and, moreover, the following monotonicity condition for $D$ holds:

(M7) $D(x)$ & $x \leq_1 x'$ imply $D(x')$.

An *$FL_e$ frame* is an FL frame which additionally satisfies:

($\mathrm{S_e}$) $R(x, y, z)$ implies $R(y, x, z)$.

An *$FL_c$ frame* is an FL frame satisfying the following stability condition:

($\mathrm{S_c}$) $R(x, x, x)$.

An *$FL_w$ frame* is an FL frame which satisfies:

($\mathrm{S_w}1$) $S(x, y, z)$ implies $z \leq_2 x$,
($\mathrm{S_w}2$) $Q(x, y, z)$ implies $z \leq_2 y$,
($\mathrm{S_w}3$) $D = \emptyset$.

A *complex algebra of an FL frame* $(X, \leq_1, \leq_2, R, S, Q, I, D)$ is an algebra $(L(X), \vee^c, \wedge^c, \otimes^c, \rightarrow^c, \leftarrow^c, 1'^c, d^c)$ such that $(X(W), \vee^c, \wedge^c)$ is a complex algebra of the doubly ordered set $(X, \leq_1, \leq_2)$, the operation $\otimes^c$ and constant $1'^c$ are defined as in Section 5, and the remaining operations and constant $d^c$ are defined as follows. For all $l$–stable sets $A$ and $B$:

$$A \rightarrow^c B = \{x \in X : \forall y, z \in X\, (R(y, x, z) \;\&\; y \in A \;\text{ imply }\; z \in B)\},$$
$$A \leftarrow^c B = \{x \in X : \forall y, z \in X\, (R(x, y, z) \;\&\; y \in B \;\text{ imply }\; z \in A)\},$$
$$d^c = lr(D).$$

**Lemma 34.** *For every FL frame $(X, \leq_1, \leq_2, R, S, Q, I, D)$ and for all $A, B \subseteq X$ the following assertions hold:*

(I) *$A \rightarrow^c B$ and $A \leftarrow^c B$ are $\leq_1$–increasing,*
(II) *if $A$ and $B$ are $l$–stable, then so are $A \rightarrow^c B$ and $A \leftarrow^c B$,*
(III) *$d^c$ is $l$–stable.*

*Proof.* (I) and (II) were proved in [15], (III) is immediate from the definition. □

**Lemma 35.** *For every FL frame $(X, \leq_1, \leq_2, R, S, Q, I, D)$ and for all $A, B, C \subseteq L(X)$ the following assertions hold:*

(I) $A \otimes^c B \subseteq C$ iff $B \subseteq A \rightarrow^c C$,
(II) $A \otimes^c B \subseteq C$ iff $A \subseteq C \leftarrow^c B$.

*Proof.* The proof is in [15]. □

**Theorem 12.** *The complex algebra of an FL frame is an FL algebra.*

*Proof.* The verification of the axioms (FL1), (FL2), (FL3), and (FL6) can be found in [7]. For the lattice axioms and (FL4) and (FL5) we follow Lemma 7 and Lemma 35, respectively. □

The analogous theorems hold for $\mathrm{FL_e}$, $\mathrm{FL_c}$, and $\mathrm{FL_w}$ frames. The proofs that the corresponding complex algebras satisfy the axioms ($\mathrm{FL_e}$), ($\mathrm{FL_c}$), and ($\mathrm{FL_w}$) can be found in [15].

Let $(W, \vee, \wedge, \otimes, \rightarrow, \leftarrow, 1', d)$ be an FL algebra. A filter-ideal pair of $W$ is a filter-ideal pair of the underlying lattice. As usual, by $X(W)$ we denote the family of all the maximal filter-ideal pairs of $W$.

A *canonical frame of an FL algebra* $(W, \vee, \wedge, \otimes, \rightarrow, \leftarrow, 1', d)$ is a structure $(X(W), \leq_1, \leq_2, R^c, S^c, Q^c, I^c, D^c)$ such that $(X(W), \subseteq_1, \subseteq_2)$ is a canonical frame of the lattice reduct of $W$, the relations $R^c$, $S^c$, $Q^c$, and $I^c$ are defined as in Section 5, and $D^c = \{x \in X(W) : d \in x_1\}$.

**Lemma 36.** *For every FL algebra, the relations of its canonical frame satisfy the monotonicity conditions (M1), …, (M4), (M7) and the stability conditions (S1), …, (S9).*

*Proof.* The proof of (M7) is immediate from the respective definitions. The remaining proofs can be found in [7]. □

We conclude that the following theorem holds.

**Theorem 13.** *For every FL algebra, its canonical frame is an FL frame.*

*Proof.* The theorem follows from Lemma 8 and Lemma 36. □

The analogous theorems hold for $\mathrm{FL_e}$, $\mathrm{FL_c}$, and $\mathrm{FL_w}$ algebras. A verification that the relations in the corresponding canonical frames satisfy the conditions $(\mathrm{S_e})$, $(\mathrm{S_c})$, $(\mathrm{S_w}1)$, and $(\mathrm{S_w}2)$ are presented in [15]. The conditions $(\mathrm{S_e})$ and $(\mathrm{S_c})$ are also mentioned in [1]. Condition $(\mathrm{S_w}1)$ is from [10].

Let $(W, \vee, \wedge, \otimes, \to, \leftarrow, 1', d)$ be an FL algebra and define $h\colon W \to 2^{X(W)}$ as $h(a) = \{x \in X(W) : a \in x_1\}$.

**Lemma 37.** *For every FL algebra $(W, \vee, \wedge, \otimes, \to, \leftarrow, 1', d)$ and for every $a \in W$ the following conditions are satisfied:*

(I) $h(d) = D^c$,
(II) $h(a \otimes b) = h(a) \otimes^c h(b)$,
(III) $h(1') = 1'^c$,
(IV) $h(a \to b) = h(a) \to^c h(b)$,
(V) $h(a \leftarrow b) = h(a) \leftarrow^c h(b)$.

*Proof.* (I) immediately follows from the respective definitions. (II) and (III) are proved in [1], see also [7]. (IV) and (V) are proved in [15]. □

**Theorem 14. (Representation theorem for FL algebras)**. *Every FL algebra is isomorphic to a subalgebra of the complex algebra of its canonical frame.*

*Proof.* The theorem follows from Theorem 2 and Lemma 37. □

Representation theorems for the algebras $\mathrm{FL_{ec}}$, $\mathrm{FL_{ew}}$, and $\mathrm{FL_{ecw}}$, obtained from the basic classes by making the corresponding joins of algebras, can be obtained by appropriately joining (and possibly simplifying) the axioms of the corresponding frames. Observe that the definitions of relations in the canonical frames are the same for all the classes FL, $\mathrm{FL_e}$, $\mathrm{FL_c}$, and $\mathrm{FL_w}$. The only differences are in the properties that the relations of each of these classes satisfy. The different properties are due to the specific axioms characteristic for the classes of algebras in question. Therefore, the representability results can be obtained in a modular way.

## 7 Conclusion

We have presented a survey of recently developed relational representation theorems for lattice-based structures which are relevant for modal logics, substructural logics and relation algebras. The theorems are obtained within a generalisation of the Urquhart-Allwein-Dunn framework. The focus in the paper has been put on the algebraic aspects of the UAD framework. A logical part of the framework which leads to a Kripke-style semantics for the logics whose algebraic semantics is provided by the classes of algebras in question will be a subject of a separate paper. Further work is also needed on an extension of the UAD framework to relation algebras based on double residuated lattices [14], where both a relative product and a relative sum (which are not dual to each other in a lattice) would get a representation.

## Acknowledgments

The cooperation for this work was partially supported by EU COST Action 274 *Theory and Applications of Relational Structures as Knowledge Instruments* (TARSKI).

Dimiter Vakarelov was partially supported by the RILA 12 project sponsored by the Bulgarian Ministry of Science and Education.

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Journal on Relational Methods in Computer Science, Vol. 1, 2004, pp. 132 - 160
Received by the editors March 30, 2004, and, in revised form, October 06, 2004.
Published on December 10, 2004.

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