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# Tarskian Algebraic Logic

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**Abstract.** This is a survey article on algebraic logic. It gives a historical background leading up to a modern perspective. Central problems in algebraic logic (like the representation problem) are discussed in connection to other branches of logic, like modal logic, proof theory, model-theoretic forcing and Gödel's incompleteness results. We focus on cylindric algebras which are natural algebras of $n$-ary relations. Relation algebras (which are algebras of binary relations) are mostly only covered insofar as they relate to cylindric algebras. Cylindric and relation algebras were introduced by Tarski, hence the title of the article.

## Algebraic Logic

Algebraic logic arose as a subdiscipline of algebra mirroring constructions and theorems of mathematical logic. It is similar in this respect to such fields as algebraic geometry and algebraic topology, where the main constructions and theorems are algebraic in nature, but the main intuitions underlying them are respectively geometric and topological. The main intuitions underlying algebraic logic are, of course, those of formal logic. Investigations in algebraic logic can proceed in two conceptually different, but often (and unexpectedly) closely related ways. First one tries to investigate the algebraic essence of constructions and results in logic, at the hope of gaining more insight that could add to his understanding, thus his knowledge. One can then study certain "particular" algebraic structures (or simply algebras) that arise in the course of his first kind of investigations as objects of interest in their own right and go on to discuss questions which naturally arise independently of any connection with logic. But often such purely algebraic results have an impact on the logic side. Throughout this article we will have occasion to deal with both types of investigations (algebraic and metamathematical) and their unexpected and indeed intriguing

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interplay and interaction. Also one of the aims of this article is to see what the ideas of Boole has led to. Have they borne their full fruit, or not yet? But let us start with some history.

**A Brief History**

In the middle of the nineteenth century, George Boole initiated the investigation of a class of algebraic structures which were subsequently called Boolean algebras. The theory of these algebras is directly related to the development of the most elementary part of mathematical logic, namely propositional logic.

As is well known, however, the theory of Boolean algebras can be developed in a purely algebraic fashion. It has at present numerous connections with several branches of mathematics - (independence results in) set theory, topology and analysis - and hence it can be understood and appreciated by mathematicians unfamiliar with the logical problems to which it owes its birth. The work of Boole was the starting point for a continuous flow of inquiries into the algebraization of *quantifier* logics (like first order logic and infinitary reducts of Keisler's logic) which through various intermediate stages led to (but did not end with) the foundation of the theory of cylindric and polyadic algebras.

The history of the subject is interesting and worth a brief review. It dates back to the nineteenth century when the pre-modern tradition of algebraic logic began. De Morgan, in a restive frame of mind, was trying to extend Aristotle's syllogistic tradition that had held sway for 2000 years to encompass more complex situations, when he came across the slightly earlier work of Boole who had come forward with a highly successful algebra of Propositions. In 1860, de Morgan published [12] thereby launching an investigation into the algebra of relations. This developed into the subject now known as algebraic logic though in the nineteenth century it was known simply as mathematical logic. This work, along with Frege's quantifier logic, became the foundation of modern logic and model theory.

So in the nineteenth century there were basically two approaches to the formalization of *quantification* in logic. The first (algebraic) approach adopted by Boole and De Morgan and taken up by Pierce and Schröder led to what we now call (following Tarski) relation algebra. The other approach due to Frege, and boosted by Russell and Whitehead in their momentous work *Principia Mathematica*, became the standard formalism of first order logic with its explicit universal and existential quantifiers. Both can express quantification, albeit in different ways.

Historians of Mathematical Logic frequently tell us that there are two traditions, the algebraic tradition of Boole, Schröder, De Morgan and Pierce, arising from the algebraization of analysis as opposed to the quantification-theoretical (or logistic) tradition of Peano, Frege and Russell arising from the development of the theory of functions. It is said that these two traditions, together with the independent set-theoretic tradition of Cantor, Dedekind and Zermelo arising out of the search for a foundation for real analysis in the work of Cauchy, Weierstrass and others, were united by Whitehead and Russel in their Principia Mathematica to create Mathematical Logic. However, the dual *algebraic* and *quantification-theoretic* traditions, as a matter of of historical fact, simply did not exist for logicians at the turn of the century. It is a false retrospective duality which derives from the Principia and is post Principia phenomenon. There was no such dichotomy in the nineteenth century, algebraic logic was simply *the* Mathematical Logic of its time.

## Cylindric Algebras

In the twentieth century first order logic was given an algebraic setting by Tarski in the framework of *cylindric algebra*. This was a natural outcome of Tarski's formalization of the notion of truth in set theory, for indeed the prime examples of cylindric algebras are those algebras whose elements are sets of sequences (i.e. relations) satisfying first order formulas.

While relation algebras constitute an algebraization of binary relations, $n$-dimensional cylindric algebras are an algebraization of $n$-ary relations. (Here $n$ is any (not necessarily finite) ordinal.) Ever since relation and cylindric algebras were defined, researchers have been investigating the connections between them. Tarski, Monk and Henkin investigated such connections, [37], [20] Thm 5.3.8. More recent references include the work of Maddux [33],[34], the work of Németi and Simon [43], [51], [5] and the work of Hirsch and Hodkinson [25].

Indeed, every cylindric algebra of dimension $> 3$, has a natural relation algebra reduct obtained by taking the 2-*neat reduct* i.e. the essentially 2-dimensional elements abstracting binary relations, and defining converse and composition using one spare dimension. Conversely, one can construct cylindric algebras of arbitrary dimension $n$, $3 < n \leq \omega$ from relation algebras that posses what Maddux calls an $n$-dimensional cylindric basis [33].

Making another leap into modern perspective, the motivation for studying the theory of relation algebra and cylindric algebra, or hereinafter algebras of relations for short, today comes from at least three areas:

(i) Logic (predicate logics, propositional multi-modal logics, dynamic logics.)
(ii) Algebra: the algebraic theory of these algebras is of interest in its own right. Indeed boolean and cylindric algebras served as a starting point for the (by now well-developed) theory of discriminator varieties.

(iii) The theory of relations: relations are of interest in themselves.

The study of algebraic logic can be further motivated by surveying its numerous applications in temporal reasoning, planning data bases, modal logic and elsewhere, cf. the survey article [40] for an exposition of such applications. Instead, we prefer to recommend the part of algebraic logic most relevant to our investigations here as a very well established mathematical procedure-*algebraization*-applied to a fundamental entity, viz. a *relation.*

For this purpose, let us focus for a while on cylindric algebras, which is an abstraction from algebras (of not necessarily binary) relations. The concept of cylindric algebras permits the use of algebraic methods in treating two related parts of mathematics. One of these is a very general kind of geometry associated with basic set-theoretic notions. Indeed, Andréka, Madarász and Németi have used their geometric intuition originating from their investigations in the theory of cylindric algebra to formulate Einstein's general theory of relativity in first order logic, an exotic, exciting and novel application of algebraic logic [6]. The other - as previously mentioned - is the theory of deductive systems in Mathematical Logic. The two parts are indeed interconnected, because models of deductive systems (like first order theories) give rise in a natural way to structures within the set-theoretic "spaces" (set algebras based on these models.)

To illustrate this connection further, let $L$ be any first order relational language, i.e. $L$ has no function symbols nor individual constants. Then $L$ has an infinite countable sequence of individual variables $x_0$, $x_1 \ldots x_n : n < \omega$, logical constants $\neg$, $\implies$, $\forall$, $\exists$, $=$, and non-logical constants-say a system $\langle R_i : i \in I\rangle$ of relation symbols, where $R_i$ is of rank $\rho_i < \omega$ for each $i \in I$. We assume as known the usual syntactical primitive notions defined in terms of $L$, e.g. the notions of a formula, a sentence (formula without free occurrences of variables), the conjunction $\phi \wedge \psi$ of formulas of $L$, the notion of a formal proof of a formula from a set of sentences, etc. Now given a set $\Gamma$ of sentences, sometimes referred to as a theory, we may call two formulas $\phi$ and $\psi$ equivalent under (or modulo) $\Gamma$, in symbols

$$\phi \equiv_\Gamma \psi$$

provided that the biconditional $\phi \longleftrightarrow \psi$ is provable from $\Gamma$, in symbols $\Gamma \vdash \phi \longleftrightarrow \psi$. Here $\vdash$ denotes a standard (complete) proof system. It is easy to check that the relation $\equiv_\Gamma$ is actually an equivalence relation on the set of formulas. In fact, it is a congruence relation compatible with the boolean operations of conjunction and negation on formulas. Even more, if we let $A_\Gamma$ denote the set of equivalence classes under $\equiv_\Gamma$, we find that certain algebraic operations can be introduced on $A_\Gamma$ which reflect the syntactical operations of building formulas:

Besides the boolean operations

$$[\phi]_\Gamma + [\psi]_\Gamma = [\phi \vee \psi]_\Gamma,$$

$$[\phi]_\Gamma . [\psi]_\Gamma = [\phi \wedge \psi]_\Gamma,$$

$$-[\phi]_\Gamma = [\neg\phi]_\Gamma,$$

we have

$$c_i[\phi]_\Gamma = [\exists x_i \phi]_\Gamma,$$

$$d_{ij} = [x_i = x_j]_\Gamma.$$

Here, of course, $[\phi]_\Gamma$ is the equivalence class of $\phi$ under $\Gamma$. Note that the congruence relation defined via provability is now compatible with the newly added operations $c_i$ and $d_{ij}$ for each $i, j < \omega$. The resulting quotient algebra $\mathcal{A}_\Gamma$ thus associated with $L$ and $\Gamma$ is one of the fundamental algebras studied in algebraic logic, and it is in fact the prime source of inspiration of cylindric algebras. It turns out that many constructions and theorems in logic can be algebraically reflected using these algebras. For example $\Gamma$ is complete and consistent if $\mathcal{A}_\Gamma$ is a simple algebra, i.e. has no proper congruences; the theorem that any consistent theory can be extended to a complete and consistent theory is mirrored by the theorem that any algebra $\mathcal{A}_\Gamma$ with $|A_\Gamma| > 1$ has a simple homomorphic image. For more on such connections between metalogical notions and algebraic ones, the reader is referred to [20] Sec. 4.3. The algebra $\mathcal{A}_\Gamma$ is referred to as the *Tarski-Lindenbaum cylindric algebra of formulas corresponding to $\Gamma$*, or simply an algebra of formulas.

### Abstract Cylindric Algebras

The notion of a cylindric algebra is obtained from algebras of formulas by a process of abstraction. Let $\alpha$ be an ordinal. A *cylindric algebra of dimension $\alpha$*, for brevity a $CA_\alpha$, is an algebra of the form

$$\mathcal{A} = \langle A, +, ., -, c_i, d_{ij} \rangle_{i,j<\alpha}$$

where $\langle A, +, ., - \rangle$ is a boolean algebra with $+$ denoting the *boolean join*, . denoting the *boolean meet* and $-$ denoting *boolean complementation.* The $c_i$'s, $i \in \alpha$, are unary operations of *cylindrifications* on $A$ (the domain of $\mathcal{A}$) and the $d_{ij}$'s, $i, j \in \alpha$ the *diagonal elements* are distinguished elements of $A$. The operation $c_i$ is an abstract version of the unary operation on first order formulas of existential quantification with respect to the $i-th$ variable $x_i$. The diagonal element $d_{ij}$ is an abstract version of the atomic identity formula $x_i = x_j$ in first order logic. The class $CA_\alpha$ of all cylindric algebras of dimension $\alpha$ is axiomatized by finitely many equational schemata that aim at capturing the essential algebraic properties of existential quantification and atomic identity formulas (cf. [20] Part I, Def 1.1.1.) Intuitively, the axioms say the following for all $i, j, k \in \alpha$.

(i) $c_i$ is an additive complemented closure operator.
(ii) $c_i c_j x = c_j c_i x$, i.e. the cylindrifications commute.
(iii) The diagonal elements satisfy the usual axioms one would expect of equality such as $d_{ij} \cdot d_{jk} \leq d_{ik}$ (transitivity).
(iv) $d_{ij} \cdot c_i x \leq x$ for all $x \leq d_{ij}$.

Axiom (iv) for example is an algebraic version of Leibniz's law of equality saying that equal objects are indistinguishable by formulas, i.e.

$$\phi \implies v_i = v_j \vdash (v_i = v_j \wedge \exists v_i \phi) \implies \phi.$$

That the abstraction from the algebras of formulas to abstract algebras is indeed sound, is established by the following logical representation Theorem, which is not very difficult to prove. Loosely speaking, the proof of Theorem 1 consists of simply checking that the postulates characterizing cylindric algebras form an adequate algebraic transcription of the axioms of any standard complete system for first order logic, cf. [20] Sec. 4.3 Thm 4.2.28(iii).

**Theorem 1.** (Tarski) *For any algebra $\mathcal{A}$ having the same signature as $CA_\omega$ the following two conditions are equivalent:*

*(i) $\mathcal{A} \cong \mathcal{A}_\Gamma$ for some set of first order formulas $\Gamma$.*
*(ii) $\mathcal{A}$ is a $CA_\omega$ such that the set $\Delta x = \{i \in \omega : c_i x \neq x\}$ is finite for every $x \in A$.*

Theorem 1 justifies the choice of equations axiomatizing $CA_\alpha$. In the infinite dimensional case; these equations force locally finite algebras satisfying them to be isomorphic to algebras of formulas. Historically that was the reason that led Tarski to stipulate the (by now) official axiomatization of cylindric algebras.

### Models and Set algebras.

The other more concrete source of cylindric algebras is that of *cylindric set algebras* which Tarski introduced as an algebraic counterpart of semantics of first order logic. Such algebras, the cylindric set algebras, arise naturally from models of first order theories, and therefore they are closely related to the algebras of formulas. To explain further the connection of cylindric set algebras to algebras of formulas, we now turn to semantical notions.

Fix a first order relational language $L$. Let $M$ be an $L$-structure. We take it as well-known what it means for a sequence $s \in {}^{\omega}M$ to *satisfy* a formula $\phi$ in $M$. We write $M \models \phi[s]$ if $s \in {}^{\omega}M$ satisfies $\phi$ in $M$. For $\phi \in L$, let $\phi^M$ be the set of all $\omega$-ary sequences, or assignments, that satisfy $\phi$ in $M$, that is

$$\phi^M = \{s \in {}^{\omega}M : M \models \phi[s]\}.$$

Then $\phi^M$ is an $\omega$-ary relation on $M$; it is a point-set in the $\omega$-dimensional space ${}^{\omega}M$, and the set of all these, i.e the set

$$A_M = \{\phi^M : \phi \in L\}$$

is the universe of a cylindric algebra of dimension $\omega$. It is easy to see that $A_M$ is a boolean field of sets; it is closed under intersections and complementation (hence under unions), for indeed

$$\phi^M \cap \psi^M = (\phi \wedge \psi)^M,$$

and

$${}^{\omega}M \smallsetminus \phi^M = (\neg\phi)^M.$$

Certain set-theoretic operations similar to the classical operations of descriptive set theory can now be introduced corresponding to the basic non-boolean operations. In harmony with the notation of Henkin, Monk and Tarski [20], we use capital letters for the interpretation of the cylindric operations in set algebras: The $i-th$ *cylindrification*, algebraizing existential quantification with respect to the $i-th$ variable, is defined as follows:

$$\begin{aligned} C_i(\phi^M) &= \{t \in {}^{\omega}M : \text{ there exists } s \in \phi^M \text{ such that } t(j) = s(j) \text{ for all } j \neq i\} \\ &= (\exists x_i \phi)^M. \end{aligned}$$

This is simply the cylinder obtained by moving $\phi^M$ parallel to the $i$-axis, hence the terminology of $i-th$ cylindrification. On the other hand, the diagonal elements, denoted by $D_{ij}$ for $i, j \in \omega$, are defined to be the following hyperplanes:

$$D_{ij} = \{s \in {}^{\omega}M : s_i = s_j\} = (x_i = x_j)^M.$$

Now let $\Gamma$ be an $L$-theory. Suppose further that $M$ is a model of $\Gamma$, i.e. that every sentence in $\Gamma$ is true in $M$. Then this is reflected algebraically by the fact that $\mathcal{A}_M$ is a homomorphic image of $A_\Gamma$ via the natural map

$$[\phi]/\Gamma \to \phi^M.$$

If $\psi$ is satisfiable in $M$, then this can be expressed algebraically by the fact that the image of $\psi$ under the above map is non-zero, for indeed $\psi^M$ contains a sequence that satisfies $\psi$ in $\Gamma$, i.e. is non-empty.

Having at hand the concrete notion of a cylindric set algebra based on models, we now make another abstraction from this notion to obtain a more general (but still concrete) set-theoretic object. There is no reason why we should restrict our attention to $\omega$-dimensional set algebras. Accordingly, relaxing the restriction of dimension, a cylindric set algebra of dimension $\alpha$, $\alpha$ an *arbitrary* ordinal, and base $U$, for short a $Cs_\alpha^U$ or simply a $Cs_\alpha$, when $U$ is clear from context, is an algebra

$$\mathcal{A} = \langle A, \cup, \cap, \setminus, C_i, D_{ij}\rangle_{i,j\in\alpha}$$

such that $A$ is a boolean field of subsets of ${}^\alpha U$ closed under each $C_i$ and containing the $D_{ij}$'s for all $i, j \in \alpha$ defined as above replacing ${}^\omega M$ by ${}^\alpha U$ and undergoing the obvious changes.

Note that set algebras can have arbitrary dimensions. It turns out that in the $\omega$-dimensional case, the cylindric set algebras corresponding to models are precisely the locally finite *regular* ones. The property of local finiteness reflects the fact that such algebras consist of relations that are essentially "finitary", though of course the rank of such relations can grow without bound. On the other hand, $\mathcal{A} \in Cs_\omega$ is regular if for all $X \in A$ and $s, t \in {}^\omega U$, whenever $s$ and $t$ agree on $\Delta X$, then $s \in X$ implies that $t \in X$. This, in turn, reflects the metalogical property that if two assignments agree on the indices of the free variables occurring in a formula, then they both satisfy the formula or none does.

For a detailed and extensive exposition of cylindric set algebras (and related structures like relativized cylindric set algebras) the reader is referred to Monk's survey article [39] and to [21].

## The Completeness and Incompleteness Theorems of Gödel algebraically.

In terms of the concept of a cylindric set algebra, a purely algebraic form of Gödel's Completenesss Theorem can now be stated. For that we need to recall that a subdirect product of a family of algebras is a subalgebra of the product of this family such that the (natural) projections are onto. A locally finite cylindric algebra $\mathcal{A} \in CA_\alpha$ is one for which $\Delta x = \{i \in \alpha : c_i x \neq x\}$ is finite for every $x \in A$.

**Theorem 2.** (Tarski) *If $\mathcal{A} \in CA_\omega$ is locally finite, then $\mathcal{A}$ is isomorphic to a subdirect product of $Cs_\omega$'s. Furthermore these $Cs_\omega$'s can be chosen to be locally finite and regular.*

The proof of Theorem 2 is somewhat deeper than that of Theorem 1. In fact, Theorem 2 is equivalent in $ZF$ (Zermelo-Fraenkel set theory without choice) to Gödel's classical Completeness Theorem. As regular locally finite set algebras of dimension $\omega$ are simple, thus Theorem 2 shows that every locally finite algebra is actually *semisimple* i.e. isomorphic to a subdirect product of simple algebras. Since, on universal algebraic grounds, the kernel of a homomorphism is a maximal ideal if and only if its range is simple, this in turn, is equivalent to that the intersection of maximal ideals in any locally finite algebra is the trivial algebra. We note that Halmos' proof that semisimplicity of (locally finite) polyadic algebras is equivalent to Gödel's Completeness Theorem was to him a revelation. In fact, Halmos in [18] was concerned with the algebraization of two of the deepest theorems in logic; the original formulations and proofs of these justly celebrated theorems are due to Gödel; these are Gödel's Completeness and Gödel's Incompleteness Theorems. The algebraic counterpart of Gödels Completeness Theorem is semisimplicity of locally finite polyadic algebras, as illustrated in Fact 2, bearing in mind that locally finite polyadic algebras are essentially the same as locally finite cylindric algebras. Now algebraizing Gödel's Incompleteness results is more intricate, and is still (to the best of our knowledge) an unfinished task. However, Halmos [19] has work in this direction, which we now briefly review. There is no difficulty in constructing locally finite polyadic algebras, and for that matter locally finite cylindric algebras, with sufficiently rich structure to mirror the axiomatic system of complex systems like for example Peano arithmetic or even the stronger Zermelo-Fraenkel set theory in detail, so that the resulting algebra is what Halmos calls a Peano algebra. A Peano algebra is adequate to mirror elementary arithmetic; so for example we can "talk about" recursive functions in a Peano algebra. Now Gödel's Incompleteness Theorem says (very roughly) that in any strong enough formal system to encode recursive functions there are propositions that are neither refutable nor provable. We recall that a simple algebra is one that has no proper congruences. The algebraic version of Gödels's incompleteness Theorem, turns out surprisingly rather simple. It states:

**Theorem 3.** (Halmos) *Not every Peano algebra is simple.*

Halmos identifies a Peano algebra with Mathematics, and rephrases his result in the following pun: "Mathematics is not simple". We refer the interested reader to [18] for a very interesting survey of the history of polyadic algebras.

In passing, we note that Tarski and Givant [59] formalized set theory in the calculus of relations, i.e in relation algebras. A similar task was done by Németi, when he formalized an essentially undecidable fragment of Peano Arithmetic due to Tarski and Robinson in the so-called class of semi-associative relation algebras of Maddux. This interpretation of essentially undecidable fragments of set theory, into the calculus of relations, was used to prove undecidability of the equational theory of several classes of relation algebras, cf. [3]. Recent results on undecidability of various classes of relation and cylindric algebras involve the more sophisticated methods of the interpretation of the (undecidable) tiling problem and the equational theory of semigroups. Conversely, Gödel's Incompleteness Theorem was used to solve problems in algebraic logic. In cylindric algebra, Gödels' Incompleteness was used by Németi to show that the (boolean reduct of the) free cylindric algebras of dimension at least 3 are not atomic. A long-standing open question here posed by Németi and Maddux is the following:

**Open question 1 (Németi-Maddux)** *Is the equational theory of the diagonal free reducts of cylindric algebras of dimension* 3*, or* $Df_3$ *for short, strong enough to encode Peano arithmetic ?*

In fact, it is conjectured by Németi that set theory (and not just Peano arithmetic) might be interpreted in $Df_3$, in the manner of Tarski-Givant [59]. The latter is also related to the "Finitization Problem" in algebraic logic to be discussed in a while. Another interesting application of Gödel's Incompleteness results is the Németi-Sági result in [42] which shows that the equational theory of Halmos' polyadic equality algebras is strong enough to "encode" second order Peano arithmetic; thus the representable polyadic *equality* algebras cannot be axiomatized by any reasonable schema, let alone a finite one, which is an interesting contrast to the classical Daigneault-Monk representation theorem for polyadic algebras *without* equality [14].

## The algebraization of first order logic, the Representation problem.

Coming back from this fascinating short detour into the algebraization of Gödel's incompleteness results, we turn our attention for a while to the class $Lf_\omega$ of locally finite $CA_\omega$'s. Soon in the development of the subject, it transpired that the class $Lf_\omega$, the algebraic counterpart of first order logic, has some serious defects when treated as the sole subject of research in an autonomous algebraic theory. In Universal Algebra one prefers to deal with *equational classes* of algebras i.e. classes of algebras characterized by postulate systems in which every postulate has the form of an equation (an identity); such classes are also referred to as *varieties*.

The reason for this preference is the fact that every variety is closed under certain general operations frequently used to construct new algebras from given ones; we mean here the operation of forming subalgebras, homomorphic images and direct products. By a well known theorem of Garrett Birkhoff, the varieties are precisely those classes of algebras that have all three of these closure properties. Local finiteness does not have the the form of an identity, nor can it be equivalently replaced by any identity or system of identities, nor indeed any set of first order axioms. This follows from the simple observation that the ultraproduct of infinitely many $Lf_\omega$'s is not, in general locally finite and a first order axiomatizable class is necessarily closed under ultraproducts.

When Alfred Tarski introduced cylindric algebras, he introduced them in the form satisfying (ii) in Theorem 1, and proved the representation Theorem expressed in Theorem 2. But some modifications in the definition of Tarski's cylindric algebras seemed desirable. The definition contains certain assumptions which considerably restrict the applicability of the definition and thus can be dispensed with. One such is the fixed dimension $\omega$. The other is local finiteness. The restrictive character of these two notions becomes obvious when we turn our attention to cylindric set algebras. We find there are algebras of all dimensions, and set algebras that are not locally finite easily constructed. For these reasons the original conception of a cylindric algebra has been extended: the restriction to dimension $\omega$ and local finiteness were removed, and the class $CA_\alpha$ was introduced.

*A central and indeed still active part of research in algebraic logic is the, vaguely posed, lengthly discussed problem concerning improvements of Theorem 2.*

This problem is referred to as *The Finitization Problem* by the Budapest group specifically by Andréka and Németi [40] while it is referred to as the *Representation Problem* by the London group specifically by Ian Hodkinson and Robin Hirsch [25]. This problem has invoked extensive amount of research, and is still a very active part of research in algebraic logic.

Let us try to make the problem a little bit more precise and tangible. Towards this end, let $RCA_\alpha$ be the class of $CA_\alpha$'s isomorphic to subdirect products of $Cs_\alpha$'s. $RCA_\alpha$ turns out to be a variety. The class $RCA_\alpha$ is indeed a plausible "natural" candidate for substituting boolean set algebras in the quest of a Representation Theorem for cylindric algebras, analogously to that of Stone. It is easily seen that every cylindric set algebra of given dimension $\alpha < \omega$ is simple (has no proper congruences) and therefore subdirectly (and directly) indecomposable in the sense of the general theory of algebras. Hence when discussing the problem as to which $CA_\alpha$'s are isomorphic to cylindric set algebras, it is natural to restrict ourselves to subdirectly indecomposable algebras.

On the other hand, as a consequence of a classical theorem of Birkhoff, every $CA_\alpha$ is isomorphic to a subdirect product of subdirectly indecomposable $CA_\alpha$'s. Therefore we are naturally led to the problem of characterizing those $CA_\alpha$'s which are isomorphic to subdirect products of set algebras. Henkin, Monk and Tarski declare that these are the *representable* algebras, thus the notation $RCA_\alpha$. $RCA_\alpha$ consists of the standard models so to speak. The definition of representability, without any change in its formulation, is extended to algebras of infinite dimension.[1] Members of $RCA_\alpha$ can be still represented as algebras consisting of genuine $\alpha$-ary relations over a disjoint union of cartesian squares, the class consisting of all such algebras is denoted by $Gs_\alpha$, with $Gs$ standing for *generalized set algebras.* Generalized set algebras thus differ from the ordinary cylindric set algebras in one respect only: the unit of the algebra i.e. the $\alpha$ dimensional cartesian space ${}^\alpha U$ is replaced everywhere in their construction by any set which is a disjoint union of arbitrary many pairwise disjoint cartesian spaces of the same dimension. The class of generalized cylindric set algebras, just as that of ordinary cylindric set algebras, has many features which make it well qualified for representing $CA_\alpha$. The construction of the algebras in this (bigger) class retains its concrete character, all the fundamental operations and distinguished elements are unambiguously defined in set-theoretic terms, and the definitions are uniform over the whole class; geometric intuition underlying the construction gives us good insight into the structures of the algebras. Thus there is (geometric) justification that $RCA_\alpha$ consists of the standard models of $CA$-theory. Its members consist of genuine $\alpha$-ary relations. Now, in elementary terms the definition of $RCA_\alpha$ runs as follows: $\mathcal{A}$ is representable if and only if for every non -zero $x \in A$ there is a homomorphism $h$ from $\mathcal{A}$ onto a $Cs_\alpha$ such that $h(x) \neq 0$. Thus Theorem 2 is equivalent to the statement that every locally finite $CA_\omega$ is representable. But it soon transpired that the $CA$ axioms (originating from the (complete) axiomatization of locally finite algebras) do not exhaustively generate all valid principles governing $\alpha$-ary relations, when $\alpha > 1$. More precisely, for $\alpha > 1$, $RCA_\alpha$ is properly contained in $CA_\alpha$. $CA_\alpha$, for $\alpha > 1$, is only an *approximation* of $RCA_\alpha$. Tarski proved that $RCA_\alpha$ is a variety. Henkin proved that $RCA_2$ is finitely axiomatizable. However for $\alpha > 2$, the class $RCA_\alpha$ cannot be axiomatized by a finite schema of equations analogous to that axiomatizing $CA_\alpha$, a classical result of Monk [38] to be recalled below. Furthermore, there is an unavoidable and inevitable degree of complexity to any (potential) axiomatization of $RCA_\alpha$, as shown by Andréka [1] for any $\alpha > 2$. The *Finitization Problem* is thus the attempt to circumvent or sidestep such complexity.

*If we look at $RCA_\alpha$ as the standard models to which the $CA_\alpha$'s aspire, the Finitization Problem can thus be rephrased as the attempt to capture the essence of the standard models by thorough "finitary" means, or else find other broader comprehensible classes of "standard models" that are sufficiently concrete and tangible and most important of all would exhaust the class $CA_\alpha$, or at worst*

[1] In this case, however, an intuitive justification is less clear since cylindric set algebras of infinite dimension are not in general subdirectly indecomposable. In fact, for $\alpha \geq \omega$ no intrinsic property is known which singles out the algebras isomorphic to set algebras among all representable $CA_\alpha$'s, as opposed to the finite dimensional case where such algebras can be intrinsically characterized by the property of being simple.

*possibly a slightly smaller class i.e. a variety that is finitely axiomatizable (by equations) over $CA_\alpha$.*

**Possible solutions, the feedback between algebraic logic, modal logic, finite combinatorics and games.** Several different strategies to get round the obstacle of the non-finite axiomatizability of the class of representable algebras were evolved.

(1) One promulgated by Tarski especially was to find elegant intrinsic conditions for representability. For example, certain comprehensible subclasses of abstractly defined $CA_\alpha$'s turn out to be representable. In this connection, examples include locally finite, dimension complemented, semisimple and diagonal algebras of infinite dimension, cf. [20] Theorem 3.2.11. Another sample of such results in this direction is the classical result of Henkin and Tarski, formulated as Thm 3.2.14 in [20], that states that any atomic $CA_\alpha$ whose atoms are rectangular is representable. This was strengthened by Andréka et all [4] by looking at dense subsets consisting of rectangular elements, that are not necessarily atoms and Venema [58] extended this result to the diagonal free case. This approach of finding simple intrinsic sufficient conditions for representability has continued to the present, and now forms an extensive field [40].

(2) Another strategy of attacking the Finitization Problem is to define variants of $RCA_\alpha$, $\alpha > 2$ that are finitely axiomatizable and are still adequate for algebraizing first order logic. Such an approach originates with William Craig [11], and is further pursued by Sain [45], [46], Simon [50] and Sayed Ahmed [56]. Here the desired class of algebras is looked for in the uncountably many reducts of Halmos Polyadic algebras, whose equational theory turns out to be extremely complex from the recursive point of view as shown by Németi and Sági [42]. The reasoning here is that maybe the negative results we already mentioned are merely a historical accident resulting from the particular (far from unique) choice of extra non-boolean operations, namely the cylindrifications and diagonal elements. This approach typically involves changing the signature of $CA_\alpha$ by either taking reducts or expansions or perhaps even changing the signature altogether but bearing in mind that cylindrifications and diagonal elements are term definable in the new-signature and broadening the notion of representability allowing representation on arbitrary subsets of an $\alpha$-ary relation, rather than just (disjoint unions of) cartesian squares. The notion of representation here though is unaltered, it is essentially the same: *set algebras*, i.e algebras whose universes are sets of sequences with set theoretic

concrete operations. In particular, the set algebra is completely defined once its universe is determined.

(3) A classical result of Resek [20] p. 101 that is relevant in this connection shows that algebras satisfying the $CA$ axioms plus the-so called *merry go round identities*, for short, can be represented as *relativized* set algebras, a primary advance in development of the theory of $CA$'s, as indicated in the introduction of [20]. Resek's result was polished and "finitized" by Andréka and Thompson [8] where they replaced the infinite $MGR$ by a finite schema, but, in the process, had to modify slightly the notion of representability. This approach of broadening the permissable units is referred to in the literature as *relativization* or the *non-square* approach. The former terminolgy comes from the Budapest group [35], while the second comes from the Amsterdam group, in fact it is due to Venema [57]. Relativization might involve adding new operations that become no longer term definable after relativization, like for example the difference operator [57]. This approach is related to *dynamic Logic*, cf. [2]. Modalizing set algebras yield variants of the $n$-variable fragment of first order logic differing from the classical Tarskian view because the unit $W$ of the set algebra in question may not be of the form of a "square" ${}^nU$, but merely a subset thereof. These mutant logics are under intensive study at the present time and we cite [2] and [36] as sources.

(4) To get positive results (i.e. finite axiomatizations of the class of representable algebras), one changes the ontology, i.e. the underlying set theory. This is done by dropping the axiom of foundation, and adopting an anti-foundation axiom saying that non-well-founded sets exist. This approach was started by a question of Maddux answered by Németi and his co-workers. In this context, Németi proves that solutions to (several versions of) the finitization problem is independent from Aczel set theory, i.e. $ZF$ (Zermelo Fraenkel set theory) minus foundation, [47] and [49], which adds a set-theoretic dimension to the problem. If anything, this suggests the richness of the problem. In particular, it is proved in [29] that the finitization problem receives a positive solution in Boffa's set theory (which is, roughly, "$ZF$ without Foundation + Boffa's anti-foundation axiom"). One of the algebras for which Németi proves a positive solution, i.e. finite axiomatizability of the true set algebras, are fork algebras suggested by e.g. Haeberer et al [15]. (These algebras are definitionally equivalent with Tarski's quasi-projective relation algebras in e.g. [59].) Sági [48] also obtains positive solutions for the representation problem in non-well-founded set theories. Among others, he proves such results for the cylindric algebraic version $CA^{\uparrow}$ of the above mentioned expansion of relation algebras. This class $CA^{\uparrow}$ of directed cylindric algebras was introduced and

studied in [41], where it turns out that $CA^{\uparrow}$ can be applied to some higher order logics, cf. also Sági [49].

(5) Another more adventurous approach is to "stay inside" $ZFC$ and so to speak the "$CA$-$RCA$ infinite discrepancy" and to try to capture the essence of (the equations holding in) $RCA_\alpha$ in as simple a manner as possible. It is not hard to show that the set of equations holding in $RCA_\alpha$ for any $\alpha$ is recursively enumerable, and indeed, using a well-known trick of Craig, several (recursive) axiomatizations of $RCA_\alpha$ exist in the literature, the first such axiomatization originating with Monk, building on work of McKenzie. Robinson's finite forcing in Model theory proves extremely potent here as shown by Hirsch and Hodkinson. The very powerful recent approach of synthesizing axioms by games due to Hirsch and Hodkinson [25], building on work of Lyndon [30], is a typical instance of giving an intrinsic characterization of the class of representable algebras by providing an *explicit* axiomatization of this class in a step-by step fashion. This approach is of a very wide scope; using Robinson's finite forcing in the form of games, Hirsch and Hodkinson [25] axiomatize, not only the variety of representable algebras, but almost all pseudo-elementary classes existing in the literature, an indeed remarkable achievement. It turns out, as discovered by Hirsch and Hodkinson, that being a representation of an algebra can be described in a first order 2-sorted language. The first sort of a model of this *defining theory* is the algebra itself, while the second sort is a representation of it. The defining theory specifies the relation between the two, and its axioms depend on what kind of representation we are considering. Thus the representable algebras are those models of the first sort of the defining theory with the second sort providing the representation. The class of all structures that arise as the first sort of a model of a two sorted first order theory is a venerable old notion in Model theory introduced by Maltsev in the forties of the 20th century, and since studied by Makkai and others. It is known as a pseudo-elementary class. We mean here a $PC_\Delta$ class in the sense of [28] but expressed in a two sorted language. The term pseudo elementary class strictly means $PC_\Delta$ when the second sort is empty, but the two notions were proved to be equivalent by Makkai. Any elementary class is pseudo elementary, but the converse is not true; the class of $\alpha$ dimensional neat reducts of $\beta$ dimensional cylindric algebras for $1 < \alpha < \beta$ is an example [52] and [55]. Another is the class of strongly representable atom structures and the completely representable ones as proved by Hirsch and Hodkinson in [23]. Many classes in algebraic logic can be seen as pseudo-elementary classes. The defining theory is usually finite and simple and essentially recursively enumerable, since we expect that a Turing machine can write down what we mean by a representation. A fairly but not completely general *definition* of

the notion of representation is just the second sort of a model of a two sorted (perhaps recursively enumerable) first order theory, the first sort of the theory being the algebra. Now model theoretic forcing as seen in Hodges in [28] and indeed the classical Completeness Theorem of Henkin, and his Neat Embedding Theorem to be recalled below, typically involves constructing a model of a first order theory by a *game.* The game builds the model step-by-step, elements of the model being produced by the second player called $\exists$, in his response to the first player's criticism called $\forall$ [22]. The approach of Hirsch and Hodkinson is basically to combine the forcing games with the pseudo elementary approach mentioned above to representations to build the second sort of a model of the defining theory whose first sort is already fixed to be the algebra whose representability is at issue. Taking the defining theory of the pseudo elementary class to be given, this defines the notion of representation to be axiomatized.

(6) Another approach initiated by van Benthem and Venema consists of viewing the class of cylindric set algebras as complex algebras of Kripke frames that have the same signature as atom structures of cylindric algebras [57], thus opening an avenue to techniques and methods coming from modal logic. This typically involves introducing Gabbay-style rules on the logic side. These extremely liberal Gabbay-style inference systems correspond to classes that are inductive i.e. axiomatized by $\forall\exists$ -formulas. An example of such classes is the class of rectangularly dense cylindric algebras, [4] and [58]. We should mention that this approach is an instance, or rather, an application of the triple duality, in the sense of Goldblatt [17], existing between abstract modal logic, Kripke frames or relational structures, and boolean algebras with operators. In this connection, we refer to the article by Venema in [36] for explaining and applying this duality to relation algebras and to [3], [35], [57], for further elaboration on this duality.

(7) There has been work in changing the notion of representation completely, i.e by *representing cylindric algebras using concrete structures other than set algebras.* This includes using quasigroups cf. [20] or sheaves [10]. Andréka and Givant use groups to represent $CA_3$'s and relation algebras. Simon [51] in an amazing result proved that any abstract 3-dimensional cylindric algebra satisfying the $MGR$ can be obtained from a $Cs_3$ by the so-called methods of *twisting and dilation* studied in [20] pp. 86-91 which adds to our understanding as to the distance between the abstract notion of cylindric algebra and its concrete one, at least in the case of dimension 3. However, Simon had to broaden Henkin's notion of dilation to exhaust the class $CA_3$. The analogous problem for higher dimension is an intriguing open problem. This problem, due to Leon Henkin is,

**Open Problem 2(Henkin-Simon)** *Can every $CA_n$, $3 < n < \omega$ be obtained from a $Cs_n$ by a finite sequence of the operations of relativization, twisting (as defined in [20]) and dilation (as defined in [51]) - not necessarily in this order?*

A form of the Representation Problem is to describe properties of the class $RCA_\alpha$ and try to give a useful characterization of it in *abstract* terms.

**Back to classical algebraic logic; Neat Reducts** An old result of Henkin which gives an abstract sufficient and necessary condition for representability fits here. An algebra $\mathcal{A}$ is in $RCA_\alpha$ if and only if for every $\beta > \alpha$, it can be embedded as a *neat subreduct* in some cylindric algebra of dimension $\beta$ - equivalently using ultraproducts - into an algebra in $\omega$ extra dimensions. This Theorem is referred to as the Neat Embedding Theorem of Henkin, or $NET$ for short. For $\alpha < \beta$, $Nr_\alpha CA_\beta$ stands for the class of $\alpha$ neat reducts of algebras in $CA_\beta$ as defined in [20] Definition 2.6.28. In symbols, for any $\alpha$ the $NET$ can be expressed more succinctly as:

$$RCA_\alpha = SNr_\alpha CA_{\alpha+\omega}.$$

Here $S$ is the operation of forming subalgebras. Infinity manifests itself in the above in the form of $\omega$, and it does so essentially in the case when $\alpha > 2$, in the sense that if $\mathcal{A}$ neatly embeds into an algebra in finitely many extra dimensions, then it might not be representable, as shown by Monk. All $\omega$ extra dimensions are needed for representability, one could not truncate $\omega$ to any finite ordinal. The $\omega$ extra dimensions play the role of added constants or witnesses in Henkin's classical Completeness proof. Therefore it is no coincidence that variations on the $NET$ lead to metalogical results concerning interpolation and omitting types for the corresponding logic; such results can be proved by Henkin's methods of constructing models out of constants [54] and [56]. Conversely, the $NET$ (conjuncted with some form of Ramsey's theorem) has been applied to prove the following classical algebraic result of Monk that established the "infinite distance" between $CA$'s and $RCA$'s. Monk's result marked a turning point in the development of the subject, and is considered one of the most, if not *the* most, important Model-theoretic result concerning cylindric algebras:

**Theorem 4.** *$rmLetomegan2andminomega$. Then $RCA_n$ is properly contained in $Nr_nCA_{n+m}$. In particular, $RCA_n$ is properly contained in $SNr_nCA_{n+m}$.*

Monk used Ramsey's Theorem to construct for each $m \in \omega$ and $2 < n < \omega$, an algebra $A_m \in Nr_nCA_{n+m}$ that is not representable. The ultraproduct of the $A_m$'s $m \in \omega$, constructed by Monk (relative to any non-principal ultrafilter on

$\omega$) is in $SNr_nCA_{n+\omega}$, hence is representable. Using elementary model theory, it follows thus that the class $RCA_n$, for $\omega > n > 2$, is not finitely axiomatizable.

The $A_m$'s are referred to in the literature as Monk's or Maddux's algebras. Both authors used them. The key idea of the construction of a Monk's algebra is not so hard. Such algebras are finite, hence atomic, more precisely their boolean reduct is atomic. The atoms are given colours, and cylindrifications and diagonals are defined by stating that monochromatic triangles are inconsistent. If a Monk's algebra has many more atoms than colours, it follows from Ramsey's Theorem that any representation of the algebra must contain a monochromatic triangle, so the algebra is not representable. We note that Monk's algebras established a very interesting connection between finite combinatorics and algebraic logic a recurrent theme in algebraic logic. A recent use - establishing this link - of Monk's algebras with a powerful combinatorial result of Erdős has been used to show that the class of the so called strongly representable atom structures of relation algebras and 3 dimensional representable cylindric algebras is not elementary [25].

## Refinements of Monk's result

Had it been otherwise, i.e. if for $\omega > n > 2$, $RCA_n$ had turned out finitely axiomatizable by a finite set of equations $\Sigma$ say, then this $\Sigma$ would have been probably taken as the standard axiomatization of $CA_n$. This turned out not to be the case. As it seemed, the hopes of workers over a hundred years starting with De Morgan and culminating in Tarski's work to produce a (simple, elegant, or at least finite) set of algebraic properties - or in modern terminology - equations that captured exactly the true properties of $n$-ary relations for $\omega > n > 2$ were shattered by Monk's result. This impasse is still invoking extensive research till the present day, in essentially two conflicting (but complementary) forms. One form, which we already discussed is to try to circumvent this negative non-finite axiomatizability result. The other form is to sharpen it. To understand the "essence" of representable algebras, one often deals with the non-representable ones, the "distorted images" so to speak. Simon's result in [51], of "representing" non-representable algebras, seems to point out that this distortion is, after all, not completely chaotic. This is similar to studying non-standard models of arithmetic. Indeed, Monk's negative result - as far as non-finite axiomatizability is concerned - stated in Theorem 5 was refined and strengthened by many authors in many directions. Biro [9] proves that $RCA_n$, $\omega > n > 2$, remains non-finitely axiomatizable if we add finitely many *first order definable* operations. Andréka [1] building on work of Jónsson for relation algebras, proves the same result in case we add other "kinds" of operations like for example *modalities*, i.e. operations distributing over the boolean join, as long as the added operations are finitely many. While Biro's result excludes axiomatizations by a finite set of equations, Andréka's, on the other hand, excludes axiomatizations involving universal formulas in which only finitely many variables occur. Sági, building on work of Lyndon, addresses the most general formulation of the problem showing that the Finitization Problem cannot be solved by adding finitely many *permutation invariant* operations in the sense of Tarski-Givant [59], as long as one hopes for particular (universal) axiomatizations involving only finitely many variables. An important result in [49] is reducing the Finitization Problem addressing certain expansions of relation algebras and for that matter cylindric algebras to working inside the class of relation and cylindric algebras, respectively. Madarász addresses the case when the (finitely many) added operations are binary and $L^3_{\infty\omega}$ definable. One general form of the Finitization Problem for both cylindric algebras and relation algebras is to the best of our knowledge still open.

**Open Problem 3: A more tangible form of the Finitization Problem (Tarski-Givant-Henkin-Monk-Maddux-Németi)**

*Can we expand the language of $RCA_n$, $\omega > n > 2$ by finitely many* **permutation invariant** *operations so that the interpretation of these newly added operations in the resulting class of algebras is still of a concrete set-theoretic nature, and the resulting class becomes a finitely axiomatizable variety or quasi-variety.*

Solutions do exist for the infinite dimensional case [45] and [56]. The requirement of permutation invariance defined in the introduction of [45] here is crucial for it corresponds to the (meta-logical) fact that isomorphic models satisfy the same formulas, a basic requirement in abstract model theory. This requirement, seems to keep the problem on the tough side [50]. Without this requirement there are rather easy solutions to the Finitization Problem for the finite dimensional case due to Biró, Maddux, and Simon, [9], [34], and [51].

No matter what, there is an unavoidable and inevitable degree of complexity to any (potential) axiomatization of $RCA_\alpha$, as shown by Andréka [1] for any $\alpha > 2$. Venema further shows that such varieties cannot be axiomatized by the so-called *Sahlqvist* equations. Hodkinson and Mikulás show that such negative non-finite axiomatizability results cannot be avoided when we pass to certain reducts, complementing Andréka's results in [1] addressing certain expansions. Hodkinson and Venema show that no axiomatization can be canonical. All this suggests that the notion of representability for cylindric algebras is unruly, subtle and difficult to capture. On the other hand, the Finitization Problem (in its algebraic form) is indeed non-trivial as the following quote from Henkin, Monk and Tarski in [20] might suggest:

"*An outstanding open problem in cylindric algebra theory is that of exhibiting a class of cylindric algebras which contains an isomorphic image of every cylindric algebra and hence serves to represent the class of all these algebras, and which* *at the same time is sufficiently concrete and simply constructed to qualify for this purpose from an intuitive point of view. It is by no means certain or even highly plausible that a satisfactory solution of this problem will ever be found! (our exclamation mark)."*

Fortunately today the situation seems to be not as drastic! However, it still involves some open questions. Indeed the three open questions we highlighted are related to the Finitization Problem. The First problem concerns the *Finitization* of Set Theory, while the second and third address the question as to

"**How far are the representable algebras - consisting of $n$-ary relations $n > 2$ - from a simple finite axiomatization"?**

# 1 Neat reducts, some recent results.

In a paper written by J. D. Monk in 1991, but published in the Logic Journal of IGPL in 2000, [39], Hajnal Andréka writes the final survey section on the subject, to update the article. It is clear from Andréka's section that among the important problems that were still open in Algebraic logic are those in Pigozzi's landmark paper on amalgamation [44] and two problems, both on neat reducts. Maddux's conjecture formulated in [33] is important, too. This conjecture was confirmed by Hirsch and Hodkinson. In [33], Maddux introduces the notion of relation algebra of every dimension. The class of relation algebras of dimension $n$ is denoted by $RA_n$. In the case of $n = 3$, the class $RA_3$ coincides with Maddux's semi-associative relation algebras. $RA_4$ is just the class of relation algebras and $RA_\omega$ is the class of representable relation algebras. In between $RA_n \subseteq RA_m$ for $n < m$ forms a (strict) hierarchy approaching the class of representable relation algebras. Maddux proves that for $3 \leq n < m < \omega$ $RA_n$ and $RA_m$ are distinct. What remained unresolved was whether $RA_{n+1}$ is finitely axiomatizable over $RA_n$, for $4 \leq n < \omega$. Maddux conjectures that the answer is no, i.e. $RA_{n+1}$ is not finitely axiomatizable over $RA_n$. Hirsch and Hodkinson confirm this conjecture [24], using the so called *Rainbow construction.* The Rainbow construction is an extremely powerful technique in providing counterexamples and has solved major open problems in the field. For example the Rainbow construction was used to show that the class of completely representable relation and cylindric algebras is not first order axiomatizable [23]. We refer the reader to [25] for many other applications of the Rainbow constructions.

All of Pigozzi's question are solved in [32]. The two problems on neat reducts are the consecutive problems 2.11 and 2.12 posed by Henkin, Monk, Tarski in [20]. Hirsch and Hodkinson solve problem 2.12. They show that for $2 < n < \omega$ and $k \in \omega$, $SNr_nCA_{n+k+1}$ is a proper subclass of $SNr_nCA_{n+k}$. Thus the decreasing sequence $CA_n \supseteq SNr_nCA_{n+1} \supseteq SNr_nCA_{n+2} \cdots$ converging to $RCA_n$ is not only, not eventually constant, as proved by Monk, but is in fact *strictly* decreasing. This shows that for every $3 \leq n < m$ there is a formula built up of $n$ variables that can be proved using $m+1$ variables, but cannot be proved using $m$ variables. Historically, one motivation for defining the class $SNr_nCA_m$, of all subalgebras of $n$-dimensional neat reducts of $m$-dimensional cylindric algebras for $n < m < \omega$, was connected to proof theory of first order logic. Algebraic logic, via the notion of neat reducts, is a useful tool for analyzing the number of variables needed in proofs.

Problem 2.11 in [20], on the other hand, asks whether the class $Nr_\alpha CA_\beta$ for $1 < \alpha < \beta$ is closed under forming subalgebras. Proved not to be closed under subalgebras by Németi, the question as to whether it is perhaps closed under elementary subalgebras appears as problem 4.4 in the monograph [20]. Németi conjectures that for $1 < \alpha < \beta$, the class $Nr_\alpha CA_\beta$ is not closed under elementary subalgebras, hence is not elementary, i.e. cannot be axiomatized by any set of first order axioms. In [52] Németi's conjecture is confirmed. In [53] it is generalized to other algebras. The notion of neat reducts is an old venerable notion. But it often happens that an unexpected viewpoint yields new insight. Indeed, the repercussions of the very seemingly innocent fact that the class of neat reducts is not closed under forming subalgebras turns out to be enormous. Indeed Henkin proves that an algebra is representable if it neatly embeds into another cylindric algebra in $\omega$ extra dimensions. In [56] and [54] it is shown that for a class of representable algebras to have the strong amalgamation property or to consist exclusively of completely representable algebras, each algebra in this class should embed neatly into another algebra in $\omega$ extra dimensions in a special way. This is further connected to metamathematical notions like interpolation and omitting types for variants of first order logics.

## 2 To sum up

In the history of the development of Algebraic logic, there have been, we believe, three important turning points. One as mentioned before was in the nineteenth century when De Morgan set out to find the laws governing binary relations launching thereby an investigation into algebras of relations. This approach was taken up by Pierce and Schröder and was put on a modern footing by Tarski.

This brings us to the other turning point. This was in the forties when Tarski and his school in Berkeley laid down the axioms for relation algebras, and in the fifties when the modern approach to algebraization of first order logic via cylindric algebras was initiated. At the same time Halmos initiated work on polyadic algebras. By 1991 when Andréka Németi and Monk edited a volume on Algebraic Logic, it was thought that a lot of the central problems were fairly understood.

Quoting Monk: “The relationships between relation algebras, cylindric algebras, and polyadic algebras is well understood. The representation theory has been carefully studied, and the relationships between various types of representable algebras such as cylindric set algebras, has been thoroughly investigated”.

Today it seems that we are at a third turning point. The subject is taking new turns that is very likely to revive it. Hirsch and Hodkinson introduced Robinson’s finite forcing in the form of games in their recent book, showing the power and elegance of game-theoretic techniques in model theory. Indeed as they illustrate in a profusion of ways, games lie very close to some of the most general and fundamental notions of model theory such as axiomatizability, step-by-step constructions and even satisfaction of formulas.

Quoting Hodges “It often happens that an unexpected view point yield dividends, and today relation algebras take their proper place as an important tool of theoretical computer science among other things.”

Following Tarski and Suppes, Andréka, Madarász and Németi applied algebraic logic to general relativity [6] an indeed fascinating application of algebraic logic to geometry and physics. It therefore seems that the ideas of Boole have not borne their full fruit, yet!

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