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Goguen Categories

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Abstract. This paper is a survey of the theory of Goguen categories which establishes a suitable categorical description of $\mathcal{L}$-fuzzy relations, i.e., of relations taking values from an arbitrary complete Brouwerian lattice $\mathcal{L}$ instead of the unit interval $[0, 1]$ of the real numbers. In particular, we concentrate on representability, the existence of crisp versions of several categorical constructions, and operations derived from suitable binary functions on the underlying lattice of scalar elements, i.e., on the abstract counterpart of $\mathcal{L}$.

1 Introduction

In a wide variety of problems one has to treat uncertain or incomplete information. Some kind of exact science is needed to describe and understand existing methods and to develop new attempts. Especially in applications of computer science, this is a fundamental problem. To handle such kind of information, Zadeh [26] introduced the concept of fuzzy sets and relations. In contrast to usual sets, fuzzy sets are characterized by a membership relation taking its values from the unit interval $[0, 1]$ of the real numbers. After its introduction in 1965 the theory of fuzzy sets and relations was ranked to be some exotic field of research. The success during the last years with even consumer products involving fuzzy methods causes a rapidly growing interest of engineers and computer scientists in this field. Nevertheless, Goguen [7] generalized this concept in 1967 to $\mathcal{L}$-fuzzy sets and relations for an arbitrary complete Brouwerian lattice $\mathcal{L}$ instead of the unit interval $[0, 1]$ of the real numbers. He described one of his motivating examples as follows:

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> A housewife faces a fairly typical optimization problem in her grocery shopping: she must select among all possible grocery bundles one that meets as well as several criteria of optimality, such as cost, nutritional value, quality, and variety. The *partial ordering* of the bundles is an intrinsic quality of this problem. (Goguen [7], 1967)

It seems to be unnatural in the example above to describe the criteria of optimality by a linear ordering as the unit interval, e.g., why should the nutritional value of a given product be described by 0.6 (instead of 0.65, or any other value from $[0, 1]$) and why should a product with a high nutritional value be better than a product with high quality since those criteria are usually incomparable?

One important notion within fuzzy theory is 0-1 crispness. The class of 0-1 crisp fuzzy sets or relations may be seen as the subclass of regular sets or relations within the fuzzy world, i.e., they are described by the property that their characteristic function supplies either the least element 0 or the greatest element 1 of the unit interval $[0, 1]$ or the complete Brouwerian lattice $\mathcal{L}$. Especially in applications, this notion is fundamental. For example, in fuzzy decision theory the basic problem is to select a specific element from a fuzzy set of alternatives. Therefore, several cuts are used $[3, 11]$. Basically, an α-cut of a fuzzy set M is a set N such that an element x is in N if and only if x is in M with a degree $\geq \alpha$. Analogously, an α-cut of a fuzzy relation R is a crisp relation S such that a pair of elements is related in S if and only if they are related in R with a degree $\geq \alpha$. Some variants of this notion may also be used. By definition, these cut operations are strongly connected to the notion of crispness. In particular, using the notion of crispness, one may define cut operations, and a cut operation naturally implies a notion of crispness. In the development of fuzzy controllers the notion of crispness is also fundamental. Usually the output of the controller has to be a 0-1 crisp value since it is used to control some non-fuzzy physical or software system. Therefore, a procedure called defuzzification is applied to transform the fuzzy output into some 0-1 crisp value. This list of examples may be continued. To sum it up, a convenient theory for $\mathcal{L}$-fuzzy relations should be able to express the notion of crispness.

Today, fuzzy theory as well as its application is usually formulated as a variation of set theory or some kind of many-valued logic (e.g., c.f. [6]). Although a lot of algebraic laws are developed, these formalizations are not algebraic themselves. But an algebraic description would have several advantages. Applications of fuzzy theory may be described by simple terms in this language. In this way, we get in some sense a denotational semantics of the application and hence a mathematical theory to reason about notions like correctness and so on. One may prove such properties using the calculus of the algebraic theory. Furthermore, a denotational semantics may be used to get a prototype of the application.

On the other hand, the calculus of binary relations has been investigated since the middle of the nineteenth century as an algebraic theory for logic and set theory [20, 21]. A first adequate development of such algebras was given by de Morgan and Peirce. Their work has been taken up and systematically extended by Schröder in [17]. More than 40 years later, Tarski started with [18] the exhaustive study of relation algebras, and more generally, Boolean algebras with operators [19].

The papers above deal with relational algebras presented in their classical form. Elements of such algebras might be called *quadratic* or *homogeneous*; relations over a fixed universe. Usually a relation acts between two different kinds of objects, e.g., between boys and girls. Therefore, a variant of the theory of binary relations has evolved that treats relations as *heterogeneous* or *rectangular*. A convenient framework to describe such kind of typing is given by category theory [4, 13, 15, 16].

There are some attempts to extend the calculus of relations to the fuzzy world. In [10] the concept of fuzzy relation algebras was introduced as an algebraic formalization of fuzzy relations with sup-min composition. These algebras are equipped with a semi-scalar multiplication, i.e., an operation mapping an element from $[0, 1]$ and a fuzzy relation to a fuzzy relation. In the standard model this is done by componentwise multiplication of the real values. Fuzzy relation algebras and their categorical counterpart [5], so-called Zadeh categories, constitute a convenient algebraic theory for fuzzy relations. Using the semi-scalar multiplication it is also possible to characterize 0-1 crisp relations. Unfortunately, there is no way to extend or modify this approach for $\mathcal{L}$-fuzzy relations since for an arbitrary complete Brouwerian lattice such a semi-scalar multiplication may not exist.

Another approach is based on Dedekind categories introduced in [13]. It was shown that the class of $\mathcal{L}$-fuzzy relations constitutes such a category. Unfortunately, the notion of 0-1 crispness causes some problems. Using the notion of scalar elements, i.e., partial identities corresponding to the lattice $\mathcal{L}$, several notions of crispness in an arbitrary Dedekind category were introduced in [5, 9]. It was shown that the notion of s-crispness as well as the notion of l-crispness coincides with 0-1 crispness under an assumption concerning the underlying lattice. This assumption is fulfilled by all linear orderings, e.g., the unit interval. Unfortunately, it was also shown that both classes of crisp relations are trivial if the underlying lattice is a Boolean lattice.

This paper is a survey of the theory of Goguen categories which establishes a suitable categorical description of $\mathcal{L}$-fuzzy relations. We will just give the corresponding theorems without a proof. The corresponding proofs can be found in [23–25]. In Section 2 we briefly introduce the notion of a Dedekind category, and in Section 3 the theory of (concrete) $\mathcal{L}$-fuzzy relations. Then, in Section 3, we introduce the notion of a Goguen category and state some its basic properties. The following three sections (Section 5, 6 and 7) focus on the representability of Goguen categories, the existence of crisp versions of several categorical constructions as relational products, relational sums and subobjects, and operations derived from suitable binary functions on the underlying lattice of scalar elements, i.e., on the abstract counterpart of $\mathcal{L}$.

We assume that the reader is familiar with the basic concepts of category theory and allegories [1, 4, 12].

2 Dedekind Categories

Throughout this paper, we use the following notations. To indicate that a morphism R of a category $\mathcal{R}$ has source A and target B we write $R : A \to B$. The collection of all morphisms $R : A \to B$ is denoted by $\mathcal{R}[A, B]$ and the composition of a morphism $R : A \to B$ followed by a morphism $S : B \to C$ by $R; S$. Last but not least, the identity morphism on A is denoted by $\mathbb{I}_A$.

In this section we recall some fundamentals on Dedekind categories [13, 14]. This kind of categories are called locally complete division allegories in [4].

Definition 1. *A Dedekind category $\mathcal{R}$ is a category satisfying the following:*

1. *For all objects A and B the collection $\mathcal{R}[A, B]$ is a complete Brouwerian lattice. The elements of $\mathcal{R}[A, B]$ are called (abstract) relations (with source A and target B). Meet, join, the induced ordering, the least and the greatest element are denoted by $\sqcap, \sqcup, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$, respectively.*
2. *There is a monotone operation $\smile$ (called converse) mapping a relation $Q : A \to B$ to $Q^\smile : B \to A$ such that for all relations $Q : A \to B$ and $R : B \to C$ the following holds: $(Q; R)^\smile = R^\smile; Q^\smile$ and $(Q^\smile)^\smile = Q$.*
3. *For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law $(Q; R) \sqcap S \sqsubseteq Q; (R \sqcap (Q^\smile; S))$ holds.*
4. *For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) such that for all $X : A \to B$ the following holds: $X; R \sqsubseteq S \iff X \sqsubseteq S/R$.*

Notice, that by convention composition binds more tightly than meet. Therefore, Axiom 3 may be written as $Q; R \sqcap S \sqsubseteq Q; (R \sqcap Q^\smile; S)$.

Corresponding to the left residual, we define the right residual by $Q\backslash R := (R^\smile/Q^\smile)^\smile$. This relation is characterized by $Q; Y \sqsubseteq R \iff Y \sqsubseteq Q\backslash R$.

Because the so-called Tarski rule

$$R \neq \perp\!\!\!\perp_{AB} \implies \top\!\!\!\top_{CA}; R; \top\!\!\!\top_{BD} = \top\!\!\!\top_{CD} \quad \text{for all objects } C \text{ and } D$$

is equivalent to a generalized version of the notion of simplicity known from universal algebra, we call a Dedekind category simple iff the Tarski rule is valid.

A (simple) Dedekind category $\mathcal{R}$ is called representable iff there is an injective homomorphism from $\mathcal{R}$ to the category of sets and binary relations.

An important class of relations is given by mappings.

Definition 2. *Let $Q : A \to B$ be a relation. Then we call*

1. *Q univalent iff $Q^{\smile}; Q \sqsubseteq \mathbb{I}_B$,*
2. *Q total iff $\mathbb{I}_A \sqsubseteq Q; Q^{\smile}$ or equivalently iff $Q;\mathbb{T}_{BC} = \mathbb{T}_{AC}$ for all objects C,*
3. *Q a map iff Q is univalent and total,*
4. *Q injective iff $Q^{\smile}$ is univalent,*
5. *Q surjective iff $Q^{\smile}$ is total,*
6. *Q an isomorphism iff Q and $Q^{\smile}$ are mappings.*

Notice, that if Q is an isomorphism we have $Q^{\smile}; Q = \mathbb{I}_B$ and $Q; Q^{\smile} = \mathbb{I}_A$.

In some sense a relation of a Dedekind category may be seen as an $\mathcal{L}$-relation. The lattice $\mathcal{L}$ may equivalently be characterized by the ideal relations, i.e., a relation $J : A \to B$ satisfying $\mathbb{T}_{AA}; J;\mathbb{T}_{BB} = J$, or by the scalar relations.

Definition 3. *A relation $\alpha_A : A \to A$ is called a scalar on A iff $\alpha_A \sqsubseteq \mathbb{I}_A$ and $\mathbb{T}_{AA}; \alpha_A = \alpha_A;\mathbb{T}_{AA}$.*

We will denote the set of scalar relations in $\mathcal{R}$ on A by $\mathrm{Sc}_{\mathcal{R}}(A)$.

3 $\mathcal{L}$-fuzzy relations

As mentioned in the introduction $\mathcal{L}$-fuzzy relations are relations taking values from an arbitrary complete Brouwerian lattice $\mathcal{L}$ instead of the unit interval $[0, 1]$ of the real numbers.

Definition 4. *Let $\mathcal{L}$ be a complete Brouwerian lattice. Then the structure of $\mathcal{L}$-fuzzy relations is defined as follows:*

1. *The objects are sets.*
2. *A relation $Q : A \to B$ between two sets A and B is function from $A \times B$ to $\mathcal{L}$.*
3. *For $Q : A \to B$ and $R : B \to C$ composition is defined by*

$$(Q; R)(x, z) := \bigsqcup_{y \in B} Q(x, y) \sqcap R(y, z).$$

4. *For $Q : A \to B$ the converse is defined by $Q^{\smile}(x, y) := Q(y, x)$.*

5. *For $Q, S : A \to B$ join and meet are defined by*

$$(Q \sqcup S)(x,y) := Q(x,y) \sqcup S(x,y), \qquad (Q \sqcap S)(x,y) := Q(x,y) \sqcap S(x,y).$$

6. *The identity, zero and universal elements are defined by*

$$\mathbb{I}_A(x,y) := \begin{cases} 0 : x \neq y \\ 1 : x = y, \end{cases} \qquad \begin{array}{l} \perp\!\!\!\perp_{AB}(x,y) := 0, \\ \top\!\!\!\top_{AB}(x,y) := 1. \end{array}$$

The structure of $\mathcal{L}$-fuzzy relation is indeed a Dedekind category.

As mentioned in the introduction crispness is a fundamental notion within fuzzy theory. An $\mathcal{L}$-fuzzy relation Q is called *0-1 crisp* iff $Q(x,y) = 0$ or $Q(x,y) = 1$ for all x and y. Obviously, the set of 0-1 crisp relations is closed under all relation algebraic operations so that we may identify those relations with regular binary relations.

The scalar elements in the Dedekind category of $\mathcal{L}$-fuzzy relations are of the form

$$\alpha_A^u(x,y) = \begin{cases} u \text{ iff } \ x = y, \\ 0 \text{ else,} \end{cases}$$

with $u \in \mathcal{L}$. Obviously, the set of scalars on A is closed under arbitrary intersections and unions and is isomorphic to $\mathcal{L}$. This isomorphism is an isomorphism of complete Brouwerian lattices since it is surjective in respect to the set of scalars on A. Notice, that the last property is not true for arbitrary Dedekind categories.

A *u-cut* of an $\mathcal{L}$-fuzzy relation is defined as the following 0-1 crisp relation

$$R_u(x,y) := \begin{cases} 1 \text{ iff } \ R(x,y) \geq u, \\ 0 \text{ else.} \end{cases}$$

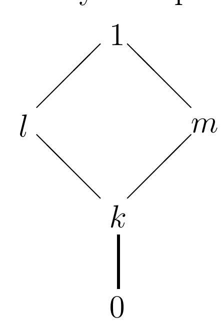
The special cut with 1 will be denoted by $R^{\downarrow}$. It is the greatest 0-1 crisp relation R contains. On the other hand, we may define

$$R^{\uparrow}(x,y) := \begin{cases} 1 \text{ iff } \ R(x,y) \neq 0 \\ 0 \text{ else.} \end{cases}$$

$R^{\uparrow}$ is the least 0-1 crisp relation containing R.

It is well-known that an $\mathcal{L}$-fuzzy relation R may be represented by the set of all its u-cuts. For example, let

$$R := \begin{pmatrix} 1 & k & l \\ 0 & k & m \\ 0 & 1 & l \end{pmatrix} \qquad \mathcal{L} :=$$



be given. The $\mathcal{L}$-fuzzy relation R is a relation on a set A with three elements, and the matrix representation above is similar to the representation of binary relations by Boolean matrices, e.g., the first element is related to the second element in R by a degree of k. Using this representation composition, for example, is computed by regular matrix multiplication using intersection (instead of multiplication) and union (instead of summation). Consider the set of u-cuts of R:

$$\underset{R_0}{\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}}\ \underset{R_k}{\begin{pmatrix}1&1&1\\0&1&1\\0&1&1\end{pmatrix}}\ \underset{R_l}{\begin{pmatrix}1&0&1\\0&0&0\\0&1&1\end{pmatrix}}\ \underset{R_m}{\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}}\ \underset{R_1}{\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\end{pmatrix}}$$

The following computation shows that the cuts above establish a representation of R.

$$\begin{aligned}\bigsqcup_{u\in\mathcal{L}} \alpha_A^u;R_u &= \alpha_A^0;R_0 \sqcup \alpha_A^k \sqcup \alpha_A^l;R_l \sqcup \alpha_A^m;R_m \sqcup \alpha_A^1;R_1\\
&= \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \sqcup \begin{pmatrix}k&k&k\\0&k&k\\0&k&k\end{pmatrix} \sqcup \begin{pmatrix}l&0&l\\0&0&0\\0&l&l\end{pmatrix} \sqcup \begin{pmatrix}m&0&0\\0&0&m\\0&m&0\end{pmatrix} \sqcup \begin{pmatrix}1&0&0\\0&0&0\\0&1&0\end{pmatrix}\\
&= \begin{pmatrix}1&k&l\\0&k&m\\0&1&l\end{pmatrix}\\
&= R\end{aligned}$$

The equation $R = \bigsqcup_{u\in\mathcal{L}} \alpha_A^u;R_u$ is valid for arbitrary $\mathcal{L}$-fuzzy relations R and is known as the α-cut theorem.

Now, the set $\{R_0, R_k, R_l, R_m, R_1\}$ may be represented by a function f from $\mathcal{L}$ to the set of crisp relations defined by

$$f(0) := R_0, \quad f(k) := R_k, \quad f(l) := R_l, \quad f(m) := R_m, \quad f(1) := R_1.$$

It is easy to verify that f is an antimorphism, i.e., that $f(\bigsqcup M) = \bigsqcap f(M)$ holds for all subsets M of $\mathcal{L}$. Consequently, the set of $\mathcal{L}$-fuzzy relations may be represented by the set of antimorphisms from $\mathcal{L}$ to the set of crisp relations.

We will study a similar representation of abstract Goguen Categories in Section 5.

4 Goguen Categories

In [23] a first order language for Dedekind categories is defined. This language is a two sorted language (relations and objects) based on the relational and the usual logical operations. Furthermore, it was shown that a relation $Q : A \to B$ fulfils a formula φ iff $f; Q; g$ fulfils φ where $f : A \to A$ and $g : B \to B$ are isomorphisms. Consider the following example:

Let $B_4 := \mathcal{P}(\{a, b\})$ be the power set of the set $\{a, b\}$, and $X = \{x\}$ and $Y = \{x, y\}$ be sets. Consider the $\mathcal{L}$-fuzzy relation $f : Y \to Y$ defined by

$$f := \begin{pmatrix} \{a\} & \{b\} \\ \{b\} & \{a\} \end{pmatrix},$$

and the 0-1 crisp relation $R : X \to Y$ defined by

$$R := \begin{pmatrix} \{a, b\} & \emptyset \end{pmatrix}.$$

A simple verification shows that f is an isomorphism. Now, suppose that φ is a formula expressing 0-1 crispness, i.e., a formula which is fulfilled exactly by 0-1 crisp relations. Then φ is fulfilled by R and by the lemma mentioned above by $\mathbb{I}; R; f$. But, this is a contradiction since

$$\mathbb{I}; R; f = \begin{pmatrix} \{a, b\} & \emptyset \end{pmatrix}; \begin{pmatrix} \{a\} & \{b\} \\ \{b\} & \{a\} \end{pmatrix} = \begin{pmatrix} \{a\} & \{b\} \end{pmatrix},$$

which shows that $\mathbb{I}; R; f$ is far from being 0-1 crisp.

The computation above shows that the theory of Dedekind categories is too weak to express crispness. This gives us the motivation to define an extended algebraic structure for $\mathcal{L}$-fuzziness. Our approach introduces an abstract version of the operations $R^{\downarrow}$ and $R^{\uparrow}$ mapping every relation R to the greatest 0-1 crisp relation R contains and to the least 0-1 crisp relation R is included in, respectively.

Definition 5. *A Goguen category $\mathcal{G}$ is a Dedekind category with $\mathbb{T}_{AB} \neq \mathbb{L}_{AB}$ for all objects A and B together with two operations $^{\uparrow}$ and $^{\downarrow}$ satisfying the following:*

1. *$R^{\uparrow}, R^{\downarrow} : A \to B$ for all $R : A \to B$.*
2. *$(^{\uparrow}, ^{\downarrow})$ is a Galois correspondence.*
3. *$(R^{\smile}; S^{\downarrow})^{\uparrow} = R^{\uparrow\smile}; S^{\downarrow}$ for all $R : B \to A$ and $S : B \to C$.*
4. *If $\alpha_A \neq \mathbb{L}_{AA}$ is a non-zero scalar then $\alpha_A^{\uparrow} = \mathbb{I}_A$.*
5. *For all antimorphisms $f : \mathrm{Sc}_{\mathcal{G}}(A) \to \mathcal{G}[A, B]$ such that $f(\alpha_A)^{\uparrow} = f(\alpha_A)$ for all $\alpha_A \in \mathrm{Sc}_{\mathcal{G}}(A)$ and all $R : A \to B$ the following equivalence holds*

$$R \sqsubseteq \bigsqcup_{\alpha_A \in \mathrm{Sc}_{\mathcal{G}}(A)} (\alpha_A; f(\alpha_A)) \iff (\alpha_A \backslash R)^{\downarrow} \sqsubseteq f(\alpha_A) \text{ for all } \alpha_A \in \mathrm{Sc}_{\mathcal{G}}(A).$$

The obvious definition of $^{\uparrow}$ and $^{\downarrow}$ introduced in the last section for $\mathcal{L}$-fuzzy relations gives us the standard model.

Theorem 1. *Let $\mathcal{L}$ be a complete Brouwerian lattice with $0 \neq 1$. Then the Dedekind category of $\mathcal{L}$-fuzzy relations together with $^{\uparrow}$ and $^{\downarrow}$ is a Goguen category.*

According to our standard model, we define crispness in an arbitrary Goguen category as follows.

Definition 6. *A relation $R : A \rightarrow B$ of a Goguen category is called crisp iff $R^{\uparrow} = R$. The crisp fragment $\mathcal{G}^{\uparrow}$ of $\mathcal{G}$ is defined as the collection of all crisp relations of $\mathcal{G}$.*

Notice, that a relation is crisp iff $R^{\downarrow} = R$ iff $R^{\uparrow} = R^{\downarrow}$. Furthermore, the next theorem shows that the crisp relation are indeed an abstract counterpart of binary relations.

Theorem 2. *Let $\mathcal{G}$ be a Goguen category. Then next $\mathcal{G}^{\uparrow}$ is a simple Dedekind category with $\mathbb{T}_{AB} \neq \mathbb{\perp}_{AB}$ for all objects A and B.*

In the remainder of this section we want to state two important properties of Goguen categories. First of all, an abstract version of the α-cut Theorem may be proved.

Theorem 3 (α-cut Theorem). *Let $\mathcal{G}$ be a Goguen category and $R : A \rightarrow B$. Then we have $R = \bigsqcup\limits_{\alpha_A \in \mathrm{Sc}_{\mathcal{G}}(A)} (\alpha_A ; (\alpha_A \backslash R)^{\downarrow})$,*

The second property shows that the elements of a Goguen category are indeed relations based on a single underlying lattice $\mathcal{L}$.

Theorem 4. *Let $\mathcal{G}$ be a Goguen category. For all objects A and B the function $f(\alpha_A) := \mathbb{I}_B \sqcap \mathbb{T}_{BA} ; \alpha_A ; \mathbb{T}_{AB}$ is an isomorphism between the complete Brouwerian lattices $\mathrm{Sc}_{\mathcal{G}}(A)$ and $\mathrm{Sc}_{\mathcal{G}}(B)$.*

In the remainder of the paper we will identify all sets of scalars and denote this set by $\mathrm{Sc}[\mathcal{G}]$. We use $\alpha, \beta, \gamma, \ldots$ to denote abstract elements from $\mathrm{Sc}[\mathcal{G}]$. The corresponding scalar on an object A is then denoted by $\alpha_A, \beta_A, \gamma_A, \ldots$ with the convention that $f(\alpha_A) = \alpha_B$, i.e., $\mathbb{I}_B \sqcap \mathbb{T}_{BA} ; \alpha_A ; \mathbb{T}_{AB} = \alpha_B$.

Consequently, we call a Goguen category $\mathcal{G}$ representable iff there is an injective homomorphism from $\mathcal{G}$ to the category of $\mathrm{Sc}[\mathcal{G}]$-fuzzy relations.

5 Representation of Goguen categories

The representation theory of Goguen categories is based on a pseudo-representation within a suitable category of antimorphisms. The set of all antimorphisms between two lattices need not be a lattice. In particular, the componentwise union of two antimorphisms is not an antimorphism. But this set forms a closure system, i.e., is closed under arbitrary intersections and contains the greatest function $\dot{1}$ defined by $\dot{1}(x) := 1$. Therefore, it induces the following closure operation:

$$\tau(f) := \bigsqcap\{h \mid f \sqsubseteq h \text{ and } h \text{ antimorphism}\}.$$

Furthermore, for all elements a in the second lattice we have the pseudo-constant antimorphism $\dot{a}$ defined by

$$\dot{a}(x) := \begin{cases} 1 \text{ iff } x = 0, \\ a \text{ otherwise.} \end{cases}$$

This leads to the following theorem about Dedekind categories of antimorphisms.

Theorem 5. *Let $\mathcal{L}$ be an arbitrary complete Brouwerian lattice, and $\mathcal{R}$ be a Dedekind category. Then the following structure $\mathcal{R}^{\mathcal{L}}$ is a Dedekind category.*

1. *The objects of $\mathcal{R}^{\mathcal{L}}$ are the objects of $\mathcal{R}$,*
2. *A morphism from A to B is an antimorphism from $\mathcal{L}$ to $\mathcal{R}[A, B]$,*
3. *The identity morphism on A, the least and the greatest element in $\mathcal{R}^{\mathcal{L}}[A, B]$ are given by $\dot{\mathbb{I}}_A$, $\dot{\perp\!\!\!\perp}_{AB}$ and $\dot{\top\!\!\!\top}_{AB}$, respectively.*
4. *Meet and conversion are defined componentwise, e.g., $(f \dot{\sqcap} g)(x) := f(x) \sqcap g(x)$.*
5. *Union and composition (denoted by $\dot{\sqcup}$ and $\dot{;}$) are defined as the closure (with respect to τ) of the componentwise definition, e.g., $f \dot{\sqcup} g := \tau(f \sqcap g)$ where $(f \sqcap g)(x) := f(x) \sqcap g(x)$.*

Now, we define the up- and down-operation in $\mathcal{R}^{\mathcal{L}}$ by

$$f^{\uparrow} := \dot{R} \text{ with } R = \bigsqcup_{y \neq 0} f(y) \qquad \text{and} \qquad f^{\downarrow} := \dot{S} \text{ with } S = f(1)$$

or componentwise by

$$f^{\uparrow}(x) := \begin{cases} \top\!\!\!\top_{AB} & \text{iff } x = 0 \\ \bigsqcup\limits_{y \neq 0} f(y) & \text{otherwise,} \end{cases} \qquad f^{\downarrow}(x) := \begin{cases} \top\!\!\!\top_{AB} & \text{iff } x = 0 \\ f(1) & \text{otherwise.} \end{cases}$$

Theorem 6. *Let $\mathcal{L}$ be an arbitrary complete Brouwerian lattice with $0 \neq 1$ and $\mathcal{R}$ be a simple Dedekind category with $\top\!\!\!\top_{AB} \neq \perp\!\!\!\perp_{AB}$ for all objects A and B. Then $\mathcal{R}^{\mathcal{L}}$ is a Goguen category.*

The converse implication of the last theorem is also valid and called the pseudo-representation theorem of Goguen categories.

Theorem 7 (Pseudo-Representation Theorem). *Let $\mathcal{G}$ be a Goguen category. Then $\mathcal{G}^{\uparrow^{\mathrm{Sc}[\mathcal{G}]}}$ is again a Goguen category and $\mathcal{G}$ and $\mathcal{G}^{\uparrow^{\mathrm{Sc}[\mathcal{G}]}}$ are isomorphic.*

The last theorem leads to the equivalence of the representation problem of Goguen categories and simple Dedekind categories.

Theorem 8. *A Goguen category $\mathcal{G}$ is representable iff $\mathcal{G}^{\uparrow}$ is representable.*

6 Equations in Goguen categories

A Goguen category may provide some relational constructions as products, sums or subobjects. Such a construction is given by an object together with a set of relations fulfilling some equations. For example, a relational product of two objects A and B is an object $A \times B$ together with two relation $\pi : A \times B \to A$ and $\rho : A \times B \to B$ such that

$$\pi^{\smile};\pi \sqsubseteq \mathbb{I}_A, \quad \rho^{\smile};\rho \sqsubseteq \mathbb{I}_B \quad \pi^{\smile};\rho = \top_{AB}, \quad \pi;\pi^{\smile} \sqcap \rho;\rho^{\smile} = \mathbb{I}_{A\times B}.$$

One may expect that the projections π and ρ as well as the pairing $Q;\pi^{\smile} \sqcap R;\rho^{\smile}$ of two crisp relations Q and R is crisp again. Notice, that the first property implies the second since the class of crisp relations is closed under the relational operations.

Especially in applications, such a property seems to be essential. For example, if the input (or output) domain of a fuzzy controller is a product of several domains with non-crisp projection there would be a fuzzification, which is not an integral part of the specification of the controller. This fuzzification arises from the specific choice of the product. In this case, reasoning about the controller using a description within Goguen categories seems to be impossible or at least difficult.

Unfortunately, there may exists such non-crisp injections or projections. Consider again $B_4 := \mathcal{P}(\{a,b\})$ and the relations

$$\pi_1 := \begin{pmatrix} \{a,b\} & \emptyset \\ \{a\} & \{b\} \\ \{b\} & \{a\} \\ \emptyset & \{a,b\} \end{pmatrix}, \quad \rho_1 := \begin{pmatrix} \{a,b\} & \emptyset \\ \{b\} & \{a\} \\ \{a\} & \{b\} \\ \emptyset & \{a,b\} \end{pmatrix},$$

$$\pi_2 := \begin{pmatrix} \{a,b\} & \emptyset \\ \{a,b\} & \emptyset \\ \emptyset & \{a,b\} \\ \emptyset & \{a,b\} \end{pmatrix}, \quad \rho_2 := \begin{pmatrix} \{a,b\} & \emptyset \\ \emptyset & \{a,b\} \\ \{a,b\} & \emptyset \\ \emptyset & \{a,b\} \end{pmatrix}.$$

Both pairs (π_1, ρ_1) and (π_2, ρ_2) constitute a product of two copies of a set with two elements, i.e., they fulfil the equations above. The first pair of relations is not crisp. But, this example also shows that there is a crisp version (π_2, ρ_2) of the product, i.e., there are crisp relations between the same objects fulfilling the same set of equations. In our example, one may require without loss of generality that the projections are crisp. In this section we want to show that under an assumption on the lattice $\mathcal{L}$ such a crisp version always exists.

Definition 7. *A subset $F \subseteq \mathcal{L}$ of a complete Brouwerian lattice $\mathcal{L}$ is called a complete prime filter iff*

1. $0 \notin F$,
2. $x \sqcap y \in F$ *iff* $x \in F$ *and* $y \in F$ *for all* $x, y \in \mathcal{L}$,
3. $\bigsqcup M \in F$ *iff* $\exists y \in M : y \in F$ *for all subsets* $M \subseteq \mathcal{L}$.

We will denote the set of all complete prime filters of $\mathcal{L}$ by $\mathcal{F}_{\mathcal{L}}$. If $\mathcal{F}_{\mathcal{L}} \neq \emptyset$ we call $\mathcal{L}$ proper.

First, we want to study the class of proper lattices.

Theorem 9. *1. Every linear ordering is proper.*
2. The class of proper lattices is closed under arbitrary products.
3. A complete atomless Boolean algebra is not proper.

Equations are defined as usual. If S is a set of equations with relational variables within $\{r_1, \ldots, r_n\}$ we denote the fact that the relations $R_1, \ldots, R_n$ fulfil S by $R_1, \ldots, R_n \models S$.

Theorem 10. *Let $\mathcal{R}^{\mathcal{L}}$ be a Goguen category with a proper lattice $\mathcal{L}$, S be a set of equations with variables within $\{r_1, \ldots, r_n\}$ and $f_1, \ldots, f_n$ be elements of $\mathcal{R}^{\mathcal{L}}$ such that $f_1, \ldots, f_n \models S$. Then there are relations $U_1, \ldots, U_n$ from $\mathcal{R}$ with $\dot{U}_1, \ldots, \dot{U}_n \models S$.*

Notice, that $\dot{U}_i$ is always crisp.

The relation U_i is given by $U_i := \bigsqcup\limits_{x \in F} f_i(x)$ with F a complete prime filter in $\mathcal{L}$.

Using our pseudo-representation theorem the last result can be extended to arbitrary Goguen categories.

Corollary 1. *Let $\mathcal{G}$ be a Goguen category with a proper underlying lattice* $\mathrm{Sc}[\mathcal{G}]$*, S be a set of equations with variables within $\{r_1, \ldots, r_n\}$ and $R_1, \ldots, R_n$ relations such that $R_1, \ldots, R_n \models S$. Then there are crisp relations $Q_1, \ldots, Q_n$ with $Q_1, \ldots, Q_n \models S$.*

Since products, sums and subobjects induced by crisp partial identities are defined by equations, we may require without loss of generality that the related relations (e.g., the projections) are crisp.

Unfortunately, we were just able to state Theorem 10 for Goguen categories $\mathcal{R}^{\mathcal{L}}$ with a proper lattice $\mathcal{L}$. The experiences we made during searching a counterexample to this theorem in general leads us to the following conjecture.

Conjecture 1. Theorem 10 is true for all Goguen categories $\mathcal{R}^{\mathcal{L}}$.

7 Derived operations in Goguen categories

Since t-norms are suitable candidates for conjunctions and t-conorms for disjunction they are widely used within applications of fuzzy theory. Usually, operations on fuzzy sets and/or relations derived from t-norms and t-conorms are the basic means to combine several parts of a fuzzy system. For example, the decision-module within a fuzzy controller, i.e., the process deciding which rule on the linguistic variables is activated with a certain degree, may be modelled by a suitable composition operator. Therefore, every theory intended on describing fuzzy systems should be able to model such operations. The corresponding notion of t-norms and t-conorms for complete Brouwerian lattices is given by complete lattice-ordered semi groups introduced in [7].

The aim of this section is to define such derived operations within arbitrary Goguen categories and state their basic properties.

Definition 8. *Let $\mathcal{L}$ be a distributive lattice with least element 0 and greatest element 1, $*$ a binary operation on $\mathcal{L}$ and $e, z \in \mathcal{L}$. Then $(\mathcal{L}, *, e, z)$ is called a lattice-ordered operator set, abbreviated loos, iff*

1. *$*$ is monotonic in both arguments,*
2. *e is a left and right neutral element for $*$, i.e., $x * e = e * x = x$ for all $x \in \mathcal{L}$,*
3. *z is a left and right zero for $*$, i.e., $x * z = z * x = z$ for all $x \in \mathcal{L}$.*

If $*$ is associative $(\mathcal{L}, *, e, z)$ is called a lattice-ordered semigroup (losg). Furthermore, if $\mathcal{L}$ is a complete Brouwerian lattice and $*$ is continuous (distributes over arbitrary unions), i.e.,

$$x * \bigsqcup_{i \in I} y_i = \bigsqcup_{i \in I} (x * y_i) \quad \textit{and} \quad (\bigsqcup_{i \in I} y_i) * x = \bigsqcup_{i \in I} (y_i * x)$$

for all nonempty sets I. $(\mathcal{L}, *, e, z)$ is called a complete lattice-ordered operator set/semigroup (cloos/closg). Finally, the structures defined above are called commutative if $*$ is.

As usual, e and z are unique, i.e., if e' (z') is another left and right neutral element (left and right zero) for $*$ then $e' = e$ ($z' = z$).

Notice, that for $\mathcal{L} = [0,1]$, $e = 1$ and $z = 0$ we get the usual definition of t-norms and for $e = 0$ and $z = 1$ of t-conorms.

$(\mathcal{L}, \sqcap, 1, 0)$ and $(\mathcal{L}, \sqcup, 0, 1)$ are commutative losg's. Furthermore, we may define the following operations

$$x \circledast y := \begin{cases} x & \text{iff } y = 1, \\ y & \text{iff } x = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad x \boxplus y := \begin{cases} x & \text{iff } y = 0, \\ y & \text{iff } x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Again, $(\mathcal{L}, \circledast, 1, 0)$ and $(\mathcal{L}, \boxplus, 0, 1)$ are commutative losg's.

Theorem 11. *Let $(\mathcal{L}, *, 1, z)$ be a loos. Then we have the following:*

1. *$z = 0$, i.e., $x * 0 = 0 * x = 0$ for all $x \in \mathcal{L}$,*
2. *$x \circledast y \sqsubseteq x * y \sqsubseteq x \sqcap y$ for all $x, y \in \mathcal{L}$,*
3. *$* = \sqcap$ iff $u * u = u$ for all $u \in \mathcal{L}$.*

If the identity 1 of $(\mathcal{L}, *, 1, z)$ in the last theorem is replaced by 0 a dual version may be proved.

Theorem 12. *Let $(\mathcal{L}, *, 0, z)$ be a loos. Then we have the following:*

1. *$z = 1$, i.e., $x * 1 = 1 * x = 1$ for all $x \in \mathcal{L}$,*
2. *$x \sqcup y \sqsubseteq x * y \sqsubseteq x \boxplus y$ for all $x, y \in \mathcal{L}$,*
3. *$* = \sqcup$ iff $u * u = u$ for all $u \in \mathcal{L}$.*

Throughout this section, unless otherwise stated, let $\mathcal{G}$ be a Goguen category and $*$ an operation such that $(\mathrm{Sc}[\mathcal{G}], *, \epsilon, \zeta)$ is a loos. Furthermore, suppose $\otimes$ is an operation on relations such that

1. $\otimes$ is defined for all pairs of relations from $\mathcal{G}[A, A]$ for all objects A and its value is within $\mathcal{G}[B, B]$ for a suitable B and if $Q \otimes R$ is defined for $Q : A \to B$ and $R : C \to D$ then $\otimes$ is defined for all pairs of relations from $\mathcal{G}[A, B]$ and $\mathcal{G}[C, D]$,
2. if $Q \otimes R$ is defined for $Q : A \to B$ and $R : C \to D$ and within $\mathcal{G}[E, F]$ then $Q \otimes \perp\!\!\!\perp_{CD} = \perp\!\!\!\perp_{AB} \otimes R = \perp\!\!\!\perp_{EF}$,
3. if $\top\!\!\!\top_{AB} \otimes \top\!\!\!\top_{CD}$ is defined and within $\mathcal{G}[E, F]$ then $\top\!\!\!\top_{AB} \otimes \top\!\!\!\top_{CD} = \top\!\!\!\top_{EF}$,
4. $\otimes$ distributes over arbitrary unions in both arguments, i.e., for all Q, Q_i, R, R_i with $i \in I$ we have

$$Q \otimes (\bigsqcup_{i \in I} R_i) = \bigsqcup_{i \in I} (Q \otimes R_i) \quad \text{and} \quad (\bigsqcup_{i \in I} Q_i) \otimes R = \bigsqcup_{i \in I} (Q_i \otimes R)$$

whenever the application of $\otimes$ is defined,

5. for all $\alpha, \beta \in \mathrm{Sc}[\mathcal{G}]$ and relations $Q : A \to B, R : C \to D$ such that $Q \otimes R$ is defined and within $\mathcal{G}[E, F]$ we have

$$(\alpha_E \sqcap \beta_E); (Q \otimes R) = (\alpha_A; Q) \otimes (\beta_C; R),$$

6. $\otimes$ is closed on $\mathcal{G}^{\uparrow}$, i.e., for all crisp relations Q, R such that $Q \otimes R$ is defined $Q \otimes R$ is crisp.

Notice that $\sqcap$ and ; satisfy the properties above. 1,2 and 4 follow immediately from the definition of a Dedekind category. For meet property 3 is trivial and for composition it follows from the fact that $\mathbb{I}$ is crisp and that the crisp relations constitute a simple Dedekind category. Property 6 is true since the crisp relations are closed under all relational operations. Finally, property 5 is shown as follows.

$$\begin{aligned}
(\alpha_A \sqcap \beta_A); (Q \sqcap R) &= (\alpha_A \sqcap \beta_A); \mathbb{T}_{AB} \sqcap Q \sqcap R \\
&= \alpha_A; \mathbb{T}_{AB} \sqcap \beta_A; \mathbb{T}_{AB} \sqcap Q \sqcap R \\
&= \alpha_A; Q \sqcap \beta_A; R, \\
(\alpha_A \sqcap \beta_A); Q; S &= \alpha_A; \beta_A; Q; S \\
&= \alpha_A; Q; \beta_B; R.
\end{aligned}$$

Now, we may define $*$ based operations as follows.

Definition 9. *Let Q, R be relations such that $Q \otimes R$ is defined. Then we define*

$$Q \otimes_* R := \bigsqcup_{\alpha,\beta \in \mathrm{Sc}[\mathcal{G}]} (\alpha * \beta); ((\alpha \backslash Q)^{\downarrow} \otimes (\beta \backslash R)^{\downarrow}).$$

The definition above corresponds to the componentwise definition in the case of $\mathcal{L}$-fuzzy relations. Notice, that $;_*$ and the composition defined in [2] coincide.

Theorem 13. *Let Q, R be $\mathcal{L}$-fuzzy relations between the sets A and B, and let S be a $\mathcal{L}$-fuzzy relation between B and C. Then we have*

1. $(Q \sqcap_* R)(x, y) = Q(x, y) * R(x, y),$
2. $(Q;_* S)(x, z) = \bigsqcup_{y \in B} Q(x, y) * S(y, z).$

Property 1. of the last theorem shows that $\sqcap_*$ and $\sqcap$ coincide for $\mathcal{L}$-fuzzy relations iff $*$ equals the operation $\sqcap$. This property can be generalized as follows.

Theorem 14. *Let Q and R be relations such that $Q \otimes R$ is defined. Then we have $Q \otimes_* R = Q \otimes R$ for all Q and R iff $*$ equals the operation $\sqcap$.*

Since $\sqcup$ is the weakest t-conorm like operation by Theorem 12 (2) we get a similar result for $\otimes$ being the operation $\sqcap$.

Theorem 15. *let $Q, R : A \to B$ be relations. Then we have $Q \sqcap_* R = Q \sqcup R$ for all Q and R iff $*$ equals the operation $\sqcup$.*

Suppose $(\mathcal{L}, *, 1, 0)$ is a loos and R is a crisp $\mathcal{L}$-fuzzy relation. Then we have

$$(Q \sqcap_* R)(x, y) = Q(x, y) * R(x, y) = Q(x, y) \sqcap R(x, y) = (Q \sqcap R)(x, y).$$

The next theorem shows that this property is true in general.

Theorem 16. *Let $\epsilon = \mathbb{I}$. Furthermore, let Q and R be relations such that $Q \otimes R$ is defined. If Q or R is crisp then we have $Q \otimes_* R = Q \otimes R$.*

Theorem 11 may be lifted to the $*$ based operation as follows.

Theorem 17. *Let $\epsilon = \mathbb{I}$. Furthermore, let Q and R be relations such that $Q \otimes R$ is defined. Then we have the following.*

1. $Q \otimes_* \perp\!\!\!\perp = \perp\!\!\!\perp \otimes_* R = \perp\!\!\!\perp$,
2. $Q \sqcap_* \mathbb{T} = Q$ *and* $\mathbb{T} \sqcap_* R = R$,
3. $Q \otimes_{\circledast} R \sqsubseteq Q \otimes_* R \sqsubseteq Q \otimes R$.

Replacing $\mathbb{I}$ by $\perp\!\!\!\perp$ we may state a kind of a dual version of the last two theorems.

Theorem 18. *Let $\epsilon = \perp\!\!\!\perp$. Furthermore, let Q and R be relations such that $Q \otimes R$ is defined. If Q or R are crisp then we have $Q \otimes_* R = (Q \otimes \mathbb{T}) \sqcup (\mathbb{T} \otimes R)$.*

In the rest of this section we want to give some basic properties of $*$ based operations and the structures induced by them. We start with the following two theorems.

Theorem 19. *Let $\otimes$ be commutative. Then $\otimes_*$ is commutative iff $(\mathrm{Sc}[\mathcal{G}], *, \epsilon, \zeta)$ is a commutative loos.*

Theorem 20. *Let $\otimes$ be associative and $(\mathrm{Sc}[\mathcal{G}], *, \epsilon, \zeta)$ complete. Then $\otimes_*$ is associative iff $(\mathrm{Sc}[\mathcal{G}], *, \epsilon, \zeta)$ is a losg.*

If $\otimes$ is composition one may ask about a categorical structure induced by $;_*$. The answer is given in the next theorem.

Theorem 21. *Let $(\mathrm{Sc}[\mathcal{G}], *, \epsilon, \zeta)$ be complete and $\mathcal{G}$ non trivial, i.e., there is an object A so that $\mathbb{T}_{AA} \neq \mathbb{I}_A$. Then $\mathcal{G}$ together with composition $;_*$ and identity morphisms ϵ is a category iff $(\mathrm{Sc}[\mathcal{G}], *, \epsilon, \zeta)$ is a losg with $\zeta = \perp\!\!\!\perp$.*

Since converse is a well-behaved operation we get the following theorem.

Theorem 22. *Let* $(\mathrm{Sc}[\mathcal{G}], *, \epsilon, \zeta)$ *be a closg. Then we have* $(Q;_* R)^{\smile} = R^{\smile};_* Q^{\smile}$ *for all* $Q : A \to B$ *and* $R : B \to C$.

Last but not least, we will focus on continuity of $\otimes_*$.

Theorem 23. *Let* $(\mathrm{Sc}[\mathcal{G}], *, \epsilon, \zeta)$ *be a cloos. Then we have*

$$(\bigsqcup_{i\in I} Q_i) \otimes_* R = \bigsqcup_{i\in I}(Q_i \otimes_* R) \quad \text{and} \quad Q \otimes_* (\bigsqcup_{i\in I} R_i) = \bigsqcup_{i\in I}(Q \otimes_* R_i)$$

for all Q, Q_i, R, R_i *with* $i \in I$ *whenever the application of* $\otimes_*$ *is defined.*

As usual, for a continuous binary operation a residuated operation may be defined.

Theorem 24. *Let* $(\mathrm{Sc}[\mathcal{G}], *, \epsilon, \zeta)$ *be a cloos. Then there are operations* $\lhd_*$ *and* $\rhd_*$ *such that*

$$\begin{array}{lccc} & Q \otimes_* X \sqsubseteq R & \iff & X \sqsubseteq Q \lhd_* R \\ \text{and} & Y \otimes_* S \sqsubseteq R & \iff & Y \sqsubseteq R \rhd_* S, \end{array}$$

whenever the application of $\otimes_*$ *is defined.*

The last theorem shows that an inclusion $Q;_* X \sqsubseteq R$ has a greatest solution in X, namely $Q \lhd_* R$. Furthermore, the equation $Q;_* X = R$ has a solution $(X = Q \lhd_* R)$ iff $Q;_* (Q \lhd_* R) = R$.

8 Conclusion

In this paper we have provided a survey of the theory of Goguen categories. The well-known theory of Dedekind categories (or locally complete division allegories) is too weak to express important properties of fuzzy relations, which naturally leads to an extended theory, e.g., the theory of Goguen categories.

Beneath some basic properties we have studied the representation theory of Goguen categories. In particular, we have indicated that the representation theories of Goguen and simple Dedekind categories are equivalent. This shows that Goguen categories are a suitable extension of the theory of binary relations to the fuzzy world. Furthermore, this result allows us to transfer representation results for Dedekind categories to the theory of Goguen categories. It also shows that there are nonstandard models.

The study of equations within Goguen categories allows us to assume (under a condition of the underlying lattice) that crisp versions of relational constructions exist. This is an important result in respect to applications of this theory to computer science, e.g., correctness considerations of fuzzy controllers.

Furthermore, we have shown that operations derived from lattice-ordered semigroups may be defined in an arbitrary Goguen category without referring to a componentwise representation of the relations, i.e., without using the coefficients from $\mathcal{L}$ of concrete $\mathcal{L}$-fuzzy relations. Again, this seems to be important in the view of applications of the theory mentioned above.

The theory of Goguen categories is based on Dedekind categories and basically equational and element-free. Therefore, it constitutes a nice categorical theory to reason about $\mathcal{L}$-fuzzy controllers and other applications of fuzziness in computer science.

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