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Representable sequential algebras and observation spaces

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Abstract. We define the concepts of representable and abstract sequential Q-algebra, which are generalizations of the (relational) Q-algebras in [10]. Just as in that paper, we then prove that the two concepts coincide. In the following section we recall the concept of observation space and note that all complex algebras of observation spaces are representable sequential algebras. Finally we give an uncountable family of representable sequential algebras that generate distinct minimal varieties (i.e. covers of the variety of one-element algebras).

1 Introduction

Representable relation algebras are collections of binary relations (on a set U) that are closed under the operations of *union* ($\cup$), *complementation* ($^-$, relative to a largest relation), *composition* (;), *converse* ($^\smile$) and contain the *identity relation* id_U. Note that the assumption of closure implies that the largest relation is an equivalence relation.

When modeling reactive systems, the converse operation is replaced by two derived operations:

$$R \lhd S = (R;S^\smile) \cap \mathbb{T} \qquad \text{and} \qquad R \rhd S = (R^\smile;S) \cap \mathbb{T}$$

where $\mathbb{T}$ is a fixed largest relation. Collections of subsets of $\mathbb{T}$ that are closed under $\cup, ^-, ;, \lhd, \rhd$ and contain id_U are called *representable sequential algebras.* Note that in this case the top relation need no longer be symmetric, but it is still reflexive and transitive. If it happens to be symmetric, then the algebra is term-equivalent to a representable relation algebra since we can recover the converse operation

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by $R^{\smile} = R \triangleright id_U$. The class of all algebras isomorphic to representable sequential algebras is denoted by RSeA. In [6] this is shown to be a variety, but it is not finitely axiomatizable [11], [6]. Lyndon [9] gave an equational basis for the variety of representable relation algebras, but this infinite list of equations is complicated to define and not easy to work with. Stebletsova and Venema [10] presented a list of 10 axiom schema for representable Q-algebras, which are expansions of representable relation algebras. The aim of this paper is to extend their result to RSeA. Representable sequential algebras can also be viewed as relativizations of representable relation algebras with respect to reflexive and transitive relations. Further motivation and results about (abstract) sequential algebras can be found in [11], [14], [3], [4]. In the latter two publications these algebras were referred to as balanced Euclidean residuated monoids (or BERMs for short).

2 *Q*-algebras and sequential *Q*-algebras

We begin by recalling the definition of a representable Q-algebra. The Q-operations are defined on $n \times n$ matrices $\underline{R} = (R_{ij})$ of binary relations on some set U by the condition

$$a \mathsf{Q}_n^{kl}(\underline{R}) b \quad \text{iff} \quad \begin{array}{c} \exists u_0, \ldots, u_{n-1} \text{ with } a = u_k,\ b = u_l, \text{ and} \\ u_i R_{ij} u_j \text{ for all } i, j < n. \end{array}$$

Definition 1. Let $\mathbb{T}$ be an equivalence relation on a set U. The *relation set* Q-*algebra on* $\mathbb{T}$ is

$$\mathsf{Q}(\mathbb{T}) = (\mathcal{P}(\mathbb{T}), \cup, ^-, id_U, \mathsf{Q}_n^{kl})_{k,l<n\in\omega}.$$

Algebras of the form $(A, +, ^-, 1', Q_n^{kl})_{k,l<n\in\omega}$ are called Q-*type* algebras. A Q-type algebra is *representable* if it can be embedded into a relation set Q-algebra. RQ denotes the class of all representable Q-type algebras.

To see that algebras in RQ are in fact expansions of representable relation algebras, one merely has to observe that

$$R^{\smile} = \mathsf{Q}_2^{01}\begin{pmatrix} \mathbb{T} & \mathbb{T} \\ R & \mathbb{T} \end{pmatrix} \qquad \text{and} \qquad R;S = \mathsf{Q}_3^{02}\begin{pmatrix} \mathbb{T} & R & \mathbb{T} \\ \mathbb{T} & \mathbb{T} & S \\ \mathbb{T} & \mathbb{T} & \mathbb{T} \end{pmatrix}.$$

Stebletsova and Venema continue by giving a list of equation schema Q1-Q10, and define a Q-algebra as a Q-type algebra that satisfies these axioms. They then prove their main result, namely that the variety Q of all Q-algebras coincides with the class RQ.

We now expand the type even further and generalize this result to sequential Q-algebras. Note that a matrix of relations on U can also be viewed as a function from $n \times n$ to $\mathcal{P}(U \times U)$. We wish to relativize this notion. Given a relation ρ on the set $n = \{0, \ldots, n-1\}$, a ρ-matrix $\underline{R}$ of binary relations on U is a function from ρ to $\mathcal{P}(U \times U)$. For $(k, l) \in \rho$, the sequential Q-operations are defined on ρ-matrices by

$$a\mathsf{Q}^{kl}_{n\rho}(\underline{R})b \quad \text{iff} \quad \begin{array}{c} \exists u_0, \ldots, u_{n-1} \text{ with } a = u_k,\ b = u_l, \text{ and} \\ u_i R_{ij} u_j \text{ for all } (i,j) \in \rho. \end{array}$$

Definition 2. Let $\mathrm{rt}(U)$ be the set of reflexive and transitive relations on the set U, and let $\mathbb{T} \in \mathrm{rt}(U)$. The *sequential set* Q*-algebra on* $\mathbb{T}$ is

$$\mathsf{SQ}(\mathbb{T}) = (\mathcal{P}(\mathbb{T}), \cup, ^-, id_U, \mathsf{Q}^{kl}_{n\rho})_{(k,l)\in\rho\in\mathrm{rt}(n), n\in\omega}.$$

Algebras of the form $(A, +, ^-, 1', Q^{kl}_{n\rho})_{(k,l)\in\rho\in\mathrm{rt}(n),n\in\omega}$ are called SQ*-type* algebras. An SQ-type algebra is *representable* if it can be embedded into a sequential set Q-algebra. RSQ denotes the class of all representable SQ-type algebras.

The class RSQ is obviously closed under taking subalgebras and isomorphic copies. It is also easily seen to be closed under products. In fact, the product of a collection of sequential set Q-algebras is (isomorphic to) a sequential set Q-algebra constructed on the disjoint union of the respective base sets of the factors. The fact that RSQ is also closed under homomorphic images is not obvious, but it follows from Theorem 13 below.

Since $\mathsf{Q}^{kl}_n = \mathsf{Q}^{kl}_{n,n\times n}$, sequential set Q-algebras are expansions of relation set Q-algebras. They are also expansions of representable sequential algebras since

$$R;S = \mathsf{Q}^{02}_{3\le}\begin{pmatrix} \mathbb{T} & R & \mathbb{T} \\ & \mathbb{T} & S \\ & & \mathbb{T} \end{pmatrix},\ R \triangleright S = \mathsf{Q}^{12}_{3\le}\begin{pmatrix} \mathbb{T} & R & S \\ & \mathbb{T} & \mathbb{T} \\ & & \mathbb{T} \end{pmatrix},\ R \triangleleft S = \mathsf{Q}^{01}_{3\le}\begin{pmatrix} \mathbb{T} & \mathbb{T} & R \\ & \mathbb{T} & S \\ & & \mathbb{T} \end{pmatrix}$$

Here $\le$ denotes the usual order on $\{0,1,2\}$. Note also that the earlier term for $R;S$ which used Q^{02}_3 would not work in the sequential case: e.g. if $\mathbb{T}$ is antisymmetric then for any $k, l < n$, the condition $a\mathsf{Q}^{kl}_n(\underline{R})b$ implies $(a,b) \in R_{kl} \subseteq \mathbb{T}$ and $(b,a) \in R_{lk} \subseteq \mathbb{T}$, hence $a = b$, which shows that for such antisymmetric sequential Q-algebras the Q^{kl}_n operations only produce sub-identity relations.

The following list of SQ identities, together with the Boolean identities for $+, -$, defines the variety SQ of SQ-algebras. As usual, $1 = -1' + 1'$, $0 = -1$ and $x \cdot y = -(-x + -y)$. Underlined letters represent ρ-matrices for the appropriate relation ρ, and for fixed $(i,j) \in \rho$ we let $\underline{x}[x_{ij}/a]$ denote the ρ-matrix $\underline{x}$ with the ij^{th} entry replaced by a. For a relation ρ on $n = \{0, 1, \ldots, n-1\}$, and a function $f : n \to m$, we let $f[\rho] = \{(f(p), f(q)) : (p,q) \in \rho\}$.

In the list, (k,l) ranges over ρ, which ranges over $\mathrm{rt}(n)$ for $n \in \omega$.

SQ1 (idmap) $Q^{01}_{2\le}\begin{pmatrix}1 & x\\ & 1\end{pmatrix} = x$

SQ2 (normal) $Q^{kl}_{n\rho}(\underline{x}[x_{ij}/0]) = 0$

SQ3 (meet) $Q^{kl}_{n\rho}(\underline{x}) \cdot y = Q^{kl}_{n\rho}(\underline{x}[x_{kl}/x_{kl} \cdot y])$

SQ4 (subid) $Q^{kl}_{n\rho}(\underline{x}) = Q^{kl}_{n\rho}(\underline{x}[x_{ii}/x_{ii} \cdot 1\text{'}])$

SQ5 (contract) $Q^{kl}_{n+1,\rho}(\underline{x}[x_{ij}/x_{ij} \cdot 1\text{'}]) = Q^{f(k)f(l)}_{nf[\rho]}(\underline{t})$
where $f : n+1 \twoheadrightarrow n$, $i \neq j$, $f(i) = f(j)$, $f[\rho] \in \mathrm{rt}(n)$ and

$$t_{pq} = \prod\{x_{p'q'} : (p', q') \in \rho,\ f(p') = p \text{ and } f(q') = q\}$$

SQ6 (rename) $Q^{f(k)f(l)}_{m\sigma}(\underline{x}) \le Q^{kl}_{n\rho}(\underline{t})$, where $f : n \to m$ is arbitrary, $f[\rho] \subseteq \sigma$, and $t_{pq} = x_{f(p)f(q)}$

SQ7 (expand) $Q^{kl}_{n\rho}(\underline{x}[x_{ij}/x_{ij} \cdot Q^{k'l'}_{m\sigma}(\underline{y})]) \le Q^{kl}_{n+m,\tau}(\underline{t})$, where $f(p) = p + n$, $\tau = (\rho \cup f[\sigma] \cup \{i, k'\}^2 \cup \{j, l'\}^2)^*$ (here * denotes the operation of reflexive transitive closure), and for $(p, q) \in \tau$

$$t_{pq} = \begin{cases} x_{pq} & \text{if } (p,q) \in \rho \\ y_{p-n,q-n} & \text{if } (p,q) \in f[\sigma] \\ 1\text{'} & \text{if } \{p,q\} = \{i,k'\} \text{ or } \{p,q\} = \{j,l'\} \\ 1 & \text{otherwise} \end{cases}$$

SQ8 (insert) $Q^{kl}_{n\rho}(\underline{x}) \le Q^{kl}_{n\rho}(\underline{x}[x_{ij}/Q^{ij}_{n\rho}(\underline{x})])$

To get a feel for the above identities, it is instructive to check that they hold in all representable SQ-algebras, i.e. $\mathsf{RSQ} \subseteq \mathsf{SQ}$. The main result of this paper is that the reverse inclusion also holds. The proof is modeled very closely on the one in [10], but as there are quite a few additional details to be checked it is not feasible to simply point out the differences. The following lemmas are needed for the proof.

Lemma 3. *Each $Q^{kl}_{n\rho}$ operation is conjugated in each argument: for any (k, l), $(i, j) \in \rho \in \mathrm{rt}(n)$, $n \in \omega$,*

$$Q^{kl}_{n\rho}(\underline{x}) \cdot y = 0 \quad \textit{iff} \quad Q^{ij}_{n\rho}(\underline{t}) \cdot x_{ij} = 0$$

where $\underline{t} = \begin{cases} \underline{x}[x_{kl}/x_{kl} \cdot y][x_{ij}/1] & \textit{if } (i,j) \neq (k,l) \\ \underline{x}[x_{ij}/y] & \textit{if } (i,j) = (k,l) \end{cases}$

Proof. Assuming $(i,j) \neq (k,l)$ and $Q^{kl}_{n\rho}(\underline{x}) \cdot y = 0$, let $\underline{s} = \underline{x}[x_{kl}/x_{kl} \cdot y]$. Then $Q^{kl}_{n\rho}(\underline{s}) = 0$ by SQ3, so by SQ8 and SQ2

$$Q^{ij}_{n\rho}(\underline{s}) \leq Q^{ij}_{n\rho}(\underline{s}[s_{kl}/Q^{kl}_{n\rho}(\underline{s})]) = Q^{ij}_{n\rho}(\underline{s}[s_{kl}/0]) = 0.$$

Now we use SQ3 to obtain $Q^{ij}_{n\rho}(\underline{s}[x_{ij}/1]) \cdot x_{ij} = Q^{ij}_{n\rho}(\underline{s}) = 0$.

For $(i,j) = (k,l)$ we use SQ3 twice to obtain

$$Q^{kl}_{n\rho}(\underline{x}) \cdot y = Q^{kl}_{n\rho}(\underline{x}[x_{kl}/x_{kl} \cdot y]) = Q^{kl}_{n\rho}(\underline{x}[x_{kl}/y]) \cdot x_{kl}. \square$$

Corollary 4. [8] *The $Q^{kl}_{n\rho}$ operations are normal and completely additive (hence monotone) in each argument. Therefore SQ-algebras are Boolean algebras with operators.*

In the abstract setting we can also define term-operations for $;, \triangleright, \triangleleft$, using the same terms as in the representable case:

$$x;y = Q^{02}_{3\leq}\begin{pmatrix} 1 & x & 1 \\ & 1 & y \\ & & 1 \end{pmatrix}, \quad x \triangleright y = Q^{12}_{3\leq}\begin{pmatrix} 1 & x & y \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad x \triangleleft y = Q^{01}_{3\leq}\begin{pmatrix} 1 & 1 & x \\ & 1 & y \\ & & 1 \end{pmatrix}$$

It is then easy to check that Lemma 3 implies the Schröder equivalences

$$(x;y) \cdot z = 0 \text{ iff } (x \triangleright z) \cdot y = 0 \text{ iff } (z \triangleleft y) \cdot x = 0.$$

For (abstract) SQ-algebras we define a weak converse by $x^{\vee} = Q^{01}_{2}\begin{pmatrix} 1 & 1 \\ x & 1 \end{pmatrix}$.

Lemma 5. *The following identities hold in* SQ.

(i) $x^{\vee} = x \triangleright 1'$
(ii) $x; 1' = x = 1'; x$
(iii) $1' \triangleright x = x$
(iv) $1'^{\vee} = 1'$

Proof. (i) $x \triangleright 1' = Q^{12}_{3\leq}\begin{pmatrix} 1 & x & 1' \\ & 1 & 1 \\ & & 1 \end{pmatrix} = Q^{10}_{2}\begin{pmatrix} 1 \cdot 1' \cdot 1 & x \\ 1 & 1 \end{pmatrix}$ by the contract law with $f = \{0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 0\}$. Using the rename law with $\{0 \mapsto 1, 1 \mapsto 0\}$ followed by the subid law, we get $Q^{01}_{2}\begin{pmatrix} 1 & 1 \\ x & 1' \end{pmatrix} = Q^{01}_{2}\begin{pmatrix} 1 & 1 \\ x & 1 \end{pmatrix} = x^{\vee}$.

(ii) We compute the first equality with the contract, subid and idmap laws as follows:

$$Q^{02}_{3\leq}\begin{pmatrix} 1 & x & 1 \\ & 1 & 1' \\ & & 1 \end{pmatrix} = Q^{01}_{2\leq}\begin{pmatrix} 1 & x \cdot 1 \\ & 1 \cdot 1' \cdot 1 \end{pmatrix} = Q^{01}_{2\leq}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = x.$$

The second equality is similar.

(iii) One can compute it directly as in (ii), but it also follows from the Schröder equivalences since $y \cdot (1' \triangleright x) = 0$ iff $x \cdot (1';y) = 0$ iff $x \cdot y = 0$.

(iv) This follows from (i) and (iii) with $x = 1'$. □

3 The representation theorem

Definition 6. Let $\mathbf{A}$ be an SQ-algebra, and $a \in A$. Φ is a *sequential ultrafilter network over* $\mathbf{A}$ *for* a if $\Phi : T_\Phi \to \text{Ultrafilters}(A)$ for some relation $T_\Phi \in \text{rt}(\omega)$ such that

(i) $a \in \Phi(i,j)$ for some $(i,j) \in T_\Phi$,
(ii) $1' \in \Phi(i,i)$ for all $i \in \omega$, and
(iii) for any $n \in \omega$, $(k,l) \in \rho \in \text{rt}(n)$, $(i,j) \in T_\Phi$ and any ρ-matrix $\underline{b}$ over A we have

$$Q^{kl}_{n\rho}(\underline{b}) \in \Phi(i,j) \quad \text{iff} \quad \exists u_0,\ldots,u_{n-1} \in \omega \text{ with } i = u_k,\ j = u_l,$$

$$(u_p,u_q) \in T_\Phi,\ \text{and } b_{pq} \in \Phi(u_p,u_q) \text{ for all } (p,q) \in \rho.$$

Lemma 7. *Let* $\mathbf{A}$ *be an SQ-algebra, and suppose that for each non-zero* $a \in A$ *there is a sequential ultrafilter network over* $\mathbf{A}$ *for* a*. Then* $\mathbf{A}$ *is representable.*

Proof. Let Φ be a sequential ultrafilter network over the SQ-algebra $\mathbf{A}$, and for $i,j \in \omega$, define $i \equiv j$ iff $(i,j) \in T_\Phi$ and $1' \in \Phi(i,j)$. We proceed to show that $\equiv$ is an equivalence relation that is compatible with Φ.

By Definition 6(ii), $\equiv$ is reflexive. Suppose that $i \equiv j$, i.e. $(i,j) \in T_\Phi$ and $1' \in \Phi(i,j)$. It follows from Lemma 5(iv) that $Q^{01}_2\begin{pmatrix} 1 & 1 \\ 1' & 1 \end{pmatrix} \in \Phi(i,j)$, so by Definition 6(iii), there exist u_0, u_1 such that $i = u_0$, $j = u_1$, $(u_1,u_0) \in T_\Phi$ and $1' \in \Phi(u_1,u_0)$. Hence $j \equiv i$.

Now suppose that $i \equiv j$ and $j \equiv k$. Then $(i,j),(j,k) \in T_\Phi$ and $1' \in \Phi(i,j) \cap \Phi(j,k)$. By transitivity of T_Φ, we have $(i,k) \in T_\Phi$, and by reflexivity $(i,i),(j,j),(k,k) \in T_\Phi$. Since each $\Phi(p,q)$ is a filter, $1 \in \Phi(i,k) \cap \Phi(i,i) \cap \Phi(j,j) \cap \Phi(k,k)$. So by Definition 6(iii),

$$Q^{02}_{3\le}\begin{pmatrix} 1 & 1' & 1 \\ & 1 & 1' \\ & & 1 \end{pmatrix} \in \Phi(i,k).$$

Hence $1';1' = 1' \in \Phi(i,k)$ by Lemma 5(iii). This proves $i \equiv k$.

Next we show that Φ is compatible with $\equiv$ in the following sense: If $i \equiv k$, $j \equiv l$ and $(i,j) \in T_\Phi$ then $\Phi(i,j) = \Phi(k,l)$. Suppose the assumptions hold. Then $(k,i),(j,l) \in T_\Phi$, so by transitivity $(k,l) \in T_\Phi$. Also 1' $\in \Phi(k,i) \cap \Phi(j,l)$ and $1 \in \Phi(p,q)$ for all $p,q \in \omega$. We show that $\Phi(i,j) \subseteq \Phi(k,l)$ then equality follows since they are ultrafilters. Let $b \in \Phi(i,j)$. By Definition 6(iii)

$$Q^{03}_{4\leq}\begin{pmatrix} 1 & 1' & 1 & 1 \\ & 1 & b & 1 \\ & & 1 & 1' \\ & & & 1 \end{pmatrix} \in \Phi(k,l).$$

By two applications of the contract law (identifying $0,1$ and $2,3$) this element is equal to $Q^{01}_{2\leq}\begin{pmatrix} 1' & b \\ & 1' \end{pmatrix}$, which in turn equals $Q^{01}_{2\leq}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ by the subid law. Thus the idmap law implies $b \in \Phi(k,l)$.

We now let $U = \omega/\equiv$ and define $\mathbb{T}_\Phi$ on U by

$$([i],[j]) \in \mathbb{T}_\Phi \quad \text{iff} \quad (i,j) \in T_\Phi$$

where $[i]$ is the equivalence class of i with respect to $\equiv$. By transitivity of T_Φ, this definition is well-defined, and clearly $\mathbb{T}_\Phi$ is again transitive and reflexive.

Next a representation function $g : \mathbf{A} \to \mathsf{SQ}(\mathbb{T}_\Phi)$ is defined by

$$g(x) = \{([i],[j]) \in \mathbb{T}_\Phi : x \in \Phi(i,j)\}.$$

The compatibility condition ensures that the definition of g is independent of the choice of representatives. We now show that g is an SQ-algebra homomorphism.

Since $\Phi(i,j)$ is an ultrafilter, it follows easily that g commutes with joins and complementation. Also $g(1') = id_U$ since

$$([i],[j]) \in \mathbb{T}_\Phi \text{ and } 1' \in \Phi(i,j) \quad \Rightarrow \quad i \equiv j \Rightarrow \quad [i] = [j]$$

and conversely $([i],[i]) \in g(1')$ follows from the reflexivity of $\mathbb{T}_\Phi$ and Definition 6(ii).

Finally, the following equivalences show that g commutes with $Q^{kl}_{n\rho}$.

$([i],[j]) \in g(Q^{kl}_{n\rho}(\underline{b}))$

iff $Q^{kl}_{n\rho}(\underline{b}) \in \Phi(i,j)$ by definition of g

iff $\exists u_0, \ldots, u_n \in \omega$ with $i = u_k$, $j = u_l$, $(u_p,u_q) \in T_\Phi$ and $b_{pq} \in \Phi(u_p,u_q)$ for all $(p,q) \in \rho$ (Definition 6(iii)) iff $\exists u_0,\dots,u_n \in \omega$ with $i = u_k$, $j = u_l$, and $([u_p],[u_q]) \in g(b_{pq})$ for all $(p,q) \in \rho$ (definition of g)

iff $\exists U_0,\dots,U_n \in U$ with $[i] = U_k$, $[j] = U_l$, and $(U_p,U_q) \in g(b_{pq})$ for all $(p,q) \in \rho$ (replacing representatives by equivalence classes)

iff $([i],[j]) \in Q^{kl}_{n\rho}((g(b_{pq})))$ (Definition 6(iii)).

Now, by assumption, for each nonzero $a \in A$ there exists a sequential ultrafilter network Φ_a which gives rise to a homomorphism $g_a : \mathbf{A} \to \mathsf{SQ}(\mathbb{T}_{\Phi_a})$. So we define $h : \mathbf{A} \to \prod_{a\neq 0} \mathsf{Q}(\mathbb{T}_{\Phi_a})$ by $h(x) = (g_a(x))_{0\neq a\in A}$. This is easily seen to be a homomorphism, and by the remark after Definition 2, h maps into (an isomorphic copy of) a sequential set Q-algebra. Since congruences in Boolean algebra with operators are determined by ideals, in order to check that h is an embedding, it suffices to show that for any nonzero $b \in A$, $h(b) \neq 0$. Fortunately, from Definition 6(i) we get some $(i,j) \in \mathbb{T}_{\Phi_b}$ such that $b \in \Phi_b(i,j)$, hence $([i],[j]) \in g_b(b)$. □

We now turn to the question of existence of sequential ultrafilter networks. Let $\mathbf{A}$ be an SQ-algebra. We will build the networks from certain chains of matrices, which may be viewed as 'partial representations'. Recall that for a relation σ on m and a function $f : m \to n$, $f[\sigma] = \{(f(p),f(q)) : (p,q) \in \sigma\}$.

Definition 8. Let $\sigma \in \mathrm{rt}(m)$ and $\rho \in \mathrm{rt}(n)$. For a σ-matrix $\underline{b}$ and a ρ-matrix $\underline{c}$, a function $f : m \to n$ is called an *embedding of* $\underline{b}$ *into* $\underline{c}$ if $f[\sigma] \subseteq \rho$ and $c_{f(i)f(j)} \leq b_{ij}$ for all $i,j < m$. We also say that $\underline{c}$ is an *extension* of $\underline{b}$, in symbols $\underline{b} \subseteq \underline{c}$, if such an f exists.

For $\rho \in \mathrm{rt}(n)$, $\underline{c}$ is said to be a *ρ-consistent* matrix if $\underline{c}$ is a ρ-matrix and $Q^{kl}_{n\rho}(\underline{c}) \neq 0$ for some $(k,l) \in \rho$.

A *first-degree defect* of a ρ-consistent matrix $\underline{c}$ is a pair $((i,j),d) \in \rho \times A$ such that $c_{ij} \not\leq d$ and $c_{ij} \not\leq d^-$. A *second-degree defect* of a ρ-consistent matrix $\underline{c}$ is a triple $((i,j),(k,l),\underline{b}) \in \rho \times \rho \times A^\sigma$ such that $c_{ij} \leq Q^{kl}_{m\sigma}(\underline{b})$ but there is no embedding $f : m \to n$ of $\underline{b}$ into $\underline{c}$ such that $f(k) = i$, $f(l) = j$.

The first observation is a simple consequence of Lemma 3, using $y = 1$ and then applying the meet law.

Lemma 9. *If* $Q^{kl}_{n\rho}(\underline{c}) = 0$ *for some* $(k,l) \in \rho$ *then* $Q^{k'l'}_{n\rho}(\underline{c}) = 0$ *for all* $(k',l') \in \rho$.

Lemma 10. *Any ρ-consistent matrix $\underline{c}$ with a first-degree defect $((i,j),d)$ has a ρ-consistent extension in which $((i,j),d)$ is not a defect.*

Proof. Consider the ρ-matrices $\underline{c}[c_{ij}/c_{ij}\cdot d]$ and $\underline{c}[c_{ij}/c_{ij}\cdot d^-]$. Clearly they are both extensions of $\underline{c}$, and since the Q-operations are additive, at least one of them is ρ-consistent. □

Lemma 11. *Let $\underline{c}$ be a ρ-consistent matrix, and $((i,j),(k',l'),\underline{b})$ a second-degree defect of $\underline{c}$. Then there exists a relation τ on $n+m-2$ and a τ-matrix $\underline{d}$ such that*

(i) *$e : n \to n+m-2$ given by $e(p) = p$ is an embedding of $\underline{c}$ into $\underline{d}$,*
(ii) *$\rho \subseteq \tau$,*
(iii) *$\underline{d}$ is τ-consistent and*
(iv) *$((i,j),(k',l'),\underline{b})$ is not a defect of $\underline{d}$*

Proof. Since $((i,j),(k',l'),\underline{b})$ is a second-degree defect of $\underline{c}$, we have $c_{ij} \le Q^{k'l'}_{m\sigma}(\underline{b})$. Let $f : m \to n+m$ be given by $f(p) = p+n$. We first construct a relation τ' which is the reflexive, transitive closure of $\rho \cup f[\sigma] \cup \{i,k'\}^2 \cup \{j,l'\}^2$, and a τ'-matrix $\underline{d}'$ of size $m+n$ by defining

$$d'_{pq} = \begin{cases} c_{pq} & \text{if } (p,q) \in \rho \\ b_{p-n,q-n} & \text{if } (p,q) \in \sigma \\ 1\text{'} & \text{if } \{p,q\} = \{i,k'\} \text{ or } \{p,q\} = \{j,l'\} \\ 1 & \text{otherwise.} \end{cases}$$

Since $\underline{c}$ is ρ-consistent, $0 \ne Q^{kl}_{n\rho}(\underline{c}) = Q^{kl}_{n\rho}(\underline{c}[c_{ij}/c_{ij}\cdot Q^{k'l'}_{m\sigma}(\underline{b})]) \le Q^{kl}_{n+m,\tau'}(\underline{d}')$ by the expand law.

We now apply the contract rule twice to $\underline{d}'$, first identifying the indices i,k' and then the indices j,l'. The matrix $\underline{d}$ is the argument on the right hand side of the second application of the contract rule, so it follows that $Q^{kl}_{n+m,\tau'}(\underline{d}') \le Q^{kl}_{n+m-2,\tau}(\underline{d})$, where $\tau = f_2[f_1[\tau']]$ and f_1, f_2 are the two contraction mappings. Hence $\underline{d}$ is τ-consistent, and (iv) holds since $g = f_2 \circ f_1 \circ f : m \to m+n-2$ is an embedding of $\underline{b}$ into $\underline{d}$ which satisfies $g(k') = i$ and $g(l') = j$.

Finally, (i) and (ii) hold since the contraction mappings are identity maps on the indices $0,\ldots,n-1$. □

We are now ready for the main lemma where the sequential ultrafilter networks are constructed inductively.

Lemma 12. *Let* **A** *be a countable SQ-algebra and a and non-zero element of* **A**. *Then there exists a sequential ultrafilter network over* **A** *for a.*

Proof. Since we are assuming that $\mathbf{A}$ is countable, we can enumerate all possible first and second-degree defects: let

$$\{C_0, C_1, \ldots\} = \omega^2 \times A \text{ and}$$
$$\{D_0, D_1, \ldots\} = \omega^2 \times \omega^2 \times \bigcup\{A^\sigma : \sigma \in \mathrm{rt}(m),\ m < \omega\}.$$

base step Let $\underline{c}^{(0)} = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$ and $\rho_0 = \{(0,0),(0,1),(1,1)\}$. Since a is assumed to be non-zero, $\underline{c}^{(0)}$ is ρ_0-consistent by the idmap law.

odd steps Assume that $\underline{c}^{(2r)}$ and ρ_{2r} have been defined for some $r < \omega$. If $\underline{c}^{(2r)}$ has no first-degree defects, let $\underline{c}^{(2r+1)} = \underline{c}^{(2r)}$. Otherwise let C_p be the first first-degree defect of $\underline{c}^{(2r)}$. By Lemma 10 there is a ρ_{2r}-consistent extension of $\underline{c}^{(2r)}$ in which C_p is not a defect, so we define $\underline{c}^{(2r+1)}$ to be this extension. In either case we take $\rho_{2r+1} = \rho_{2r}$.

even steps As in the previous case, but using second-degree defects and Lemma 11 with $\rho_{2r+2} = \tau$.

This process gives a chain of relations $\rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \ldots$ and a chain of matrices $\underline{c}^{(0)} \subseteq \underline{c}^{(1)} \subseteq \underline{c}^{(2)} \subseteq \ldots$ such that each $\underline{c}^{(r)}$ is ρ_r-consistent. By construction we have $c_{ij}^{(s)} \leq c_{ij}^{(r)}$ whenever $r \leq s$ and $(i,j) \in \rho_r$. We now define the sequential ultrafilter network Φ on $T_\Phi = \bigcup_{r<\omega} \rho_r$ by

$$\Phi(i,j) = \{b \in A : (i,j) \in \rho_r \text{ and } c_{ij}^{(r)} \leq b \text{ for some } r < \omega\}.$$

It remains to check that Φ satisfies Definition 6. First we show that $\Phi(i,j)$ is an ultrafilter for each $(i,j) \in T_\Phi$. Suppose $b, d \in \Phi(i,j)$. Then there exist $r, s < \omega$ such that $c_{ij}^{(r)} \leq b$ and $c_{ij}^{(s)} \leq d$. We may assume that $r \leq s$, whence $c_{ij}^{(s)} \leq b \cdot d$, so $b \cdot d \in \Phi(i,j)$. Also, $0 \notin \Phi(i,j)$ since $c_{ij}^{(r)} \neq 0$ for all $(i,j) \in \rho_r$ by the normal law and ρ_r-consistency. Clearly $\Phi(i,j)$ is upward closed, so we have shown $\Phi(i,j)$ is a proper filter.

Suppose for some $d \in A$, $d \notin \Phi(i,j)$. By construction of T_Φ and the sequences of matrices and relations, there is a stage $r < \omega$ such that $(i,j) \in \rho_r$ and the first-degree defect $((i,j),d)$ does not occur in $\underline{c}^{(r)}$ (since every defect is eventually repaired). Hence $c_{ij}^{(r)} \leq d$ or $c_{ij}^{(r)} \leq d^-$. Since we assumed $d \notin \Phi(i,j)$, we have $c_{ij}^{(r)} \leq d^-$, so by definition of Φ, it follows that $d^- \in \Phi(i,j)$. Hence $\Phi(i,j)$ is an ultrafilter.

Definition 6(i) is built in at the base step. To see that (ii) holds, observe that $Q_{n\rho_r}^{kl}(\underline{c}^{(r)}) \neq 0$ for all $r < \omega$, by ρ_r-consistency. Since $\Phi(i,i)$ is an ultrafilter, it contains either 1' or 0'. But the latter is impossible because the subid and normal law imply that $c_{ii}^{(r)} \cdot 1' \neq 0$.

Finally, we need to check (iii). Let $m \in \omega$, $(k,l) \in \rho \in \mathrm{rt}(m)$, $(i,j) \in T_\Phi$ and consider a ρ-matrix $\underline{b}$. For the forward direction, suppose $Q^{kl}_{m\rho}(\underline{b}) \in \Phi(i,j)$. Then there exists $r \in \omega$ such that $c^{(r)}_{ij} \leq Q^{kl}_{m\rho}(\underline{b})$. By construction, there exists $s \geq r$ such that the triple $((i,j),(k,l),\underline{b})$ is not a defect of $\underline{c}^{(s)}$. Let n be the size of $\underline{c}^{(s)}$. It follows from the definition of second-degree defect that there is an embedding $f : m \to n$ of $\underline{b}$ into $\underline{c}^{(s)}$ with $f(k) = i$, $f(l) = j$. This means $f[\rho] \subseteq \rho_s$ and $c^{(s)}_{f(p)f(q)} \leq b_{pq}$ for all $(p,q) \in \rho$, hence $(f(p),f(q)) \in \rho_s \subseteq T_\Phi$ and $b_{pq} \in \Phi(f(p),f(q))$ for all $(p,q) \in \rho$ as required.

For the reverse direction, suppose there exist $u_0,\ldots,u_{m-1} \in \omega$ with $i = u_k$, $j = u_l$ such that $(u_p,u_q) \in T_\Phi$ and $b_{pq} \in \Phi(u_p,u_q)$ for all $(p,q) \in \rho$. Then, by construction of Φ, there exists $r < \omega$ such that $c^{(r)}_{u_p u_q} \leq b_{pq}$ for all $(p,q) \in \rho$ (r is the maximum of the r's that exists for each $(p,q) \in \rho$).

Let n be the size of $\underline{c}^{(r)}$ and $d = Q^{ij}_{n\rho_r}(\underline{c}^{(r)})$. We claim that $d \in \Phi(i,j)$. If not, then $d^- \in \Phi(i,j)$, since $\Phi(i,j)$ is an ultrafilter. So there would exist an $s < \omega$ such that $c^{(s)}_{ij} \leq d^-$ and we may assume that $s > r$. Hence $0 = c^{(s)}_{ij} \cdot d = Q^{ij}_{n\rho_r}(\underline{c}^{(r)}[c^{(r)}_{ij}/c^{(s)}_{ij}])$ by the meet law. But

$$0 \neq Q^{ij}_{n'\rho_s}(\underline{c}^{(s)}) \leq Q^{ij}_{n\rho_r}(\underline{c}^{(r)}[c^{(r)}_{ij}/c^{(s)}_{ij}])$$

by ρ_s-consistency, the rename law and monotonicity (here n' is the size of ρ_s, and in the rename law we use $f : n \to n'$ given by $f(p) = p$). This contradiction establishes the claim.

Using the rename law again with $f(p) = u_p$, and monotonicity, we have $Q^{ij}_{n\rho_r}(\underline{c}^{(r)}) \leq Q^{kl}_{m\rho}(\underline{b})$. Hence $Q^{kl}_{m\rho}(\underline{b}) \in \Phi(i,j)$ as required. □

Theorem 13. *The variety* SQ *of all SQ-algebras coincides with the class* RSQ *of all representable SQ-algebras.*

Proof. Lemma 7 and Lemma 12 show that countable SQ-algebras are representable. The following standard argument extends this result to all SQ-algebras.

Let $\mathbf{A}$ be any SQ-algebra, and define $\mathcal{L}_\mathbf{A}$ to be the first-order language with binary predicate symbols P_a for each $a \in A$. Consider the theory $\mathcal{T}_\mathbf{A}$ given by the following axioms, where a, b range over A, $\underline{a} \in A^{n\times n}$, $(k,l) \in \rho \in \mathrm{rt}(n)$, $n \in \omega$.

1. $\forall v(P_1(v,v))$, $\forall uvw(P_1(u,v) \wedge P_1(v,w) \to P_1(u,w))$
2. $\forall vw(P_{a\cdot b}(v,w) \leftrightarrow P_a(v,w) \wedge P_b(v,w))$
3. $\forall vw(P_1(v,w) \to [P_{a^-}(v,w) \leftrightarrow \neg P_a(v,w)])$
4. $\forall vw(P_{1'}(v,w) \leftrightarrow v = w)$
5. $\forall vw(P_{Q^{kl}_{n\rho}(\underline{a})}(v,w) \leftrightarrow$
$$\exists u_0 \ldots u_{n-1}(v = u_k \wedge w = u_l \wedge \textstyle\bigwedge_{(i,j)\in\rho} P_{a_{ij}}(u_i,u_j))).$$

A model of $\mathcal{T}_{\mathbf{A}}$ corresponds to a representation of $\mathbf{A}$, so by the compactness theorem it suffices to show that each finite subset of $\mathcal{T}_{\mathbf{A}}$ has a model.

Let $\mathcal{F}$ be a finite subset of $\mathcal{T}_{\mathbf{A}}$ and consider the subalgebra $\mathbf{F}$ of $\mathbf{A}$ that is generated by the elements a of A for which the predicate symbol P_a occurs in $\mathcal{F}$. Since SQ-algebras have only countably many operations, $\mathbf{F}$ is countable. Therefore it is representable, whence its theory $\mathcal{T}_{\mathbf{F}}$ has a model. To see that this is also a model of $\mathcal{F}$, it is enough to check that $\mathcal{F} \subseteq \mathcal{T}_{\mathbf{F}}$. But any sentence in $\mathcal{F}$ contains only predicate symbols that are associated with generators of $\mathbf{F}$, hence such a sentence also occurs in $\mathcal{T}_{\mathbf{F}}$. ☐

Corollary 14. *An algebra* $\mathbf{A} = (A, +, ^-, ;, \triangleright, \triangleleft, 1')$ *is a representable sequential algebra iff it is a subreduct of an SQ-algebra.*

If an SQ-algebra satisfies $1^\vee = 1$ then it is term-equivalent to a Q-algebra. Thus we can deduce the Stebletsova-Venema result (with a slightly different equational basis) from the above theorem.

4 Complex algebras of observation spaces

Hoare and von Karger [14] have argued for a general theory of observations that underlies many different models of process semantics. Extensive discussion, motivation and examples can be found in [12] [13]. What these models all have in common is an associative partial composition operation and two unary operations which map each observation to a left and right unit observation respectively.

Definition 15. An *observation space* is a partial algebra of the form $\mathbf{O} = (O, ;, \overrightarrow{\ }, \overleftarrow{\ })$, where $\overrightarrow{\ }, \overleftarrow{\ }$ are total unary operations and for all $x, y, z \in O$

(i) $x;y$ is defined iff $\overrightarrow{x} = \overleftarrow{y}$
(ii) $\overrightarrow{\overrightarrow{x}} = \overrightarrow{x} = \overleftarrow{\overrightarrow{x}}$, $\overrightarrow{\overleftarrow{x}} = \overleftarrow{x} = \overleftarrow{\overleftarrow{x}}$
(iii) $\overleftarrow{x};x = x$ and $x;\overrightarrow{x} = x$
(iv) $\overleftarrow{x;y} = \overleftarrow{x}$ and $\overrightarrow{x;y} = \overrightarrow{y}$ if $x;y$ is defined
(v) $(x;y);z = x;(y;z)$ whenever both sides are defined
(vi) $x;y = \overleftarrow{x}$ implies $y;x = \overleftarrow{y}$

Conditions (i)-(v) are the axioms of a small category, and (vi) states that every retraction is an isomorphism. Note that if $x;y = \overleftarrow{x} = x;y'$ then $y;x = \overleftarrow{y}$, so $\overrightarrow{x} = \overrightarrow{y;x} = \overrightarrow{\overleftarrow{y}} = \overleftarrow{y}$ and similarly $\overrightarrow{x} = \overleftarrow{y'}$. It follows that $y = \overleftarrow{y};y = y;x;y = y;x;y' = \overleftarrow{y};y' = \overrightarrow{x};y' = \overleftarrow{y'};y' = y'$, hence there is a unique y with the property that $x;y = \overleftarrow{x}$. If such a y exists, it is called the inverse of x and is denoted by x^{-1}.

A *pair observation space* is of the form $\mathbf{T} = (T, ;, \overleftarrow{\ }, \overrightarrow{\ })$, where T is a reflexive transitive relation on a set U, and for all $(s,t),(u,v) \in T$,

$$(s,t);(u,v) = \begin{cases} (s,v) & \text{if } t = u \\ \text{undefined} & \text{otherwise} \end{cases}$$

$\overleftarrow{(u,v)} = (u,u)$ and $\overrightarrow{(u,v)} = (v,v)$.

The *conjugated complex algebra* of an observation space $\mathbf{O}$ is

$$\mathrm{Cm}^c(\mathbf{O}) = (\mathcal{P}(\mathbf{O}), \cup, ^-, ;, \triangleright, \triangleleft, \mathbf{1'})$$

where for $R, S \subseteq O$

$$\begin{aligned} R;S &= \{x;y : \overrightarrow{x} = \overleftarrow{y}, x \in R, y \in S\} \\ R \triangleright S &= \{z \in O : x; z = y \text{ for some } x \in R, y \in S\} \\ R \triangleleft S &= \{z \in O : z; y = x \text{ for some } x \in R, y \in S\} \\ 1' &= \{\overrightarrow{x} : x \in O\} \end{aligned}$$

Axiom (ii) of observation spaces implies that $1' = \{\overleftarrow{x} : x \in O\}$. It is almost immediate that the conjugated complex algebra of any pair observation space is a representable sequential algebra. The following generalization of this result is proved by the same approach as Theorem 2 in [5].

Theorem 16. *The conjugated complex algebra of any observation space is a representable sequential algebra. Hence* $\mathrm{Var}(\mathrm{Cm}^c(\mathsf{Obs})) = \mathsf{RSeA}$.

Proof. Let $\mathbf{O}$ be an observation space and define $E = \{\overrightarrow{a} \mid a \in O\}$ and $D = O \backslash E$. For a binary relation R, let $\mathrm{dom}R = \{\langle u,u\rangle \mid \langle u,v\rangle \in R\}$ and $\mathrm{rng}R = \{\langle v,v\rangle \mid \langle u,v\rangle \in R\}$.

We would like to find a set U and a collection $\{R_a \subseteq U^2 \mid a \in O\}$ of pairwise disjoint nonempty binary relations on U such that $R_a \circ R_b = R_{a;b}$, $\mathrm{dom}R_a = R_{\overleftarrow{a}}$, $\mathrm{rng}R_a = R_{\overrightarrow{a}}$, and $\mathrm{id}_U = \bigcup_{a \in O} R_{\overrightarrow{a}}$.

The set U and the relations R_a are defined step-by-step using transfinite induction. A detailed discussion of this method for representing relation algebras can be found in [1] or [2]. Our setting of sequential algebras requires some modifications, and we take a rather informal approach here.

Suppose at the κth step we have an "approximate embedding", by which we mean a collection of disjoint relations $R_{a,\kappa}$ on a set U_κ such that $R_{a,\kappa} \circ R_{b,\kappa} \subseteq R_{(a;b),\kappa}$, $\mathrm{dom}R_{a,\kappa} = R_{\overleftarrow{a},\kappa}$, $\mathrm{rng}R_{a,\kappa} = R_{\overrightarrow{a},\kappa}$, and $R_a \cap \mathrm{id}_U = \emptyset$ for $a, b \in D$.

Using the well-ordering principle, we list all the pairs in $R_{(a;b),\kappa} \setminus (R_{a,\kappa} \circ R_{b,\kappa})$ for all $a, b \in D$, and proceed to extend U_κ and the $R_{a,\kappa}$ so as to eventually obtain $R_a \circ R_b = R_{a;b}$, where R_a is the union of all the $R_{a,\kappa}$ constructed along the way.

For all $a, b \in D$, and all $\langle u, v\rangle \in R_{(a;b),\kappa} \backslash (R_{a,\kappa} \circ R_{b,\kappa})$, choose $w \notin U_\kappa$ and let

$U_{\kappa+1} = U_\kappa \cup \{w\}$

$R'_z = \bigcup\{R_{x,\kappa} \circ \{\langle u, w\rangle\} \mid x; a = z\} \cup \bigcup\{\{\langle w, v\rangle\} \circ R_{y,\kappa} \mid b; y = z\}$

$R_{a,\kappa+1} = R_{a,\kappa} \cup R'_a \cup \{\langle u, w\rangle\}$ $(\cup\{\langle w, u\rangle\}$ if $a = a^{-1})$

$R_{b,\kappa+1} = R_{b,\kappa} \cup R'_b \cup \{\langle w, v\rangle\}$ $(\cup\{\langle v, w\rangle\}$ if $b = b^{-1})$

$R_{a^{-1},\kappa+1} = R_{a^{-1},\kappa} \cup R'_{a^{-1}} \cup \{\langle w, u\rangle\}$ if $a \neq a^{-1}$ exists

$R_{b^{-1},\kappa+1} = R_{b^{-1},\kappa} \cup R'_{b^{-1}} \cup \{\langle v, w\rangle\}$ if $b \neq b^{-1}$ exists

$R_{\overrightarrow{a},\kappa+1} = R_{\overrightarrow{a},\kappa} \cup \{\langle w, w\rangle\}$

$R_{z,\kappa+1} = R_{z,\kappa} \cup R'_z$ if $z \in O \setminus \{a, b, a^{-1}, b^{-1}, \overrightarrow{a}\}$.

For limit ordinals λ, we let $U_\lambda = \bigcup_{\kappa<\lambda} U_\kappa$ and $R_{x,\lambda} = \bigcup_{\kappa<\lambda} R_{x,\kappa}$ for all $x \in O$.

It remains to check that the new relations are still an approximate embedding. In the limit ordinal case this is immediate. In the successor ordinal case, the relations are pairwise disjoint by construction. For $z \in E$, $R'_z = \emptyset$ hence $\text{dom}R_{x,\kappa+1} = R_{\overleftarrow{x},\kappa} = R_{\overleftarrow{x},\kappa+1}$ unless $x = a = a^{-1}$. In the latter case, $\overleftarrow{a} = \overrightarrow{a}$ implies $R_{\overleftarrow{a},\kappa+1} = R_{\overrightarrow{a},\kappa+1} = \text{dom}R_{a,\kappa} \cup \{\langle w, w\rangle\} = \text{dom}R_{a,\kappa+1}$. The argument for $\text{rng}R_{x,\kappa+1} = R_{\overrightarrow{x},\kappa+1}$ is similar.

Checking the inclusion $R_{c,\kappa+1} \circ R_{d,\kappa+1} \subseteq R_{(c;d),\kappa+1}$ involves several cases, depending on whether $c, d \in \{a, b, a^{-1}, b^{-1}\}$. Since they are similar, we consider only the case $c, d \notin \{a, b, a^{-1}, b^{-1}\}$. Let $\langle p, q\rangle \in R_{c,\kappa+1} \circ R_{d,\kappa+1}$. Then there exists $r \in U_{\kappa+1}$ such that $\langle p, r\rangle \in R_{c,\kappa+1}$ and $\langle r, q\rangle \in R_{d,\kappa+1}$. If $r \in U_\kappa$ then the conclusion follows from the assumption that $R_{z,\kappa}$ is an approximate embedding. So we may assume $r = w$ (the unique element in $U_{\kappa+1} \setminus U_\kappa$). By construction $\langle p, u\rangle \in R_{x,\kappa}$ for some x such that $x; a = c$ and $\langle v, q\rangle \in R_{y,\kappa}$ for some y such that $b; y = d$. Since $\langle u, v\rangle \in R_{(a;b),\kappa}$ it follows that $\langle p, q\rangle \in R_{x;(a;b);y,\kappa}$. By associativity we have $R_{x;(a;b);y,\kappa} \subseteq R_{(x;a);(b;y),\kappa+1} = R_{(c;d),\kappa+1}$, as required.

Finally, to start the construction take $U_0 = D \cup D'$ where $D' = D \times \{0\}$, and for each $a \in D$, define $a' = \langle a, 0\rangle$. If $a = a^{-1}$ exists, take $R_{a,0} = \{\langle a, a'\rangle, \langle a', a\rangle\}$, $R_{\overrightarrow{a},0} = \{\langle a, a\rangle, \langle a', a'\rangle\}$ and otherwise take $R_{a,0} = \{\langle a, a'\rangle\}$, $R_{\overleftarrow{a},0} = \{\langle a, a\rangle\}$ and $R_{\overrightarrow{a},0} = \{\langle a', a'\rangle\}$. It is straightforward to check that this is an approximate embedding. □

From the preceding result and Corollary 14 we deduce the following.

Corollary 17. *The conjugated complex algebra of any observation space is a subreduct of an SQ-algebra.*

5 Uncountably many minimal varieties of representable sequential algebras

In this section we construct an uncountable family of reflexive and transitive relations on $\mathbb{Z}$, and show that the residuated complex algebras of these pair observation spaces generate pairwise distinct minimal varieties.

Let S be any subset of negative integers, and define a binary relation T_S on $\mathbb{Z}$ by $T_S = \{(m,n) \mid 0 \leq m \leq n\} \cup \{(m,-n) \mid 0 \leq m \leq n \in S\} \cup id_S$. It is easy to check that T_S is a partial order, hence it is the universe of a pair observation space. Let $\mathbf{A}_S$ be the conjugated complex algebra of this structure.

We first note that each $\mathbf{A}_S$ is 0-generated, i.e. has no nontrivial subalgebras. This follows from the observation that the domain operator $x^\delta = (1 \lhd x)1$' and range operator $x^\rho = (x \rhd 1)1$' do indeed compute the domain and range of any subrelation of T_S. So we have $id_{\mathbb{N}} = id^{-\delta}$, $\{(0,0)\} = T_S^{\rho-}$, $\{(1,1)\} = (T_S \setminus \{(0,0)\}; T_S)^{\rho-} \cdot id_{\mathbb{N}}$, and so on. Since the structure of each $\mathbf{A}_S$ can be described by first-order formulas (involving only ground terms), $\mathbf{A}_S$ is a subalgebra of any nontrivial member of $\mathbf{HSP}_u(\mathbf{A}_s)$, hence $\mathrm{Var}(\mathbf{A}_S)$ is a minimal variety.

Using results from [4] it is also straightforward to check that $\mathbf{A}_S$ is a discriminator algebra. Finally, for distinct subsets S and S', there exist equations (with no variables) that hold in $\mathbf{A}_S$, but not in $\mathbf{A}_{S'}$. Thus we conclude with the following result.

Theorem 18. *The lattice of subvarieties of representable sequential discriminator algebras has continuum many atoms.*

This is in contrast with the well-known corresponding result about relation algebras, where there are only three minimal subvarieties [7].

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